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# DESIGN OF CONCRETE STRUCTURES


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DESIGN OF CONCRETE STRUCTURES

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# PREFACE

This edition, like the previous ones, aims to give sufficient development of the theory of concrete design with illustrative examples to ensure the beginner a thorough understanding of the fundamentals. Complete designs of some of the more common structures are given in order to furnish a vehicle for bringing together all the fundamental theory involved. It presents the material for a first course and one or two advanced courses on the design of reinforced concrete structures covering the entire field of of the customary reinforced concrete design with the exception of some highly specialized and involved unusual structures, such as shell construction and the like.

In order to take into consideration the many new developments and changes in design codes in the last fifteen years, the book has been rewritten, while still retaining the basic method of presentation and the general arrangement of material. Greater emphasis has been given to a thorough discussion of basic structural performance, fundamental mechanics, and physical explanation.

The treatment of rigid frames has been completely revised to include, besides a new treatment of slope deflection, an extensive discussion of the moment-distribution method, including side sway as well as the customary abbreviated methods of analyzing tier buildings for vertical and horizontal loads. As now constituted, the section on rigid frames can be used without reference to other texts to provide all or nearly all the information on indeterminate analysis in connection with an advanced course in reinforced concrete design.

The great attention given in the last few years to ultimate design is of sufficient importance so that the student even in an elementary course should be cognizant of this trend. The same is true of the development of prestressed concrete. Therefore a new section covering these features has been added.

In general, the provisions of the American Concrete Institute's Building Code or those of the American Association of State Highway Officials have been followed, although examples have been included not completely following those provisions, since all building and highway codes do not conform to these standards in all respects. The new system of numbering bar sizes has also been used throughout.

Owing to the untimely death of Professor C. E. O'Rourke, the coaut. of the first four editions, the present fifth edition has been p ·pared L. C. Urquhart and George Winter.

Acknowledgment is made to Professor W. McGuire of Cornell Uni versity for his extensive contributions in revising Sections 8 and 11 anc to H. Jack Leonard of East Orange, N.J., for his assistance on Section 11

L. C. URQUHART
GEORGE WINTER

# CONTENTS

PREFACE . . . . . v

SECTION 1

PLAIN CONCRETE . . . . . 1

Concrete and Cements, 1. Aggregates, 3. Proportioning, 9. Design of Mortar Mixtures, 10. Water-Cement Ratio, 11. Yield and Strength, 13. Concrete Mixtures, 14. Design of Concrete Mixtures, 15. Control of Concrete Mixtures, 19. Mixing and Placing, 20. Curing, 21. Effect of Freezing, 22. Admixtures, 23. Compressive, Tensile, Transverse, and Shearing Strengths, 25. Elasticity, 26. Plastic Flow, 27. Contraction and Expansion, 28. Bond, 29. Weight, 29. Resistance to Fire, 29. Weathering Qualities, 30. Abrasive Resistance, 30. Effect of Various Substances on Hardened Concrete, 31.

SECTION 2

GENERAL PROPERTIES OF REINFORCED CONCRETE . . . . . 33

Types of Reinforcement, 33. Grade of Steel, 34. Coefficient of Expansion, 35. Modulus of Elasticity, 35. Bond between Concrete and Steel, 35. Development of Full Strength of Reinforcement in Bond, 39. Reinforced Concrete in Tension, 40.

SECTION 3

BEAMS AND SLABS . . . . . 41

Stresses in Homogeneous Beams, 41. Assumptions in the Theory of Flexure, 42. Plain Concrete Beams, 43. Flexure Analysis of Reinforced Concrete Beams, 44. Application to Design and Review, 48. Approximate Design Moments, 49. Placing the Reinforcement, 51. Allowable Unit Stresses, 52. Tables for Rectangular Beams, 56. The Transformed Section, 58. Slabs Supported on Two Sides, 63. Temperature Reinforcement, 64. Slabs Supported on Four Sides, 69. Diagonal Tension, 77. Reinforcement for Diagonal Tension, 83. Design of Web Reinforcement, 87. Spacing of Vertical Stirrups, 88. Arrangement of Bent-up Bars, 89. Bond Stresses, 91. Anchorage of Web Reinforcement, 92. T Beams, 93. Flexure Analysis of T Beams, 95. Shearing Strength of T Beams, 97. Beams Reinforced for Compression, 106. Formulas for Design, 107. Formulas for Review, 109. Diagrams for Review of Beams Reinforced for Compression, 112. Continuous T Beams, 116.

SECTION 4

COLUMNS . . . . . 122

Columns with Spiral and Longitudinal Reinforcement, 122. Columns with Longitudinal Reinforcement and Lateral Ties, 127. Composite, Combination, and Pipe Columns, 128. Unsupported Length and Limiting Dimensions, 129. Flexural Stresses in Columns, 130. Column Tables, 131.

SECTION 5

BENDING AND AXIAL STRESS . . . . . 134

General Theory, 134. Rectangular Sections, 136. Circular Sections, 138. Allowable Stresses, 140. Design of Columns, 141. Illustrative Examples, 145.

SECTION 6

CONTINUOUS BEAMS AND FRAMES . . . . . 150

Continuity of Reinforced Concrete Structures, 150. Continuous Beams, 152. Slope-deflection Method, 154. Tables of Fixed-end Moments, 163. Sidesway, 169. Moment-distribution Method, 173. With Sidesway, 181. Building Frame, Vertical Loads, 186. Building Frame, Horizontal Loads, 197. Effective Spans, Moments of Inertia, Support Conditions, 201. Preliminary Design, 204.

SECTION 7

FOOTINGS . . . . . 207

Bearing Capacity of Soils, 207. Proportioning of Footing Areas, 209. Wall Footings, 215. Single-column Footings, 217. Multiple-column Footings, 224. Strap Footings, 235. Footings on Piles, 240.

SECTION 8

REINFORCED CONCRETE BUILDINGS . . . . . 243

Floor and Roof Loads, 244. Floor and Roof Systems, 245. Walls and Partitions, 259. Stairways, 264. Design of Beam-and-Girder Floor, 266. Design of Ribbed Floor, 278. Flat-slab Floors, 287. Design of Flat-slab Building, 298.

SECTION 9

RETAINING WALLS . . . . . 327

Earth Pressure, 328. External Stability, 334. Drainage and Other Details, 337. Design of Gravity Wall, 338. Design of Cantilever Wall, 340. Design of Counterfort Wall, 346.

SECTION 10

ARCHES . . . . . 350

Types, Uses, Advantages, 350. Theory of Two-hinged Arches, 354. Theory of Fixed Arches, 359. Evaluation of Integrals, 366. Preliminary Design, 367. Shape of Axis, 367. Tables of Moments and Reactions in

Roof and Bridge Arches, 369. Hinges in Reinforced Concrete Structures, 375. Design of a Two-hinged Roof Arch, 377.

SECTION 11

SLAB, BEAM, AND GIRDER BRIDGES . . . . . 387

Types of Bridges, 387. Loads and Design Specifications, 389. Abutments, 397. Construction Details, 400. Design of Slab Bridge, 402. Design of Deck-girder Bridge, 404. Design of Through-girder Bridge, 413. Composite Bridge Construction, 418.

SECTION 12

MODERN DEVELOPMENTS IN CONCRETE DESIGN . . . . . 421

Ultimate or Plastic Design, 424. Stresses in Beams, 426. Jensen-PCA Method, 428. Whitney Method, 434. Practical Significance of Ultimate Design, 438. Prestressed Concrete, 441. Principles and History, 441. Stresses on Beams, 444. Design and Construction of Pretensioned Beams, 446. Design and Construction of Posttensioned Beams, 450. Additional Information, 454.

APPENDIX A. Standard Notations . . . . . 457

APPENDIX B. Allowable Unit Stresses (ACI Code) . . . . . 459

APPENDIX C. Flat Slabs with Square or Rectangular Panels (ACI Code) . 461

APPENDIX D. Tables and Diagrams . . . . . 468

INDEX . . . . . 501

# SECTION 1

# PLAIN CONCRETE

**1.1. Introductory.** Since the beginning of the twentieth century, concrete has taken its place as one of the most useful and important structural materials. Owing to the comparative ease with which it can be molded into any desired shape, its structural uses are almost unlimited; therefore wherever portland cement and suitable aggregates can be obtained, it has, for certain classes of work, displaced older materials. This apparent ease with which concrete may be prepared has led to its being employed by anyone who feels that the material is suited to his particular purpose. In many instances, proper knowledge of the substance and skill in its manufacture are not available, so that the resultant concrete is little more than a bulky, heavy material, lacking the strength and other properties which it should have attained, and often failing to fulfill that purpose for which it was intended.

To the individuals who obtain such results, concrete is merely a shoveled-together mass of cement, sand, stone, and water, which in a short time attains a varying degree of hardness and an uncertain strength. To the engineer who is more or less familiar with the many factors and variables entering into its manufacture, the process of making concrete does not appear quite so elementary. Experience shows that the quantity and quality of cement, aggregates, and water and the process of mixing and curing are all involved in the production of concrete. Results are dependent upon all these variables. It is, therefore, the problem of the engineer so to study and control these factors that a concrete of the desired quality may be obtained.

**1.2. Cements.** Portland cement is the product obtained by finely pulverizing the clinker obtained by calcining to incipient fusion an intimate and properly proportioned mixture of argillaceous and calcareous materials with no additions subsequent to calcination except water and calcined or uncalcined gypsum. All portland cement used in reinforced concrete construction should pass such standard specifications as those of the American Society for Testing Materials.[1] The color and rapidity

[1] ASTM C 150-49.

of hardening of different brands of cement vary considerably and may be elements requiring special attention for the particular work involved.

Portland cement is manufactured by either the *dry* process or the *wet* process.[1] In the dry process the calcareous material is usually limestone and in the wet process marl. In all cases its content is principally calcium carbonate. The argillaceous material may be shale, clay, argillaceous limestone, or blast furnace slag which provides the silica and alumina.

In the dry process the raw materials are ground separately, mixed in the proper proportions, pulverized, and burned in a kiln. The resulting clinker is cooled and seasoned, materials are added to control the rate of setting, and the clinker is ground to a fine powder. In the wet process the procedure is similar except that the marl is stored in the form of a thin mud in vats and the argillaceous material in powder form is mixed with it before grinding and burning.

The finished product contains from 25 to 61 per cent of tricalcium silicate, from 7 to 44 per cent of dicalcium silicate, about 10 per cent of tricalcium aluminate, about 8 per cent of tetracalcium alumino ferrite, and about 3 per cent of calcium sulfate. The tricalcium silicate and the tricalcium aluminate are the predominating influences in producing strength up to 28 days.[2] The former continues to be effective in producing strength beyond that period, but the effect of the latter rapidly diminishes after 28 days and becomes zero or negative in 1 or 2 years. The dicalcium silicate contributes very little to the strength during the early stages but is the principal strength-producing compound beyond the 28-day period.

*High-early-strength portland cements* are made from mixtures having a high lime content and are ground to a greater degree of fineness than the ordinary portland. The finished product contains a large proportion of tricalcium silicate, the percentage by weight usually being at least 65 and often as high as 75, with a corresponding decrease in the amount of dicalcium silicate. With such a chemical content these cements harden and gain strength very rapidly and their use is economical where their extra cost is justified by the desire for high strength in a short period of time.

*High-alumina cements* contain about 40 per cent of lime, 40 per cent of alumina, and 15 per cent of ferric oxides. While they set at about the same rate as portland, they gain strength much more rapidly on account of the high alumina content, so that in 24 hr a concrete made with a

[1] For description of cement manufacture, see J. B. Johnson's "Materials of Construction," 8th ed., p. 310, John Wiley & Sons, Inc., New York.

[2] For effect of cement composition on mortars and concretes, see *Proc. ASTM*, vol. 34, part II, p. 244.

cement of this type will reach a strength equal to or greater than a concrete made with standard portland will achieve in 28 days. However, as the age of such a concrete increases, there is a tendency toward a slight retrogression in strength.

Other special cements, such as *sand cement, tufa cement, pozzuolana cement,* and *natural cement,* have been used in various projects, but in general they do not produce as strong or as durable a concrete as portland cement. Reference to cement in future discussions means portland cement.

**1.3. Fine Aggregate.** Fine aggregate should consist of sand, stone screenings, or other inert materials with similar characteristics, or a combination thereof, having clean, hard, strong, durable, uncoated grains, and free from injurious amounts of dust, lumps, soft or flaky particles, shale, alkali, organic matter, loam, or other deleterious substances. In general, all particles passing a No. 4 sieve (4 meshes per linear inch) are considered as fine aggregate. Most specifications, however, allow some degree of latitude from this requirement.

It is, in general, advantageous that the fine aggregate be well graded from fine to coarse, but considerable latitude must be allowed in any grading specified in order that economical use of available aggregates may be made. Extremely fine particles, if present in any great amount, are not beneficial to the strength of the resultant concrete, since they furnish so great an excess of surface area to be covered by the cement-water mix.

*Sand.* All sands are derived from rocks which have broken down or disintegrated through the operation of physical agencies. In some cases, in addition to disintegration, there has been more or less decomposition involving the formation of new compounds. The principal disintegrating agencies are temperature changes, weathering, and abrasion. The former cause cracking and a spalling off of particles of the constituent rock, because of the unequal expansion and contraction of the component minerals. The latter may be caused by the flow of water, wind, or glacial action. Chemical decomposition is brought about through the solvent power of water, aided often by the presence of various chemically active substances, such as acids, which are carried by the water.

*Quartz* is the mineral which makes up the bulk of the particles of most sands. This is due to the fact that only the harder constituents of rocks survive disintegration and decomposition, and quartz is a common constituent of most rocks and capable of resisting these destructive agencies. Not all quartz sands, however, make suitable concrete aggregates, for comparatively small amounts of organic impurities will render the sand unfit for use. Sandstone is a common source of quartz sand. Here the character of the binder of the original rock will determine the quality of

the sand, since the individual particles of the sand are made up of still smaller particles of quartz, bound together by silica, iron oxide, lime carbonate, or clay.

*Pyroxene* and *hornblende* are complex silicates possessing a degree of hardness, strength, and durability slightly inferior to that of quartz. Hornblende has inferior weathering qualities. Feldspars are considerably less strong and durable than quartz. Mica is a very objectionable constituent of sand, being soft, low in strength, and because of its laminated structure, offering opportunity for the percolation of water.

Sand deposits, being usually the result of stream or glacial action, and also of such character that the percolation of surface water through the beds is very easy, are often contaminated by matter of organic origin carried in suspension by the water. Thus the coating of the grains by such substances as tannic acid, sewage, manure, sugar, etc., is frequently encountered Such a coating on the sand grains appears not only to prevent the cement from adhering but also affects it chemically. Thus the effect of such substances is extremely detrimental, but at the same time their presence is hard to detect, a fact which increases the importance of carefully testing concrete sands.

*Screenings.* Limestone screenings or crusher dust are sometimes used as fine aggregates, but the concrete made therefrom is usually inferior in quality to that made with an average sand and is apt to be or become more permeable.

**1.4. Coarse Aggregate.** Coarse aggregate should consist of crushed stone, gravel, or other approved materials of similar characteristics or combinations thereof, having clean, hard, strong, durable uncoated particles free from injurious amounts of soft, friable, thin, elongated, or laminated pieces, alkali, and organic or other deleterious matter.

*Crushed Stone.* The quality of an aggregate of this type obviously depends upon the character of the original rock. The principal classes of rocks from which aggregates are derived are granites, traprocks, limestones, and sandstones. Granite is an igneous rock whose principal mineral constituents are quartz and feldspar, with varying amounts of mica, hornblende, and other materials. The structural qualities of granite vary greatly, but granites as a class rank among the hardest, strongest, and most durable stones. Traprock includes basalt, diabase, and a number of other igneous rocks possessing similar chemical and physical properties. The principal mineral constituents of most of these rocks are pyroxene and feldspar. They are generally rather fine grained, hard, tough, and durable. Limestone is a sedimentary rock which contains carbonate of lime, calcite, or carbonate of lime together with a double carbonate of lime and magnesia, dolomite, as the essential constituent. Sand and clay are common impurities, some varieties of which contain

large amounts of shells and other fossils. Limestones vary greatly in structure, strength, hardness, and durability, and, although there are some limestones which have superior structural qualities, the average limestone is inferior to average granites and traprocks as a concrete aggregate. Sandstones consist of grains of varying sizes, chiefly quartz, bound together by various cementing agencies or binders. A silicious binder produces a sandstone of the greatest structural strength, while an iron oxide or lime carbonate binder is much less efficient. A sandstone whose binder is clay is the least valuable of all as a concrete aggregate.

The maximum size of coarse aggregate advisable depends upon the character of the work. Since the stone is one of the strongest constituents of concrete, it is desirable to have as many and as large particles as possible. The greater the size of the particles, the less surface area there is to be coated, and the smaller amount of cement required for a concrete of given strength. When, however, the maximum size is comparatively large, it is very important that it be well graded down to the minimum size in order to make a dense, compact mass. In small reinforced concrete members the maximum size usable is as small as $\frac{3}{4}$ in. in diameter, and rarely in any reinforced work is a diameter greater than $1\frac{1}{2}$ in. advisable. On the other hand, for large, massive work, with no structural reinforcement, much larger sizes may be used to advantage.

*Gravel.* Gravel of good quality makes a suitable concrete aggregate. Gravel is nothing more than pieces of natural rock broken away from the parent ledges and worn down by stream or glacial action. Its strength as an aggregate depends upon the rock from which it came, provided it has not become decayed or coated with objectionable organic matter. Too much emphasis cannot be given to the necessity of determining the cleanness of the gravel. A clayey coating is easily detectable, but the transparent organic coatings which prevent adhesion are not so easily discerned, without chemical analysis, so that many weak and inferior concretes result from the use of an apparently clean but really "dirty" gravel as an aggregate.

Natural gravel may have a large proportion of particles so small as to be classed as "fine aggregate." These may be screened out before using, or, a sieve analysis of the natural gravel being made, the proper amount of additional fine aggregate (if any) to add to obtain the desired proportions may be determined.

*Slag.* Slag from blast furnaces is a hard though porous material of high compressive strength, which in some localities can be obtained much more cheaply than stone of good quality. It offers a very rough surface for the adhesion of the cement, and, provided the sulfur content is low, it may make an excellent aggregate for massive concrete construction.

Generally it should not be used in thin sections exposed to any action from water on account of its porosity.

*Cinders.* Cinders as an aggregate have the advantage of making a resultant concrete considerably lighter in weight than that made from stone or gravel. Formerly it was thought that well-burned cinders made a more fire-resisting concrete than other aggregate, but more recent experiences have shown that cinder concrete is little, if any, better in this respect. Cinder concrete is inferior in strength to other concrete, and on account of the danger from the probable sulfur content, it is not used where any great structural strength is required. Its principal use occurs for filling where no great strength is necessary. When used, cinders should be free from unburnt coal or fine ashes.

**1.5. Selection and Tests of Aggregates.** For most concrete, economical considerations dictate that the aggregates be selected from those locally available and the choice often lies between natural deposits of gravel with or without the addition of sand or crushed material and sand or screenings. For work of sufficient importance to justify design and control of the mixture, the physical characteristics, uniformity, and grading of the aggregates must be determined preliminary to such a design.

*Grading.* The grading and the uniformity of grading are important in that a favorable grading reduces the quantity of cement paste required to produce concrete of a given strength, while uniformity of grading ensures the uniform workability of different batches. Grading is evaluated by the sieve analysis,[1] from which are recorded the percentages of a sample passing or retained on a graded series of sieves, No. 100, 50, 30, 16, 8, and 4 for fine aggregates and No. 4, $\frac{3}{8}$, $\frac{3}{4}$, 1.5, 3, etc., for coarse aggregates.

*Grading of Fine Aggregate.* In ASTM C 33-52T it is specified that fine aggregate be graded from coarse to fine within the limits shown in the following table.

| *Sieve size* | *Total passing, % by weight* |
|---|---|
| $\frac{3}{8}$ in. | 100 |
| No. 4 | 95–100 |
| No. 8 | 80–100 |
| No. 16 | 50–85 |
| No. 30 | 25–60 |
| No. 50 | 10–30 |
| No. 100 | 2–10 |

The wide range of grading indicated in the above table should be permitted only when it is economically impracticable to obtain aggregates meeting more restrictive requirements, as a decrease in the percentage passing each sieve will result in a more favorable grading. However, such a restriction should not be carried to extreme, and a range of about

[1] ASTM C 136-49.

20 per cent or less for the No. 16 sieve, about 15 per cent or less for the No. 50 sieve, and about 5 per cent for the No. 100 sieve is the greatest restriction that should be specified.

*Grading and Size of Coarse Aggregate.* In general, the most economical concrete is one which contains the largest proportion of coarse aggregate which can be incorporated in the mixture without undue harshness. It also follows that the larger the maximum size of the coarse aggregate, the smaller will be the required amount of cement paste to surround the surface area of the aggregate. In mass concrete it has been found economical to use aggregate as large as 6 or 9 in., and in structures where watertightness is not important even larger pieces where available prove economical.

In reinforced concrete, however, the size of the member and the distance between reinforcing bars dictate the maximum size of the coarse aggregate that may be used without causing wedging and consequent probable honeycombing in the bottom and sides of the member. It follows that the maximum size of the aggregate should not exceed about one-fifth of the minimum dimension of the form and no larger than three-fourths of the minimum clear spacing between reinforcing bars. The common designation for coarse aggregate is based on the maximum size, *i.e.*, a 1-in. aggregate contains no pieces that will not pass a 1-in. sieve and is graded downward to the No. 8 sieve, with very small percentages passing that sieve. The grading of the various sizes of coarse aggregate specified in ASTM C 33-52T is given in the following table.

| Size no. | Nominal size (sieves with square openings) | Amounts finer than each laboratory sieve (square openings), per cent by weight | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 4 in. | 3½ in. | 3 in. | 2½ in. | 2 in. | 1½ in. | 1 in. | ¾ in. | ½ in. | ⅜ in. | No. 4 | No. 8 |
| 1 | 3½ to 1½ in. | 100 | 90-100 | ... | 25–60 | ..... | 0–15 | ..... | 0 5 | | | | |
| 2 | 2½ to 1½ in. | . | ..... | 100 | 90- 100 | 35-70 | 0 15 | ..... | 0–5 | | | | |
| 357 | 2 in. to No. 4 | ... | ..... | ... | 100 | 95 100 | ..... | 35–70 | ..... | 10–30 | ..... | 0–5 | |
| 467 | 1½ in. to No. 4 | ... | ..... | ... | ..... | 100 | 95 100 | ..... | 35–70 | ..... | 10–30 | 0–5 | |
| 57 | 1 in. to No. 4 | ... | ..... | ... | ..... | ..... | 100 | 95-100 | ..... | 25–60 | ..... | 0–10 | 0 5 |
| 67 | ¾ in. to No. 4 | ... | ..... | ... | ..... | ..... | ..... | 100 | 90-100 | ..... | 20–55 | 0–10 | 0–5 |
| 7 | ½ in. to No. 4 | ... | ..... | ... | ..... | ..... | ..... | ..... | 100 | 90–100 | 40–70 | 0–15 | 0–5 |
| 3 | 2 to 1 in. | ... | ..... | ... | 100 | 95- 100 | 35-70 | 0–15 | ..... | 0–5 | | | |
| 4 | 1½ to ¾ in. | ... | ..... | ... | ..... | 100 | 90–100 | 20 55 | 0–15 | ..... | 0–5 | | |

*Uniformity of Grading.* Once a concrete mixture has been designed from the samples of the aggregate to be used, it is important that all aggregate furnished for that particular project have gradings similar to the original samples, as any change in the grading affects the workability of the resultant concrete.

One of the best measures of the grading as it affects the workability of the mixture is the *fineness modulus.* The fineness modulus is an abstract number which is the summation of the percentages *retained* on the standard series of sieves, enumerated above, divided by 100. It is a definite indication of the surface area of the aggregate. Many combinations of different-sized particles may possess similar fineness moduli but for any particular type of aggregate similar fineness moduli will give reasonable assurance of similar plastic properties when incorporated in a concrete mixture. Typical sands, used as fine aggregate for concrete mixtures, have a fineness modulus varying from 2.4 to 3.2, with an average value of about 2.9. In general, sands with a fineness modulus less than 2.5 should not be used for concrete. It is often specified that any shipment of fine aggregate, made during the progress of the work, which shows a variation in fineness modulus greater than $0.20\pm$ from the original sample be rejected or that a complete redesign of the concrete mixture be required.

*Organic Impurities and Deleterious Substances.* Organic impurities in sand, usually coatings deposited by water filtering through sand deposits, may be detected by the colorimetric test.[1] Foreign substances such as clay lumps, coal, and lignite, if present in any great amount, make the sand which contains them an undesirable aggregate. Any aggregate having more than $1\frac{1}{2}$ per cent by weight of clay lumps or more than 1 per cent by weight of coal and lignite should be rejected. The amount of coal and lignite may be determined by the test specified in ASTM C 123-44, while the amount of clay lumps may be determined by examining the various fractions which remain after the test for grading. Any particles that can be broken up with the fingers shall be classified as clay lumps, and the total percentage by weight of all the clay lumps shall be determined as a percentage of the original weight of the sample.

Extremely fine particles, such as will pass a No. 200 sieve, are not desirable in any great amount and it is recommended that aggregates containing more than 5 per cent of such particles for all concrete mixtures be rejected, with a limit of 3 per cent for those concretes which are subject to abrasion. The amount in any aggregate may be determined by the test specified in ASTM C 117-49.

*Mortar Strength.* A final test for a sand as a fine aggregate may be made by comparison with standard Ottawa sand.[2] It is generally specified that mortar specimens made with the fine aggregate when tested in

[1] ASTM C 40-48.

[2] Standard Ottawa sand is a natural sand obtained at Ottawa, Ill., passing a screen having 20 meshes and retained on a screen having 30 meshes per linear inch, prepared and furnished by the Ottawa Silica Co., Ottawa, Ill.

accordance with the mortar-strength test[1] shall have an average compressive strength of at least 90 per cent of the strength of similar specimens made with the same cement and Ottawa sand when tested at the age of not less than 7 days.

*Soundness.* Where the finished concrete is to be exposed to the weather or other destructive agencies, it is important that the aggregates be sound and hard so that the varying effects of the weathering or other destructive agencies will not act upon the exposed surface to produce pitting and disruptive action. Where such exposure is anticipated, the aggregates should be subjected to the accelerated soundness test as specified in ASTM Designation C 88-46T. When an aggregate is subjected to five alterations of this test the loss of weight should not exceed 8 per cent and in no case exceed 12 per cent for a fine aggregate, with corresponding limits of 12 and 15 per cent for a coarse aggregate.

**1.6. Water.** Water used for concrete should be clean and free from oil, acid, alkali, organic matter, or other deleterious matter. Sea water is not so desirable as fresh water, although where it has been used in structures subject to the weathering action of sea water no greater disintegration has resulted.

**1.7. Proportioning of Ingredients.** Haphazard and careless proportioning of the ingredients of concrete was formerly the rule rather than the exception, but the number of resultant failures is surprisingly small. In the early years of concrete construction little attention was paid to any of the ingredients except the cement. Specifications might require a clean, sharp sand and a coarse aggregate of a specified crushed stone, but there was little emphasis on the grading of the aggregates or on the amount of water to be used. On the other hand, on almost all work of any magnitude, frequent and exhaustive tests of the cement were required. Under those conditions the practice grew of specifying certain arbitrary proportions for the mix, such as 1:2:4, or 1:3:6. When a concrete of high structural strength was desired, a rich mix was specified, and mixtures as rich as 1:1:2 have occasionally been used, while for less important or more massive work, mixtures as lean as 1:4:8 have been specified. Unfortunately this practice of arbitrary proportioning has persisted to some extent to the present day, although it is now generally recognized that the relative consistency (amount of water used) has some relation to the strength of the resultant concrete.

The measuring of the quantities of the constituent materials is done either by volume or by weight. The latter method is much more accurate, but the former method requiring less equipment and having been in use for so many years was hard to replace. However, in the last few

[1] ASTM C 87-52.

years the uniformly better results which have been obtained where measurement by weight has been used has caused the construction industry to accept it without question for projects of any magnitude or importance. The development of the ready-mixed concrete industry in the last few years has contributed greatly to adoption of measurement by weight. As each ready-mix plant distributes concrete to a large number of projects, the cost of the weighing equipment and its accompanying facilities is readily amortized.

The proportions referred to above are proportions by volume, *i.e.*, a 1:2:4 mixture indicates 1 cu ft of cement, 2 cu ft of fine aggregate, and 4 cu ft of coarse aggregate. Since the volume occupied by a given weight of cement varies by as much as 30 per cent for different degrees of compactness, it is necessary that the degree of compactness when used be known. For this reason it is usual to specify that one bag of cement weighing 94 lb shall be considered as 1 cu ft. Cement is sold by the bag, or in barrels of four bags each, or in bulk. Where used in bulk, means must be provided for weighing the loose cement in order to ensure the proper proportions.

The space occupied by a given number of sand grains varies considerably with the moisture content. Increases in volume of 25 per cent or more, caused by the addition of water to dry sand, are not uncommon.[1] Natural sand as it comes from the bank ordinarily contains from 2 to 4 per cent moisture by weight. Sand used in the laboratory is often entirely free from moisture, and sand used on the work may have either more or less moisture content than natural sand. For accurate proportioning, the moisture content must be taken into consideration when the measuring is done by volume. The bulking factor at the usual degree of moisture must be ascertained and proper allowances made, while at the same time the degree of moisture must be maintained as nearly constant as possible.

**1.8. Design of Mortar Mixtures.** In selecting the proportions of materials that are to go into a mortar it is often necessary and usually desirable to be able to predict the strength, permeability, and the amount of mortar that will result from a given proportion of the ingredients. If there are no air voids in a wet mortar (and tests show that this is practically true for consistencies used in practice), the resultant volume[2] of the mortar is equal to the absolute volume of the cement plus the absolute volume of the sand, plus the volume of water used. In the average cubic foot of cement (1 sack of 94 lb) there is 0.487 cu ft of solids. In 1 cu ft of sand the amount of solids depends upon the grading and the

[1] See *Proc. ASTM*, vol. 20, part II, p. 147.

[2] The absolute volume is the volume of the solid material, or the gross volume minus the volume of voids.

size of the grains. In beach sand there are from 0.60 to 0.63 cu ft and in commercial building sands about 0.68 cu ft of solids. The exact amount may be determined for any sand by obtaining its weight per cubic foot and its specific gravity.[1]

*Strength.* The water in a mixture chemically reacts upon the cement and produces a colloidal coating on the particles. The finest particles of cement become completely colloidal, while some of the larger particles are never completely penetrated. This colloidal coating is the whole cementing action, and once sufficient water is present to produce this reaction, additional water tends to weaken the cementing material. Therefore, it is reasonable to expect that the strength of a mortar at any given age will vary with the amount of mixing water per unit quantity of cement, and the larger the amount of water, the weaker the mortar.

In Fig. 1.1 are shown the results of studies of several investigators. The increased strength shown by the later studies is due to the stronger cements produced today in comparison with those of 30 or more years ago. Figure 1.1 brings out clearly the direct relation between strength and *water-cement ratio.* Figure 1.2 shows the same relation, but here the water-cement ratio is expressed in gallons of water per sack of cement. This relation was first stated by Abrams[2] in 1918, whose curve of that date is shown in the figure. A more recent curve[3] by the same investigator shows the improvement in cement in later years, and since the strength of plastic mixtures is independent of the amount and gradation of the aggregates, this later curve is applicable to the design of mortars for strength which use present-day portland cements. This curve cannot be used unless damp curing conditions are maintained.

In the measurement of water in a mixture all water outside the solid particles is included. Most sand is delivered in a more or less damp condition and the moisture that is carried must be determined and allowance made in determining the amount of water to add.

*Permeability.* The permeability of a mortar is measured by the rate at which water under a given pressure will pass through a given thickness of the material. When the mortar is mixed, not all the water reacts with the cement, and a large portion of it is really free water necessary in the mix to make it workable. As the mortar hardens, this free water gradually evaporates, thus providing air voids. The longer the mortar is cured in a damp condition, the greater the amount of water in permanent combination with the cement, and the less the amount of air voids. Excess water in the original mix will, of course, increase the air voids and make the mortar more permeable. Repeated tests have shown that a

[1] ASTM C 128-42.

[2] *Eng. News-Record,* May 2, 1918.

[3] *Proc. ACI,* 1931, p. 1330.

reasonably watertight mortar cannot be obtained with a water-cement ratio greater than 7½ gal of water per sack of cement. In addition to a low water-cement ratio and prolonged curing, as dense a mixture as possible should be used.

*Density.* Other things being equal, a coarse-graded sand will produce a denser mortar than fine-graded sand. For any given proportion of

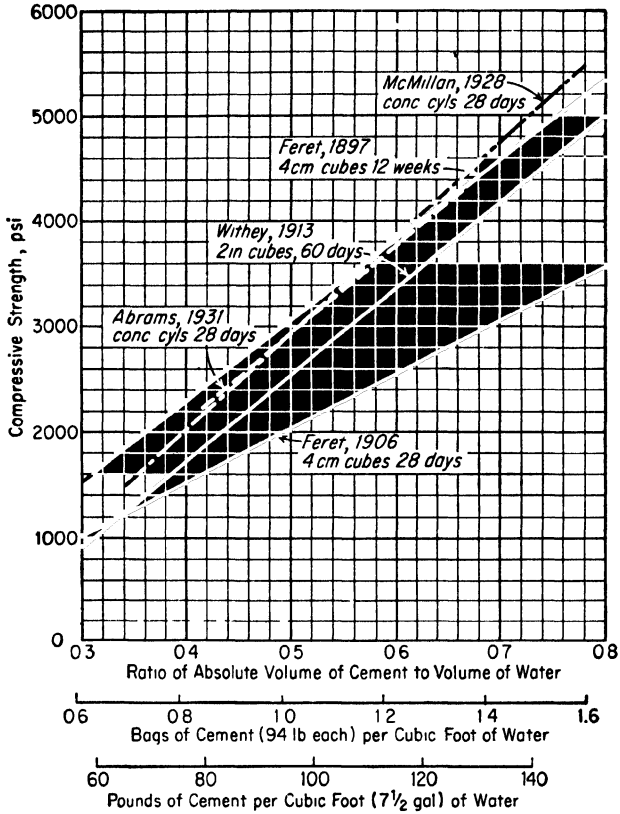



FIG. 1.1. Relation between compressive strength and absolute volume of cement to volume of water

sand to cement, the voids in the mortar will vary with the volume of the mixing water used. In a very dry mortar the percentage of voids may be greater than in a wetter mix so that the voids decrease as water is added until the point of minimum voids (minimum volume of mortar) is reached. The addition of more water increases the voids. The exact proportions of cement, sand, and water that will produce the densest mortar vary for each sand, depending upon its gradation. In general, coarse sands take less water and less cement than fine sands.

*Workability.* There are so many factors affecting the workability of a mortar that no satisfactory method of measuring it has ever been developed. Gradation, shape of particles, and proportions all have their effect. A mortar allowed to stand for 1 or 2 hr and then reworked often is found more workable than the original mortar. Fine sand makes a more workable mortar than coarse sand, but since its water requirement is higher, the strength is decreased. In providing a workable mortar, care must be taken that strength and durability are not sacrificed.

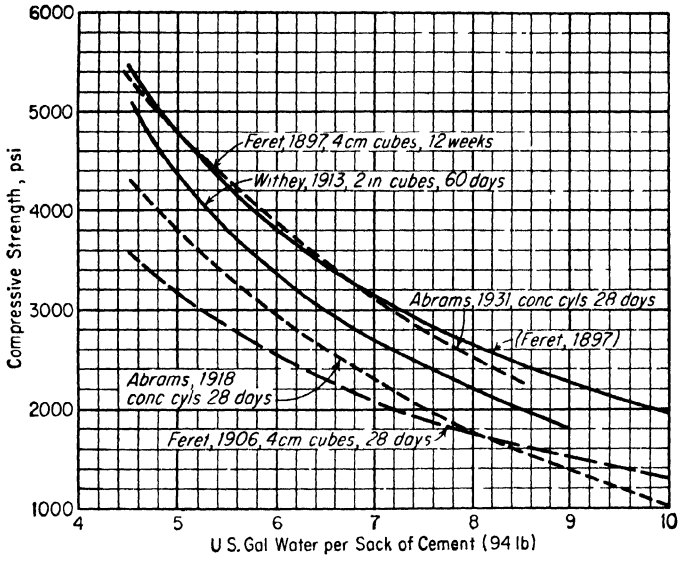



FIG. 1.2. Relation between compressive strength and water-cement ratio.

**1.9. Determination of Yield and Strength.** It is desired to supply 100 cu yd of 1:3 mortar which will reach a strength of 3000 psi in 28 days. The sand available has 35 per cent voids, weighs 100 pcf when dry, and contains 2 per cent of moisture by weight, in which condition it weighs 83 pcf. What quantities of materials are required?

Reference to Abram's curve in Fig. 1.2 shows that a 3000-psi mortar, cured damp for 28 days, should have 7.2 gal of water per sack of cement. A one-sack batch will yield

$$0.487 + 3 \times 0.65 + 7.2 \times 0.134 = 3.40 \text{ cu ft} = 0.126 \text{ cu yd}$$

In 100 cu yd there will be

$$100 \div 0.126 = 793 \text{ one-sack batches}$$

Therefore the quantities required are

Cement: 793 sacks
Sand: $3 \times 793 \div 27 = 88.1$ cu yd dry
Water: $7.2 \times 793 = 5710$ gal

However, since the sand contains 2 per cent moisture, the number of cubic yards in its moist condition is

$$\frac{100}{83} \times 88.1 = 106 \text{ cu yd}$$

If this mortar were to be mixed in approximately 1-cu-yd batches (8 sacks), the sand and water for each batch would be as follows:

Sand: $3 \times 8 \times \frac{100}{83} = 28.9$ cu ft
Water: $7.2 \times 8 - (0.02 \times 28.9 \times 83) \div 8.35 = 52$ gal

In actual work some of the water in the mix is lost either by absorption by adjacent materials or by evaporation so that the actual yield is reduced from 2 to 4 per cent. This should be considered in making estimates of quantities of materials required for a given volume of mortar.

**1.10. Concrete Mixtures.** Concrete may be considered as a mortar mixture into which the coarse aggregate particles have been mixed, or it may be considered as a combination of the separate ingredients. In any case the factors governing the strength of a mortar hold true for a concrete, provided a suitable coarse aggregate is used. Since the water in the mixture has to wet the surfaces of the coarse aggregate particles as well as those of the cement and sand grains, it follows that the larger the size of the coarse aggregate, the less the water that will be required for a given consistency; hence greater strength and durability.

Design specifications have not kept pace with the improvement in cement in recent years. For this reason caution should be exercised in designing for low specified strengths. With fair curing conditions a compressive strength of 2000 psi in 28 days may be obtained with a very lean mixture, but the resulting concrete will be neither watertight nor capable of withstanding ordinary weather conditions. Experience has shown that concrete exposed to the elements should not have a water content of more than 7½ gal per sack of cement.

A concrete must have a certain degree of workability, but, as for mortars, no method has yet been developed which is an accurate measure of workability. In the case of a concrete, this quality is influenced by the nature and amount of the coarse aggregate, as well as by the amounts of cement and water, and the amount and nature of the fine aggregate. A method of measuring the workability, or more properly the consistency, is the *slump test.*[1] A conical shell of 16-gauge galvanized metal with a base 8 in. in diameter, a top 4 in. in diameter, and a height of 12 in. is filled to overflowing with concrete rodded into the shell in three

[1] ASTM C 143-52.

separate layers, each receiving 25 strokes of a ⅝-in. bullet-end rod 24 in. in length. The excess is carefully struck off and the mold is at once lifted slowly vertically. The amount of drop of the top of the mass below the original 12-in. height, measured in inches, is known as the slump. [Mass concrete and highway mixtures are workable with a slump of from 1 to 3 in. Concretes for reinforced beams and columns require a greater degree of workability, with slumps of from 4 to 6 in.]

As in mortar, excess water in concrete dries out as the concrete ages, so that a watertight concrete should not have a water content of more than 7½ gal per sack of cement. Careful grading of aggregates to obtain as dense a mixture as possible also tends to decrease the permeability, but the waste and consequent additional expense rarely warrant such a procedure.

**1.11. Design of Concrete Mixtures.** The determination of the relative proportions of cement, fine aggregate, coarse aggregate, and water constitutes the design of a concrete mixture. The requirements to be met are usually specified in terms of strength and workability (or slump). When severe exposure to the elements or other disintegrating agencies are to be expected, the strength requirement usually is superseded by the necessity of designing a concrete of greater strength than that required structurally. The workability or slump specified will be dependent upon the type and size of the members in the structure. In small members or those containing closely spaced reinforcement, a concrete should be designed with a comparatively large slump, while in mass concrete, large members, and lightly reinforced slabs, a concrete with very little slump will prove the most economical.

The *trial-batch* method of design is the one most widely used. From a table or from a curve similar to Fig. 1.3 is selected the water-cement ratio corresponding to the strength desired. To a water-cement paste or slurry of these proportions is added fine and coarse aggregate, until the appearance and successive slump tests indicate that the mixture has reached the specified workability. Normally such a trial batch is hand-mixed, the total weight being from 40 to 50 lb. The aggregates should previously have been brought to a saturated surface-dry condition[1] in order that they neither add nor extract water from the mixture.

From the relative weights of cement, water, and aggregates of the trial batch, it is possible to compute the yield, *i.e.*, the number of cubic feet per sack of cement, the cement factor, *i.e.*, the number of sacks of cement per cubic yard of concrete, and the actual quantities of each material to be incorporated in the size of batch to be used on the project.

The amount of water to be used in the project batch must be corrected for absorption by the aggregates during the period of mixing, if the aggregates are dry, and for the free water in the aggregates, if they are wet and

[1] ASTM C 128-42

carry free moisture. Moisture tests[1] on representative samples of the aggregates as the work progresses furnish the necessary information for this correction.

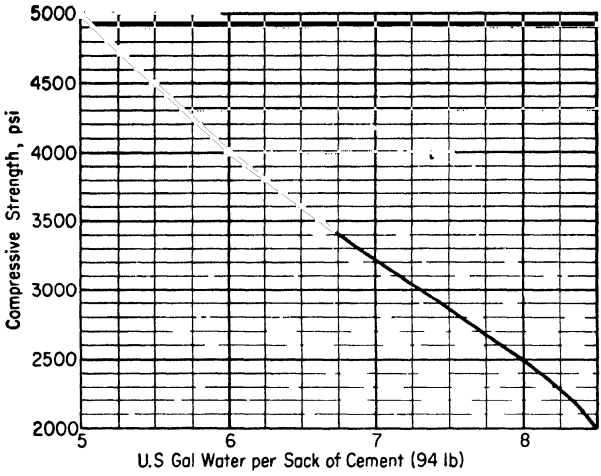



FIG. 1.3. Relation between water-cement ratio and compressive strength at 28 days. (*Portland Cement Association Bull. T12*, 1945.)

**1.12. Illustrative Design by the Trial-batch Method.** With accepted aggregates it is required to design a concrete mixture having a 4-in. slump and a 28-day standard-cured[2] compressive strength of 3000 psi. The specific gravity of both fine and coarse aggregate is 2.65. By reference to Fig. 1.3 it is indicated that $7\frac{1}{4}$ gal of water per sack of cement should produce a 3000-psi concrete. Five pounds of cement is arbitrarily selected for the trial batch. The amount of water to be used is therefore $\frac{5}{94} \times 7.25 = 0.386$ gal $= 3.22$ lb. By successive additions of fine and coarse aggregate and occasional slump tests after thorough mixing it is found that 12.6 lb of saturated surface-dry fine aggregate and 21.4 lb of saturated surface-dry coarse aggregate produce the mixture desired. By multiplying the respective specific gravities by 62.4 the solid volumes in the batch are computed as follows:

| | Weight in batch, lb | Solid weight of 1 cu ft, lb | Solid volume, cu ft |
|---|---|---|---|
| Cement (sp. gr 3.15) | 5 00 | 196 6 | 0 026 |
| Water.. | 3 22 | 62 4 | 0 052 |
| Fine aggregate | 12 6 | 165 4 | 0 076 |
| Coarse aggregate | 21 4 | 165 4 | 0 129 |
| Total weight and solid volume | 42 2 | | 0 283 |

[1] ASTM C 70-47.
[2] ASTM C 192-49.

The space occupied by the freshly mixed concrete consists of the solid constituents and the voids filled with the total amount of free water and the entrapped air. Unless an air-entraining agent is used (see Art. 1.18), the amount of entrapped air is so small that it may be neglected. The solid volume of the trial batch as computed above is 0.283 cu ft, and dividing the solid volumes of the constituents by 0.283, the relative solid volume of each material in a unit batch of any size is obtained as follows:

| | | |
|---|---|---|
| Cement: | 0.026 ÷ 0.283 | = 0 092 |
| Water: | 0.052 ÷ 0 283 | = 0 183 |
| Fine aggregate | 0.076 ÷ 0.283 | = 0 269 |
| Coarse aggregate: | 0.129 ÷ 0.283 | = 0 456 |
| | | 1 000 |

The weight of each material in a 1-cu-ft or 1-cu-yd batch can then be obtained as follows:

| | | *Per cu ft, lb* | | *Per cu yd, lb* | |
|---|---|---|---|---|---|
| Cement: | 0.092 × 196.6 = | 18 1 | × 27 = | 489 | = 5.20 sacks |
| Water: | 0.183 × 62.4 = | 11 4 | × 27 = | 308 | = 36.9 gal |
| Fine aggregate: | 0.269 × 165.4 = | 44 6 | × 27 = | 1205 | |
| Coarse aggregate: | 0.456 × 165.4 = | 75 6 | × 27 = | 2011 | |
| | | 149 7 | | 4043 | |

This concrete would be termed a 5.20 bag mix,[1] *i.e.*, the cement factor is 5.20, as 5.20 sacks of cement are required per cubic yard of concrete. The yield, *i.e.*, the quantity of concrete for one sack of cement, is

$$27 \div 5.20 = 5.19 \text{ cu ft}$$

*Proportioning by Volume.* The bulk unit weight[2] of the fine aggregate has been determined to be 104.4 pcf and that of the coarse aggregate 105.0 pcf. The quantities required for 1 cu yd of concrete are

| | | *Cu ft* | |
|---|---|---|---|
| Cement: | 489 ÷ 94 = | 5 20 | = 5 20 sacks |
| Water: | 308 ÷ 62.4 = | 4.94 | = 36.9 gal |
| Fine aggregate: | 1205 ÷ 104.4 = | 11.54 | |
| Coarse aggregate: | 2041 ÷ 105.0 = | 19.44 | |

The proportions by volume are 5.20:11.54:19.44, or 1:2.22:3.73.

*Adjustments for Water Content of Aggregates.* In the above calculations from the trial batch the aggregates were assumed to be in the saturated surface-dry condition. In the field this condition rarely exists,

[1] Often referred to the nearest ¼ bag, in which case this would be a 5¼-bag mix.
[2] ASTM C 29-42.

and the amount of water in the aggregates as they are actually used may, if not considered in the design, seriously affect both the workability and the strength of the resultant concrete.

During the period of mixing and placing the aggregates will absorb a certain amount of water. It is usual to assume that saturated surface-dry aggregates will absorb approximately 1 per cent by weight during this period. For the 1-cu-yd mix designed above, therefore, the fine aggregate will absorb $0.01 \times 1205 = 12.05$ lb and the coarse aggregate $0.01 \times 2041 = 20.41$ lb, a total of 32.46 lb, or 3.9 gal. Thus the mixing water for the 1-cu-yd batch should be increased from 36.9 gal to 40.8 gal. However, such a correction rarely is necessary in the field as the aggregates are very seldom bone dry, but usually contain free water.

The aggregates may carry considerable free moisture, and where batch proportioning is by weight the mixture will, if uncorrected, contain not only an excess amount of water but also lack the proper amount of aggregate. If the accepted aggregates, as used in the field, are found to contain[1] 3.5 per cent of free water and 1.0 per cent of free water in the fine and coarse aggregates, respectively, the weights of the moist aggregates must then be increased and become:

Fine aggregate: $1.035 \times 1205 = 1247$ lb, of which 1205 is fine aggregate and 42 lb is free water.

Coarse aggregate: $1.01 \times 2041 = 2061$ lb, of which 2041 lb is coarse aggregate and 20 lb is free water.

The original weights are now corrected for field use as follows:

| | Saturated surface-dry, lb | Corrections, lb | Revised weights, lb |
|---|---|---|---|
| Cement | 489 | 0 | 489 |
| Fine aggregate | 1205 | +42 | 1247 |
| Coarse aggregate | 2041 | +20 | 2061 |
| Water | 308 | −62 | 246 = 29.5 gal |

Without the above correction for the free water in the aggregates the total amount of water in the mix would have been $308 + 62 = 370$ lb, or 44.1 gal. This is equivalent to $44.1 \div 5.20 = 8.50$ gal per sack of cement with a consequent reduction in strength to 2000 psi.

As mentioned in Art. 1.7, the bulking effect on fine aggregate containing free water must be considered if proportioning is to be done by volume. In this example the fine aggregate containing 3½ per cent of free water has a weight of 88.1 pcf, while this same quantity when thoroughly dried has a weight of 84.3 lb. Dividing the bulk unit weight of the fine

[1] ASTM C 70-47.

aggregate (104.4 lb) by 84.3 lb, a bulking factor of 1.24 is indicated. Therefore the volumetric proportions shown on page 17 must be corrected in order that sufficient fine aggregate is provided in the mixture, *i.e.*, the amount of fine aggregate must be increased from 11.54 cu ft to $11.54 \times 1.24 = 14.31$ cu ft and the proportions by volume become 5:20:14.31:19.44, or 1:2.75:3.73, instead of 1:2.22:3.73. Unless the coarse aggregate has a grading which contains many small particles, the bulking effect is small and may be neglected.

The correction in the amount of mixing water must be made the same as for proportioning by weight. The fine aggregate contains

$$14.31 \times 84.3 \times 0.035 = 42 \text{ lb of water}$$

and the coarse aggregate (neglecting the slight bulking factor)

$$19.44 \times 105.0 \times 0.01 = 20 \text{ lb}$$

Therefore the amount of mixing water is reduced to 29.5 gal per cu yd, the value determined previously for weight proportioning.

**1.13. Control of Concrete Mixtures.** Where a small quantity of concrete is to be placed, elaborate examination and inspection of each batch are not warranted and economy is best served by keeping the water-cement ratio within the specified limit.

On the other hand, where large quantities of concrete are to be placed in important structures, special considerations are desirable to ensure economy and a uniform quality of concrete. Once the proportions of the constituent materials are selected, as long as the aggregates come from the same source, little trouble should be encountered in providing in each batch the proper amount of fine aggregate. The coarse aggregate may segregate in the stock pile and in such cases care must be taken to ensure uniformity. The percentage of moisture in the aggregates in the stock pile may have a considerable variance, and, as shown in Art. 1.12, this may have considerable effect on the strength and workability of the concrete. Therefore periodic determinations of the amount of moisture in the aggregate must be made. Whether measurement is to be made by weight or volume, and considerable variance of the amount of water in the aggregates is anticipated, a "job curve" for each aggregate and the amount of mixing water to be added is warranted. Such a series of curves shows the weight or volume of each aggregate and the amount of mixing water for the range of percentages of moisture expected in the aggregates.

In addition, tests of the quality of the concrete are made by taking representative specimens either at the mixer or at the forms for testing, usually at ages of 7 or 28 days after having been standard cured.[1] Job-

[1] ASTM C 39-49.

cured specimens, cured as nearly as possible as the concrete in the job is cured, are also sometimes advisable both to ascertain the quality of the concrete and to determine the safe time of form removal and the ability of the concrete to sustain a portion or all of its design load.

**1.14. Mixing.** Practically all concrete is machine-mixed. The ordinary mixer consists of a rotating drum, in the interior of which are blades so placed that, as the drum rotates, they lift the ingredients, which in turn slide off the blades and drop to the mass then at the bottom. Mixers vary from small laboratory machines of 1 or 2 cu ft capacity and small job one-bag mixers up to large mixers of 5 or 6 cu yd capacity.

A batch of concrete is normally mixed from 1 to 4 min after all materials are in the mixer, the longer periods being for the larger batches. Overmixing, if not carried to extreme, is objectional only because of the added cost and the grinding action on the fine aggregate, which tends to stiffen the mix.

Mixing for longer periods is the rule where concrete is delivered from a central mixing plant by means of a mixer mounted on a truck known as a "transit mixer." Experience has shown that these transit mixers may continue to rotate the concrete for an hour or more with no injurious effects. However, it is usual to specify that the concrete must be discharged into the forms within 1½ hr from the time of the beginning of the mixing.

**1.15. Placing.** For many years it was the usual practice either to hoist the fresh concrete to a high tower for chuting or to locate the mixer above the job. The more fluid the mix, the greater the lengths of chutes that could be used, and the larger the area that could be covered from one location. This practice encouraged overwet mixtures, but with the acceptance of the water-cement ratio this practice was gradually discontinued. Placement of most building concrete is now made on horizontal runways by wheelbarrows or buggies that have been brought to the horizontal runways by hoists or elevators. The concrete should be deposited in horizontal even layers and not dumped all in one place and allowed to flow horizontally. It should then be tamped, spaded, or vibrated thoroughly before another layer is deposited. It is especially important that thorough spading or vibrating be done near the face of the form in order to ensure a smooth nonhoneycombed surface.

Excess working of the concrete brings too much of the water and fine aggregate to the upper portion of the layer, which produces a weak chalky layer of material known as laitance. Where continuous deposition is impossible, any such deposit of laitance should be scraped off and the concrete beneath roughened and wetted before placing is resumed. Where the deposition is continuous, which is of course desirable, it is good practice to decrease gradually the water-cement ratio toward the

top of the lift in order to offset the water gain on the surface of the several layers as the work progresses.

**1.16. Curing.** The principal variations in curing conditions which affect the process of hardening and the strength of the concrete are variations in moisture and temperature conditions. While it is important that the amount of water used in mixing be controlled so that the consistency be kept as nearly normal as possible, it is just as important that the concrete be not allowed to dry out immediately, if the maximum strength is to be obtained. All concrete should be protected against premature drying out for at least 1 week, and for a longer time when low temperatures prevail. This may be accomplished in several ways, the more usual of which follow:

1. Covering with an impervious paper, to minimize evaporation losses, and wetting down under the paper at short intervals to keep surfaces moist.
2. Covering surfaces with burlap which is kept wet by sprinkling at frequent intervals.
3. Ponding with earth dykes holding the water.
4. Covering with earth or straw which is kept moist.
5. Sprinkling at intervals.
6. Applying a liquid which forms a membrane which prevents evaporation losses. This is often colored to ensure full coverage.

The importance of keeping the concrete moist while hardening cannot be too strongly emphasized. Tests show that a concrete allowed to dry out immediately will usually reach a strength of not more than 50 per cent of the strength of similar concrete kept moist over the entire period of curing. Figure 1.4 shows this relation[1] graphically. All test specimens were tested at 4 months, having had various intervals of storage in damp sand. Each value is the average of 24 tests (four each for six consistencies).

The relation between the mean temperature during the curing period and the strength of concrete is illustrated by Fig. 1.5.[2] The tests from which the curves were plotted covered a wide range of temperature conditions, and the results were fairly consistent. A knowledge of the effect of the mean temperature upon the strength is very necessary in determining the time when forms may be removed and loads applied, and a careful study of Fig. 1.5 will furnish the necessary information for determining the relative length of time the forms should be kept in place under different temperature conditions.

By combining high temperatures with a saturated condition of the atmosphere, it would follow that accelerated hardening of the concrete

[1] Taken from *Lewis Inst. Structural Materials Research Lab. Bull.* 2.

[2] Taken from *Univ. Ill. Eng. Expt. Sta. Bull.* 81.

would be obtained. These conditions are brought into being by the application of live steam to concrete while hardening. This method is especially useful in the manufacture of concrete blocks, tile, small pipe, etc., where the saving in forms, storing space, and time is important. By placing the concrete products in a confined space, and applying the steam under pressure, a still more rapid increase in strength will be attained.

Pressures up to 150 psi with temperatures as high as 350° have resulted in the concrete obtaining in a few hours the strength which it would have attained under normal curing conditions in a period of several weeks.

**1.17. Freezing.** The effect of low temperatures in delaying the hardening of concrete is shown in Fig. 1.5. When water reaches a tempera-

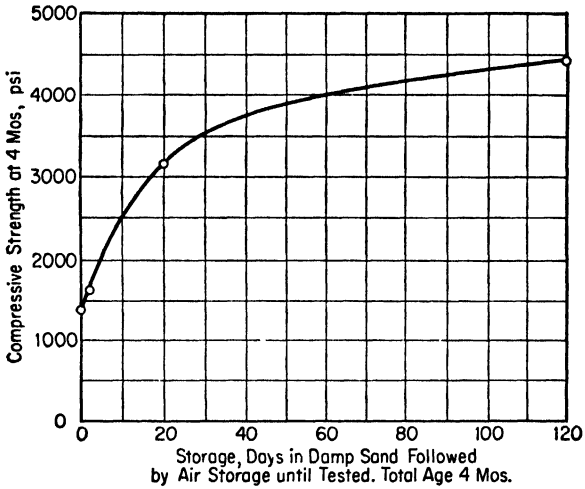



FIG. 1.4

ture of 39°F, some subtle change occurs which decreases its chemical ability for combination. This change becomes more marked as the freezing point is approached, and concretes placed with the temperature near the freezing point take several times as long to obtain a final set as concretes cured at normal temperatures. In case of dry atmospheric conditions much of the water may evaporate before the final set takes place, and insufficient water be left to combine chemically with the cement. In case the temperature falls below the freezing point before final set, the expansion of the water while freezing exerts a force sufficient to destroy the cohesion between the particles of the green concrete.

The injurious effect of freezing is lessened by two factors, namely, that concrete is a very poor conductor of heat, and that the chemical action of setting and hardening generates a certain amount of heat to combat the freezing action of the atmospheric conditions. Thus the serious

injury is usually confined to the surface of the concrete, and rarely penetrates more than an inch or two in depth. In massive members this may not seriously impair the strength, but be harmful only to the appearance. In the smaller members, however, a large percentage of the strength may be lost.

Various methods are used to prevent the freezing of concrete, such as heating the constituent materials and covering the green concrete. In building construction the fresh concrete is often housed in by canvas

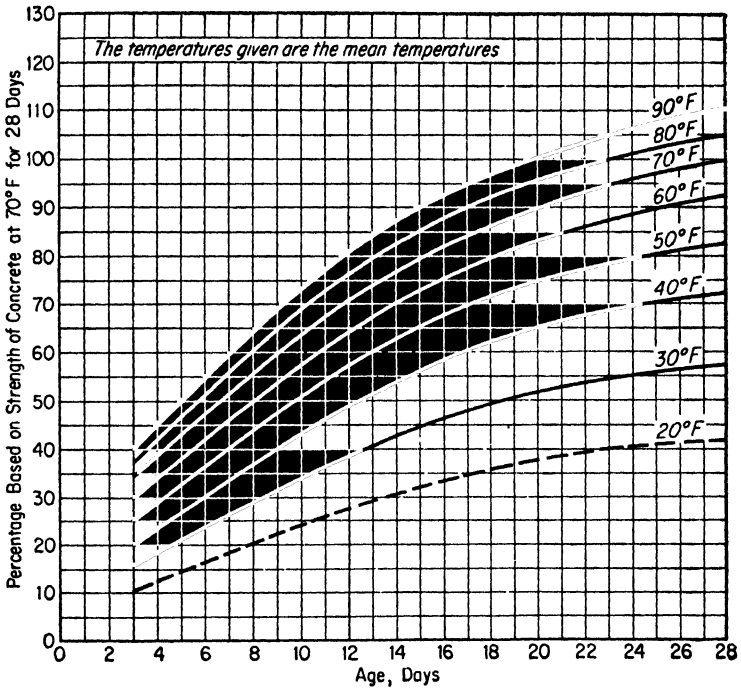



FIG. 1.5

curtains and the interior heated by stoves or "salamanders" to maintain above-freezing temperatures. In such cases it is important that sufficient moisture be supplied by sprinkling or the interjection of live steam in order that the artificial heat not dry out the concrete.

Generally it is advisable to protect the surface of the freshly poured concrete for a period of about a week and in all cases for a period of at least three days against freezing temperatures, keeping the surface temperature of the concrete well above the freezing point.

**1.18. Effect of Admixtures.** *Air-entraining Agents.* About 15 years ago it was discovered that air bubbles of microscopic proportions in the hardened concrete add greatly to its resistance to deterioration under

alternate freezing and thawing. Air-entraining agents are either introduced as additions interground with the clinker in the manufacture of the cement or introduced later as an admixture. The most commonly used agents are Vinsol resin and Darex. The usual increase in voids by the presence of the entrained air ranges from 3 to 6 per cent. When such an agent is used, the mixture becomes more workable and a higher ratio of coarse to fine aggregate may be used. The amount of water may also be reduced, usually by a percentage equal to about two-thirds of the percentage of the entrained air. Even with this reduction in water-cement ratio the compressive strength of rich mixtures is slightly reduced, while that of lean mixtures is slightly increased. However, the increase in the durability of concrete pavement against alterations of freezing and thawing where such an agent is present has led to its wide use, and it has also often been used for less exposed structures because of the favorable effect on the workability and texture of the concrete.

*Accelerators.* Calcium chloride is the most widely used accelerator. When used in the ratio of about 2 per cent of the weight of the cement, it increases the early strength of the concrete and provides a certain protection against freezing temperatures. If larger amounts are used, the ultimate strength of the concrete is substantially reduced.

*Hydrated Lime.* The use of hydrated lime in quantities up to 15 per cent of the weight of the cement has been advocated by various authorities on the theory that it improved the workability of the concrete or increased its strength and watertightness. In lean mixtures it is true that the addition of hydrated lime does have a marked effect in producing a more plastic and better working concrete. In the richer mixes this effect is less pronounced. Some tests have shown a slight increase in strength with the use of a small percentage of hydrated lime. It appears that if the hydrated lime is added without decreasing the amount of cement or increasing the amount of water, such an increase usually occurs, but if some of the cement is replaced by hydrated lime, a reduction in strength can be expected.

*Water-repelling Agents.* Various waterproofing compounds in powdered or liquid form are sometimes used to make a more impervious concrete. They are either added to the mixing water, mixed with the cement on the job, or added to the cement during its manufacture. Their function is to fill the voids or pores of the concrete with a more or less soapy, insoluble filler, and thus prevent the percolation of water through the concrete. The results obtained are varied. Some practically impervious concrete has been produced, while on other work the waterproofing has not been successful. Practically all the compounds in use detract from the compressive strength of the concrete. Fully as impervious concrete can generally be obtained by using a slightly richer

mix, well-graded fine aggregate, as stiff a consistency as possible, and thoroughly puddling the concrete as it is placed.

## PROPERTIES OF CONCRETE

**1.19. Compressive Strength.** The ultimate strength of a concrete normally increases with age. This increase proceeds very rapidly for the first few days after the concrete is placed, but becomes more gradual as time goes on, though continuing at a more reduced rate for an indefinite period. The compressive strength of concrete at the age of 28 days, as determined by test specimens made and cured in accordance with ASTM C 192-49, is generally used as a measure of the quality of the concrete. This assumes proper mixing and placing and suitable curing conditions.

The ultimate compressive strength expressed in pounds per square inch is used as a basis for determining the unit stresses to be used in design, for it has been found that practically all the other structural properties of a concrete are proportional to the compressive strength.

**1.20. Tensile Strength.** The tensile strength of concrete is a property of little importance, because it is so low in comparison with the compressive strength that it is usually neglected altogether in the design of reinforced concrete structures. It may roughly be estimated as having a value of about 10 per cent of the compressive strength.

**1.21. Transverse Strength.** The transverse, or flexural, strength of concrete is low as compared with its compressive strength, but much greater than the strength in pure tension. The transverse strength is measured by the stress developed in beam action. In a reinforced concrete member this strength is usually disregarded, and steel reinforcement is placed in the member to develop the flexural stresses on the tension side. Load tests, however, on reinforced concrete structures have shown that the transverse strength of the concrete contributes to a marked degree in increasing the capacity of the structure. Tests made by Duff A. Abrams[1] indicate that, at 28 days, the transverse strength varies from 26 per cent of the compressive strength for a 1000-psi concrete to 15 per cent for a 4000-psi concrete.

**1.22. Shearing Strength.** The shearing strength of concrete is important in that failure by shear on a diagonal plane often occurs in short compression specimens. The direct shear must not be confused with the combination of shear and diagonal tension that occurs in the web of a beam. The resistance of concrete to direct shear is difficult to determine, as it is almost impossible to eliminate the effect of bearing, diagonal tension, and other stresses, so that different series of tests show quite a variation from one another. For most concrete the shearing

[1] See *Lewis Inst. Structural Materials Research Lab. Bull.* 11.

strength is at least 60 per cent of the compressive strength, and will need to be considered in design only in exceptional cases.

**1.23. Elasticity.** Concrete is not a perfectly elastic material, there being a slight decrease in the ratio of stress to strain as the stress increases. Concrete also shows a permanent set under the smallest loads, but within working limits there is a fairly constant relation between temporary stress and strain, which may be considered as the modulus of elasticity of the concrete.

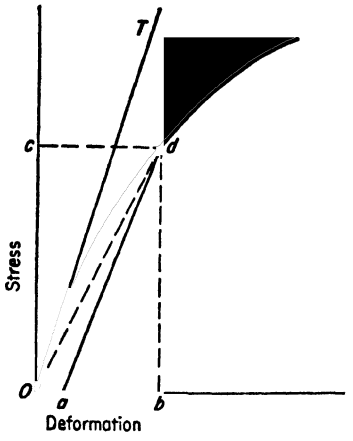



FIG. 1.6

Since the stress-strain line is curved almost from the beginning, the method of calculating the modulus of elasticity needs to be considered. Figure 1.6 is a typical stress-strain diagram with the curvature somewhat exaggerated. A load producing a stress $Oc$ has been applied and removed a sufficient number of times until the permanent set $Oa$ shows no further appreciable increase, and all points on the stress-strain line $ad$ fall on an approximately straight line. The deformation measured from the original position, for a stress $Oc$, is $Ob$, and the slope of the line $Od$ is called the "secant modulus." The slope of the tangent to the curve, represented by $OT$, is known as the "initial modulus" or "initial tangent modulus."

In reinforced concrete design the principal use of the modulus of elasticity is to determine the value of the relative stresses carried by the steel and concrete, assuming that there is perfect bond between the two materials. For such a computation the deformation should be measured from the original position, and the secant modulus should be used. In the case of a beam where the stress in the concrete varies according to the "straight-line" theory, the secant modulus, while not exactly representing the conditions, is nearer to the exact conditions than the initial modulus. The relation between the secant modulus and the initial modulus is not a constant, but for all but the smallest loads the former is the smaller.

There is a relation between the modulus of elasticity of a concrete and its compressive strength, but this relation is not linear. Stanton Walker[1] expresses this relation for usual concretes as $E = 33{,}000\ S^{5/6}$, where $E$ is the initial modulus and $S$ the compressive strength of the concrete. The tests from which this relation was determined covered a wide range of

[1] See *Lewis Inst. Structural Materials Research Lab. Bull.* 5.

consistencies, mixes, times of mixing, curing conditions, and age at time of test. These tests also showed that the modulus of elasticity increases as the aggregate becomes coarser (within certain limits), that it increases with age, the richness of the mix, and time of mixing, and is less for wet than for dry consistencies.

The value for this initial modulus for concrete at the age of 28 days varies from about 1,500,000 to 5,500,000 psi with a somewhat narrower range for the usual concretes. The values most often used in design are generally somewhat smaller, the ACI Code specifying that the modulus of elasticity of the steel be taken as 30,000,000 psi and that of the concrete as $1000f'_c$, where $f'_c$ is the ultimate compressive strength determined as described in Art. 1.19.

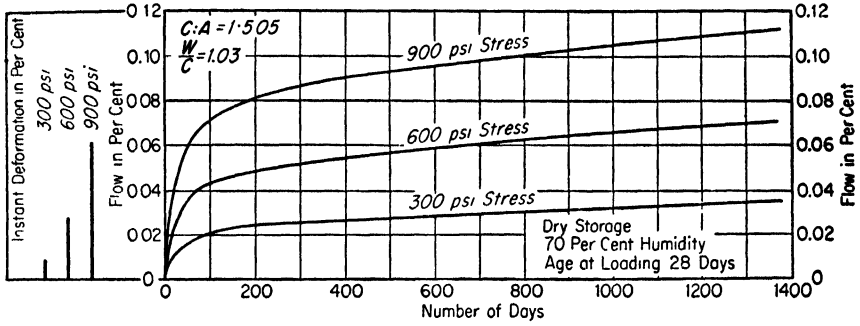



FIG. 1.7

**1.24. Elastic Limit.** For the same reasons as those given at the beginning of the previous section, there can be no *elastic limit* in the true sense of the term. There appears to be a stress, however, below which repetition of the same load does not cause appreciable increase in set, while beyond this stress, repetition of load causes increased set indefinitely, and final failure far below the normal ultimate strength. This stress may be considered as the *elastic limit*. Tests show quite a range of values, varying from 25 to 90 per cent of the ultimate compressive strength, but for the average concrete it is probably in the neighborhood of from 40 to 60 per cent of the ultimate compressive strength.

**1.25. Plastic Flow.** Tests have shown that all concrete under a sustained load continues to deform or "plastically flow" for a long period of time. This deformation is independent of that due to shrinkage caused by the decrease in moisture content. The results of tests by R. E. and H. E. Davis[1] on 4- by 14-in. cylinders of 1:5.05 concrete with a water-cement ratio of 1.03, loaded at 28 days, and stored in air of 70 per cent humidity are shown in Fig. 1.7. For specimens stored under water the deformations due to flow were much smaller.

[1] *Proc. ACI*, vol. 27, p. 837, 1931.

The effect of this plastic flow in a reinforced concrete column is, under sustained load, to relieve gradually the compressive stress in the concrete until the yield point of the steel is reached. Tests of reinforced concrete columns, sponsored by the American Concrete Institute, made at Lehigh University and at the University of Illinois,[1] have shown increases in the steel stress of from two to four times that obtained under the initial loading.

In beams and slabs the effect of the shrinkage and plastic flow is to move the neutral axis toward the tensile reinforcement, thus decreasing the stress in the concrete and increasing that in the steel.

In all cases the removal of the load shows some recovery or backward flow, which often continues for an appreciable period, but such values are always less than the deformations under the sustained load.

**1.26. Contraction and Expansion.** Concretes expand as the temperature is raised, and contract as the temperature is lowered. The coefficient of expansion per degree of temperature change increases somewhat with the richness of the mix, but the range of values is small. Tests made in the laboratories of Cornell University gave a range of values of from 0.00000677 for a 1:1½:3 concrete to 0.00000537 for a 1:3:6 concrete, with an average for all tests of 0.00000604. Other tests have shown a close agreement. The value generally used is 0.000006 per degree Fahrenheit.

Concretes expand in volume if kept wet or immersed in water, and contract if exposed to air. This property is not confined to freshly placed concrete but is characteristic of concretes of many years' service. A concrete which dries out in air may be expected to contract from 0.02 to 0.05 per cent, and when immersed in water may expand at least one-half of this amount.

This tendency to change in volume with different moisture conditions and changes in temperature does, of course, set up stresses of both tension and compression in a restrained reinforced concrete structure. The tensile stresses often exceed the amount that the concrete can sustain, and cracks result.

**1.27. Bond.** The adhesion of new concrete to work previously placed becomes an important consideration in certain classes of construction. Very few tests of this property, however, have been made. Tests made by Hector St. George Robinson in 1912 indicate that a thorough cleaning of the old surface is beneficial to bond. When the old surface was roughened, cleaned, either treated with hydrochloric acid or coated with cement grout, a joint of about 80 per cent efficiency was obtained. Merely wetting the surface gave an efficiency of about 40 per cent, and wetting and roughening something more than 50 per cent. If the old

[1] *Proc. ACI*, vol. 27, pp. 677, 761, 1931; vol. 28, p. 157, 1932.

concrete is not thoroughly wetted, it will draw the moisture from the new concrete, often leaving not enough in the latter for a normal consistency, and resulting in weak concrete near the joint as well as producing a joint of low efficiency. (For bond between concrete and steel, see Art. 2.6.)

**1.28. Weight.** The weight of a concrete varies somewhat with the proportions of the mix, the consistency, and the character of the aggregate. The richer concretes are slightly heavier, and the wetter consistencies are lighter, except when porous lightweight aggregates are used as the coarse aggregate. A stone or gravel concrete will usually weigh between 140 and 150 pcf, with an average of about 145 pcf. In reinforced concrete the steel adds from 3 to 5 pcf, and the weight of reinforced concrete (including the steel) is usually taken as 150 pcf. The weight of some structural lightweight concrete may be as low as 100 pcf.

**1.29. Resistance to Fire.** Concrete is not only incombustible, but also a poor conductor of heat. Hence it is a splendid fire-resisting and fireproofing material. Clay products and building stones are equally noncombustible, but they possess greater conductivity and a higher coefficient of expansion. The low coefficient of expansion lessens the tendency to crack when heated, and the low conductivity prevents the transference of the heat of the fire to the interior of the mass and to the reinforcing steel. Tests of conductivity have shown that when the surface of a mass of concrete is exposed for hours to a high heat, the temperature at a depth of 1 in. beneath the surface is considerably lower, while at a depth of 3 in. or more the rise in temperature is very slight.

The low thermal conductivity of concrete is to a large extent due to voids in the material. Neat cement, with a void content about twice that of the average concrete, shows a corresponding decrease in its conductivity. It is also partly due to the absorption of the heat of vaporization by the water of combination in the hardened cement. The absorption of heat by the surface material as it becomes dehydrated retards the dehydration of the concrete beneath. The surface concrete which is injured by heat, but which remains in place, affords protection for the material farther in, as it is a poorer conductor than the original concrete.

The experience gained from some of the great fires, for example, those of Baltimore and the Edison Plant, etc., has shown that concrete exposed to high heat for a considerable length of time becomes calcined to a depth of from $\frac{1}{4}$ to $\frac{3}{4}$ in. but shows no tendency to spall off except at exposed corners and edges.

For fire-resistant construction it is required that all metal reinforcement should be protected against fire by at least $1\frac{1}{2}$ in. of concrete in beams, girders, and columns and at least $\frac{3}{4}$ in. of concrete in slabs. This minimum requirement applies only when the aggregates consist of

materials which change in volume a relatively small amount when exposed to high temperature. For other aggregates such as granite, quartzite, siliceous gravel, sandstone, gneiss, and cinders containing from 25 to 40 per cent of combustible material, the protection should be increased to 2 in. for beams, girders, and columns and to 1 in. for slabs.

An idea of the severe test which concrete may be expected to pass as a fire-resisting material may be obtained from the following specification of the Building Code of the City of New York for fireproof partition walls.

A vertical panel of not less than 14 ft. long and 9 ft. high shall be subjected to a fire continuous for not less than 1 hour at an average temperature of 1700 deg. Fahrenheit during the latter half hour, followed by an application for not less than 2½ minutes of a hose stream from a 1⅛-in. nozzle at 30 lb. nozzle pressure, without passage of flame during the test.

**1.30. Weathering Qualities.** The principal weathering agencies affecting the durability of concrete are variation in temperature, wind, rain, and variation in moisture conditions. Changes in temperature and moisture conditions cause more or less expansion and contraction in concretes, which in turn are apt to cause cracking that may result in ultimate failure. Cracking due to variations in temperature is likely to be confined principally to the surface of a structure, and may be made less harmful by the use of steel reinforcement so placed that a multitude of small cracks, which often are not visible to the naked eye, replace a few large and deep cracks. Expansion and contraction due to moisture changes are oftentimes more serious, as the moisture may penetrate the concrete farther and cause dangerous stresses to be introduced. The expansion and contraction of rich mixes are considerably more than those of the leaner mixtures, when moisture and temperature conditions vary. This circumstance is often responsible for the cracking off of a rich surface coat floated or plastered on a leaner base. The surface material not only tends to expand and contract more on account of its comparative richness, but it protects the underlying material from going through the extensive temperature and moisture changes which it itself is experiencing. To prevent the ultimate spalling off of this surface layer, as lean and as thin a surface coat as possible should be used, and where practicable, it should be applied before the leaner base has set so as to make the bond between the two as strong as possible.

**1.31. Abrasive Resistance.** The extensive use of concrete in the construction of roads, pavements, and floors makes its resistance to wear or abrasion an important consideration. In general, a concrete of high compressive strength will have a high resistance to abrasive action. Abrasion either wears away the cement and sand grains or it pulls the

sand grains out of the cement matrix. It follows, therefore, that with soft aggregates more cement will be needed in order to keep the wear low, while with hard and durable aggregates just sufficient cement is needed to hold the aggregate against the abrasive action. The quantity of mixing water used, however, the length of time of the mixing, and the curing conditions have more effect on the abrasive resistance than the hardness of the aggregate, and a good wearing surface can be produced with inferior aggregates if other conditions are favorable. Provided the proper precautions are taken, and a good-quality concrete produced, the actual wear on the surface of either a pavement or a floor will not be of any serious amount, no matter how heavy the traffic.

**1.32. Effect of Various Substances on Hardened Concrete.** *Sea Water.* Almost invariably, specifications forbid the use of sea water in mixing concretes. The detrimental effect to the concrete itself is usually not so large except in very lean mixtures, but in reinforced concrete construction where sea water is used in mixing, the corrosion of the steel is likely to be serious, and may eventually result in the complete destruction of the work.

The reliability of concrete and reinforced concrete when exposed to the action of sea water is variable, but, under favorable conditions and with proper care, structures comparing favorably in durability with those of timber or steel can be constructed. A richer and denser mix should be used than for ordinary construction, in order to ensure against the infiltration of water into the concrete, which causes, in the case of plain concrete, complete disintegration due to chemical reaction, or in the case of reinforced concrete, failure due either to electrolysis or to rusting of the steel. Just enough fresh water should be used in mixing so that the concrete settles around the reinforcing rods with light tamping. The forms should be oiled to ensure a smooth surface, and if practical, a richer mortar coat should be applied next the forms at the same time the other concrete is placed. Reinforcing material should be protected by at least 3 in. of concrete. Construction joints should be avoided, and the concrete should not be exposed to the sea water until it is thoroughly hardened.

Reinforced concrete in sea water subjected to intermittent saturation should have a water-cement ratio not greater than 5 gal of water per sack of cement, and where complete continuous submergence in sea water is to be realized, the water-cement ratio should not exceed 6 gal per sack of cement. All reinforcement should be protected by at least 3 in. of concrete.

*Alkali Waters.* Alkali waters have the same effect on concrete as sea water and the same precautions should be exercised as for sea water, where the concrete is to be subject to this exposure.

*Oils, Acids, and Sewage.* Concrete thoroughly hardened is unaffected by mineral oils such as ordinary petroleum or engine oils. Various animal and vegetable oils may slightly weaken and disintegrate a concrete, but such cases are rare.

Acids which seriously injure other materials will also injure concrete. The condition where this is most likely to occur is in the discharge of acids in sewage. Strong sulfuric acid in contact with the concrete converts the carbonate of lime into sulfate of lime, which is soft and easily corroded. Two factors, however, tend to make this effect less marked: (1) the likelihood of the acid being so much diluted by the water of the sewage as to be practically harmless, and (2) the greasy substance which is usually found to coat the perimeter of a sewer under the water line prevents the full action of the acid upon the cement.

become inoperative and the entire tension force in the beam is resisted by the steel reinforcement. The cracks propagate to a level only slightly below the neutral axis. They do not cross the axis, since they are caused only by the low resistance of concrete to tension and hence will not propagate into the compression zone of the beam.

These cracks are directed perpendicular to the tension stresses by which they are caused. Hence, in the central portion of the beam of Fig. 3.1 where pure bending prevails and tension stresses are horizontal, the cracks are vertical in direction. In the outer portions of the beam, where shear combines with moment, the cracks form approximately perpendicular to the inclined tension stresses $t$ (see Art. 3.1) and, hence, are inclined to the horizontal by an amount approximately equal to $90° - \alpha$.

It is evident from this discussion that cracks will form on the tension side of reinforced concrete beams under normal working loads. These

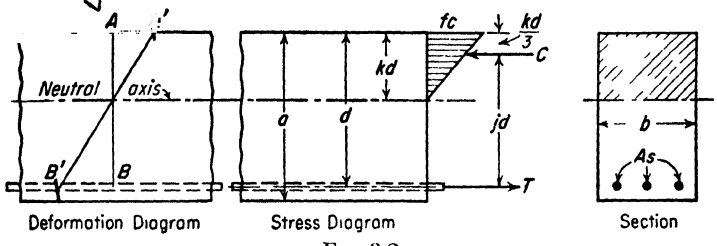



FIG. 3.2

cracks, in well-designed beams, are so small that they usually escape detection, and are known as hairline cracks. They are an inevitable occurrence and have no detrimental effect on the performance of the beam.

In a section through a crack, such as $aa$ in Fig. 3.1, the portion of the beam which carries stresses consists of the part of the concrete above the neutral axis, which is in compression, and the tension steel. The crack, which actually stops slightly below the neutral axis, for the sake of simplicity is assumed to have propagated to the axis itself.

For this reason the stress analysis that follows is based only on the uncracked part of the concrete section, that is, on the shaded portion of the section shown in Fig. 3.2.

Figure 3.2 represents a portion of a rectangular reinforced concrete beam. Let $AB$ represent any cross section before the load is applied to the beam and $A'B'$ the same cross section after the load is applied. The upper fibers of the beam (the compression fibers) will tend to shorten, and the lower fibers will lengthen. According to Assumption 1, Art. 3.2, the deformation at any horizontal plane in the beam is proportional to the distance from that plane to the neutral axis. With proper interpre-

tation, the distance $AA'$ may be considered to represent the shortening of the extreme upper fibers for a unit length of beam, and $BB'$ the unit elongation of the steel. In the stress diagram, $f_c$ represents the unit compressive stress in the extreme fiber at the section $AB$. The total compressive $C = (\frac{1}{2}f_c kd)b = \frac{1}{2}f_c kbd$, and the total tension, neglecting that in the concrete, is $T = A_s f_s$, in which $f_s$ is the unit stress in the steel and $A_s$ the cross-sectional area of the steel. For the general notation used in the following discussion, see Appendix A.

For equilibrium, the total compressive force in a beam must equal the total tensile force. From Fig. 3.2,

$$\tfrac{1}{2}f_c kbd = A_s f_s \tag{a}$$

From the assumption that deformations vary as the distance from the neutral axis,

$$\frac{AA'}{BB'} = \frac{kd}{d - kd} \tag{b}$$

Since $E = \dfrac{\text{unit stress}}{\text{unit deformation}}$, it follows that

$$AA' = \frac{f_c}{E_c} \quad \text{and} \quad BB' = \frac{f_s}{E_s}$$

Hence

$$\frac{AA'}{BB'} = \frac{E_s}{E_c}\frac{f_c}{f_s} = \frac{nfc}{f_s} \tag{c}$$

Equating $(b)$ and $(c)$,

$$\frac{nf_c}{f_s} = \frac{kd}{d - kd} \tag{d}$$

from which

$$f_s = \frac{nf_c(1 - k)}{k} \tag{3.1}$$

or

$$f_c = \frac{f_s k}{n(1 - k)} \tag{3.1a}$$

Equations (3.1) and (3.1$a$) give the relation between the actual simultaneous stresses in the steel and the concrete in any beam at any stage of loading, provided the value of $k$ is known. The equation for $k$ is derived in the following paragraph.

The actual ratio of the area of the steel to the effective cross-sectional area of the concrete is[1]

[1] In this definition the effective area of the cross section is taken as $bd$ rather than $ba$ for the following reason: Since the concrete below the neutral axis is cracked, its only function is to hold the steel bars in place, *i.e.*, to fix the distance $d$. Any con-

$$p = \frac{A_s}{bd}$$

Hence, from Eq. (*a*),

$$k = \frac{pf_s}{\frac{1}{2}f_c} \tag{e}$$

Substituting in Eq. (*e*) the value of $f_s$ from Eq. (3.1),

$$k = \frac{2pn(1 - k)}{k}$$

Solving for $k$,

$$k = \sqrt{2pn + (pn)^2} - pn \tag{3.2}$$

This value of $k$ is independent of the unit stresses in the steel and concrete but is dependent upon the proportion of steel in the beam and the ratio of the moduli of elasticity of the two materials. It is to be used in reviewing, *i.e.*, in calculating unit stresses or resisting moments of a beam whose dimensions and amount of reinforcement are known.

From Fig. 3.2, the lever arm $jd$ of the internal stress couple is

$$jd = d - \frac{kd}{3}$$

hence

$$j = 1 - \frac{k}{3} \tag{3.3}$$

The resisting moment of a beam is dependent upon the strength of either the steel or the concrete. The resisting moment of each is equal to the total force in each, *i.e.*, compression in concrete and tension in steel, multiplied by the lever arm $jd$ of the couple, or

$$M_c = (\tfrac{1}{2}f_c kbd)jd = \tfrac{1}{2}f_c kjbd^2 \tag{3.4}$$

$$M_s = A_s f_s jd \tag{3.5}$$

Since $A_s = pbd$, Eq. (3.5) may also be written as follows:

$$M_s = pf_s jbd^2 \tag{3.5a}$$

The actual internal moments as expressed by Eq. (3.4) and (3.5) are each equal to the external bending moment at all stages of loading (for equilibrium), but if the maximum allowable value of the resisting moment of the concrete $M_c$ is reached before that of the steel $M_s$, it means that the beam will be overstressed on the compression side before the maximum allowable fiber stress in the steel is reached; *i.e.*, the beam has more steel than is theoretically required—it is overreinforced.

crete below the level of the bars is, therefore, ineffective structurally and is provided only to furnish insolation and protection for the bars.

In designing a reinforced concrete beam, it is desirable to place in the beam an amount of steel such that the limiting unit stresses or limiting resisting moments as expressed by Eqs. (3.4) and (3.5) or (3.4) and (3.5*a*) shall be reached simultaneously. If this ideal steel ratio is obtained

$$M_c = M_s = \tfrac{1}{2}f_c kjbd^2 = A_s f_s jd = pf_s jbd^2 \qquad (f)$$

or

$$M = Kbd^2 \qquad (3.6)$$

where $K = \tfrac{1}{2}f_c kj$ or $pf_s j$.

If the ratio $f_s/f_c = r$, Eq. (*d*) reduces to the form $n/r = k/(1 - k)$, and solving for $k$,

$$k = \frac{n}{n + r} \qquad (3.7)$$

This value of $k$ depends only upon the unit stresses in the steel and in the concrete and upon the value of the ratio $n$. Therefore Eq. (3.7) cannot be used in review, since the simultaneous values of $f_s$ and $f_c$ are not known.

An expression for the ideal steel ratio may be obtained as follows: since with this ideal percentage of steel $\tfrac{1}{2}f_c kj = pf_s j$ [Eq. (*f*)], it follows that $p = k/2r$, and since for any given values of $f_s$ and $f_c$, $k = n/(n + r)$, the equation for $p$ becomes

$$p = \frac{n}{2r(n + r)} \qquad (3.8)$$

This ratio is known as balanced reinforcement.

The values of $M_c$ and $M_s$ will be equal to each other only when the amount of steel placed in the beam is such that the actual steel ratio, $p = A_s/bd$, is equal to the value given by Eq. (3.8). On account of the commercial sizes of reinforcing steel in use, the actual ratio will usually be greater or less than the ideal. In the former case $M_s$ will be greater than $M_c$ and the strength of the beam will be limited by that of the concrete. For underreinforced beams, in which the actual steel ratio is less than the ideal ratio, the reverse will be true.

**3.5. Application of Equations to Design and Review Problems.** The equations previously developed are summarized below.

$$f_s = \frac{nf_c(1 - k)}{k} \qquad (3.1)$$

$$f_c = \frac{f_s k}{n(1 - k)} \qquad (3.1a)$$

$$k = \sqrt{2pn + (pn)^2} - pn \text{ (review)} \qquad (3.2)$$

$$j = 1 - \frac{k}{3} \qquad (3.3)$$

$$M_c = \tfrac{1}{2}f_c kjbd^2 \qquad (3.4)$$

$$M_s = A_s f_s jd \tag{3.5}$$

$$= pf_s jbd^2 \tag{3.5a}$$

$$M = Kbd^2 \tag{3.6}$$

$$k = \frac{n}{n + r} \text{ (design only)} \tag{3.7}$$

$$p = \frac{n}{2r(n + r)} \tag{3.8}$$

To design a beam, either Eq. (3.4) or (3.6) may be used to determine the cross section required to ensure against crushing of the concrete under any given bending moment, and Eq. (3.5) may be used to determine the area of steel necessary to develop the full strength of the concrete in compression. Values of $k$ and $j$ are obtained from Eqs. (3.7) and (3.3).

Economic and constructional considerations are usually best served when the cross section of a rectangular beam is so proportioned that $b$ is from one-half to three quarters of $d$. Construction limitations make it undesirable to select $d$ in multiples of less than $\frac{1}{2}$ in., and sometimes 1-in. multiples are used. Also, in order to keep the amount of mill or carpenter work as small as possible, the width of beam $b$ should be chosen so that a plank of standard width may be used for the bottom form. In fulfilling this requirement, most designers consider the nominal width of the plank and hence proportion all beams for widths of even integral inches. Some designers, however, prefer to consider the actual width of the lumber available after dressing and to proportion their beams accordingly. The nominal width is used in all the problems in this text.

To determine the resisting moment of a given beam of known dimensions and reinforcement, Eqs. (3.4) and (3.5) or (3.4) and (3.5$a$) should be solved for $M_c$ and $M_s$; the smaller value is the required moment. To determine the maximum unit stresses $f_s$ and $f_c$ in a beam of known dimensions and condition of loading, Eqs. (3.4) and (3.5), or (3.4) and (3.1), or (3.5) and (3.1$a$) can be used, substituting for $M_s$ or $M_c$ the maximum external bending moment $M$. In any review problem, values of $k$ and $j$ are obtained from Eqs. (3.2) and (3.3).

**3.6. Approximate Design Moments.** The value of the external bending moment varies with the form of support and the type of loading. For beams resting on two supports, one at each end, and not restrained in any way, with a uniformly distributed load the moment is equal to $\frac{1}{8}wl^2$. A partially continuous beam (continuous over one support only) has a moment of approximately $\frac{1}{10}wl^2$ (negative over support and positive at the center of span), and a fully continuous beam (continuous over two or more supports) has a moment of approximately $\frac{1}{12}wl^2$ (negative over support and positive at center of span), where $w$ is the load per unit of length and $l$ is the distance center to center of supports. Concen-

trated loads, partial loads, or irregular uniform loads cause a variety of moment coefficients, depending upon the number of spans and the amount of continuity to be considered.

When reinforced concrete construction was in its infancy, design followed the procedure outlined above. However, most reinforced concrete building construction consists of an arrangement of slab, beams, and columns built monolithically with slab and beam supports, often of considerable width, so that the critical negative moment section is not at the center of the support but at the edge of the support. With the critical negative-moment section at this point, the maximum positive moment at the center of the span (or elsewhere) is also reduced, as the effective span length becomes the distance between faces of supports and the restraint at the support further reduces the positive moment.

In order to take the above conditions into consideration, the ACI Code specifies that a satisfactory approximate method of design for buildings of the usual type of construction, spans, and story heights is as follows.

In case of two or more approximately equal spans (the larger of two adjacent spans not exceeding the shorter by more than 20 per cent) with loads uniformly distributed, where the unit live load does not exceed three times the unit dead load, design for the following moments and shears is satisfactory:

| | |
|---|---|
| Positive moment at center of span: | |
| End spans* | $\frac{1}{14}\ wl'^2$† |
| Interior spans | $\frac{1}{16}\ wl'^2$ |
| Negative moment at exterior face of first interior support: | |
| Two spans | $\frac{1}{9}\ wl'^2$ |
| More than two spans | $\frac{1}{10}\ wl'^2$ |
| Negative moment at other faces of interior supports | $\frac{1}{11}\ wl'^2$ |
| Negative moment at face of all supports for (*a*) slabs with spans not exceeding 10 ft and (*b*) beams and girders where ratio of sum of column stiffnesses to beam stiffness exceeds 8 | $\frac{1}{12}\ wl'^2$ |
| Shear in end members at first interior support | $1.15\frac{wl'}{2}$ |
| Shear at other supports | $\frac{wl'}{2}$ |

* Where a condition such as (*a*), below, exists, it is recommended that the positive moment at the center of the end span be taken as $\frac{1}{11}\ wl'^2$.

† $l'$ is the clear span for positive moment and shear and the average of two adjacent clear spans for negative moment.

It will be noted that the Code does not specify a negative-moment coefficient for an exterior support. Three conditions may exist at such a support: (*a*) the beam or slab may rest freely on the support or be bricked in a supporting wall, (*b*) the beam or slab may be constructed integrally with a supporting beam, (*c*) the beam may be constructed integrally with a supporting column. Where a condition such as (*a*)

exists, the negative moment is zero or so small that it may be neglected. For the condition described in (*b*), the amount of negative moment developed depends upon the torsional resistance of the supporting beam. It is recommended for this condition that the negative moment at the face of the exterior support be taken as $\frac{1}{24}wl'^2$. For the condition described in (*c*), due to the greater stiffness of the column support, a larger negative moment can be expected and the negative moment recommended for consideration at the face of the support is $\frac{1}{16}wl'^2$, unless the ratio of the sum of the column stiffnesses to the beam stiffness exceeds 8, in which case the moment coefficient as given above is used.

The span length $l$ of members that are not built integrally with their supports is, however, taken as the clear span plus the depth of the slab or beam, or the distance center to center of supports, whichever is the lesser.

In order not to complicate the introduction of continuity into the design of reinforced concrete beams, the majority of the illustrative problems of this chapter will consider that beams and slabs are not built integrally with their supports, and the span length will be taken as the distance center to center of supports.

The moment values given here are reasonably accurate and somewhat conservative for average conditions, *i.e.*, for uniformly distributed loads and for nearly equal spans. They should not be used except for these conditions. For all other situations it is necessary to find moments and shears by detailed analysis. Since most reinforced concrete structures are continuous, *i.e.*, do not consist of individual simple beams, methods of analysis of statically indeterminate structures must be used for determining moments, shears, and forces. Such statically indeterminate analysis has become an integral part of reinforced concrete design. The most important pertinent methods are briefly discussed in Sec. 6. For a more detailed treatment of this important topic, the reader is referred to any of a number of good texts on indeterminate structures.

**3.7. Placing the Reinforcement.** In placing the reinforcement, three general requirements must be fulfilled. First, there must be sufficient space between the bars to permit proper placing of the concrete around them; second, there must be sufficient concrete in the plane of the bars properly to transmit the stresses of tension and shear; third, there must be sufficient concrete below the steel to afford ample protection for the steel against moisture and fine damage.

The ACI Code specifies that the clear distance between parallel bars shall be not less than (*a*) 1 in.; (*b*) the nominal diameter of the bars; (*c*) $1\frac{1}{3}$ times the maximum size of the coarse aggregate.

The ACI Code also specifies that the concrete protective covering for reinforcement at surfaces not exposed directly to the ground or weather

shall be not less than ¾ in. for slabs and walls and not less than 1½ in. for beams, girders, and columns. If the concrete surface is to be exposed to the weather or in contact with the ground, a protective covering of at least 2 in. is required, except that if the concrete is to be poured in direct contact with the ground, without the use of forms, a covering of at least 3 in. must be furnished.

In general, the centers of bars in beams should be at least 2½ in. from the bottom surface of the beam, in order to furnish at least 1½ in. of clear insulation below the bars and the stirrups that are used for web reinforcement (see Fig. 3.3). If no stirrups are required, this distance from the center of the bar to the surface can be made 2 in., provided that the bars are not larger than 1 in. in size. In slabs, 1 in. to the center of the bars is ordinarily sufficient to give the required ¾ in. insulation. These conditions are illustrated in Fig. 3.3. Although the distances 2½, 2, and

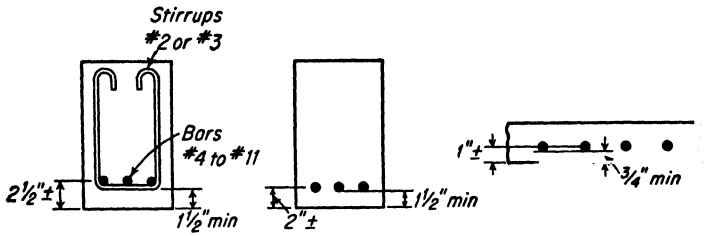



FIG. 3.3

1 in., shown in Fig. 3.3, are not the exact distances required to furnish the clear insulation specified in the Code, they will in practically all cases satisfy the minimum requirements and are sufficiently exact for all design purposes. Total depths of beams should be taken in multiples of not less than ½ in., and preferably 1 in., so that effective depths ($d$) will also be in multiples of $^1{}_2$ in., or 1 in., if the above-mentioned distances to the centers of the bars are used.

**3.8. Allowable Unit Stresses.** The allowable unit working stresses in the concrete, as specified in the ACI Code, are given in Appendix B. These stresses are given as a percentage of the ultimate compressive strength, $f'_c$, of the concrete at the age of 28 days (see Art. 1.19). Thus, the safe working unit stress in flexure is 0.45 $f'_c$, which gives for 2500-psi concrete a value of 0.45 × 2500, or 1125 psi. A somewhat lower unit stress in flexure is specified in many municipal building codes. To emphasize this fact, some of the following problems are solved with assumed working stresses which do not agree with the recommendations as given in Appendix B. Where such exception is taken, the assumed allowable unit stresses are given in the data of the problem.

Concretes used in ordinary building construction are proportioned so as to develop ultimate compressive strengths varying usually from 2500 to 3750 psi. The lower-strength concretes are normally used for the floor systems and for footings, while the higher-strength concretes are used for the columns. In special cases, greater-strength concretes may be used.

**3.9. Illustrative Examples.** I. A rectangular simply supported reinforced concrete beam has a total cross section of 12 by 24 in. and a length of 20 ft-0 in. It is reinforced with four No. 8 bars in one row, the centers of the bars being 2½ in. above the lower surface of the beam. Assuming a 2500-psi concrete, structural-grade steel, and following the specifications of the ACI Code, Appendix B, what is the resisting moment of the beam?

$$p = \frac{A_s}{bd} = \frac{3.14}{12 \times 21.5} = 0.0122$$

$$k = \sqrt{2 \times 0.0122 \times 12 + (0.0122 \times 12)^2} - 0.0122 \times 12 = 0.414$$

$$j = 1 - \frac{0.414}{3} = 0.862$$

From Eqs. (3.4) and (3.5),

$$M_c = \tfrac{1}{2} \times 1125 \times 0.414 \times 0.862 \times 12(21.5)^2 = 1{,}114{,}000 \text{ in.-lb}$$
$$M_s = 3.14 \times 18{,}000 \times 0.862 \times 21.5 = 1{,}047{,}000 \text{ in.-lb}$$

The beam is therefore underreinforced, the strength of the steel governs, and the resisting moment is 1,047,000 in.-lb.

II. Use the beam of the preceding problem, and compute the value of the unit stress in the steel ($f_s$) and in the concrete ($f_c$), if a uniform live load of 1400 lb per lin ft is placed upon it.

$$\text{Weight of beam} = \frac{12 \times 24}{144} \times 150 = 300 \text{ lb per ft}$$

$$\text{Total load} = 1400 + 300 = 1700 \text{ lb per ft}$$

$$M = \tfrac{1}{8} \times 1300 \times 20^2 \times 12 = 1{,}020{,}000 \text{ in.-lb}$$

From Example I, $k = 0.414$, and $j = 0.862$. Substituting in Eq. (3.5),

$$1{,}020{,}000 = 3.14 \times f_s \times 0.862 \times 21.5$$
$$f_s = 17{,}500 \text{ psi}$$

From Eq. (3.1$a$),

$$f_c = \frac{17{,}500 \times 0.414}{12(1 - 0.414)} = 1030 \text{ psi}$$

The value of $f_c$ could also have been obtained from Eq. (3.4) as follows:

$$1{,}020{,}000 = \tfrac{1}{2} \times f_c \times 0.414 \times 0.862 \times 12 \times (21.5)^2$$
$$f_c = 1030 \text{ psi}$$

III. If the beam of Example I were to support a single concentrated load at the mid-span, what would be the maximum safe load that could be so placed? From the solution of Example I, the resisting moment is 1,047,000 in.-lb. The weight of the beam is 300 lb per ft, and the moment caused by this dead load is $\frac{1}{8} \times 300 \times 20^2 \times 12 = 180{,}000$ in.-lb. The maximum moment available for the concentrated load is

$$1{,}047{,}000 - 180{,}000 = 867{,}000 \text{ in.-lb}$$

The maximum moment caused by the concentrated load is $M = \frac{1}{4}Pl$; hence

$$\frac{1}{4}P \times 20 \times 12 = 867{,}000$$

from which

$$P \text{ (maximum)} = 14{,}700 \text{ lb}$$

It should be noted that if the concentrated load were not at the mid-span, the maximum moments due to the dead load and the concentrated load would not occur in the same cross section. If the concentrated load were near the mid-span, it is very probable that the section of maximum moment would coincide with the section at which the concentrated load was placed. The dead-load moment at this section, instead of the maximum dead-load moment, should be deducted from the resisting moment of the beam in order to obtain the maximum moment available for the concentrated load (see Example III, Art. 3.11).

IV. Determine the cross section of concrete and area of steel required for a simply supported rectangular beam with a span of 18 ft-0 in. which is to carry a uniform live load of 1400 lb per lin ft. A 2500-psi concrete is to be used, and the allowable unit stresses are to be as specified in the ACI Code. The allowable unit stress in the steel is 18,000 psi.

Assuming that the weight of the beam is 200 lb per lin ft, the total load to be carried is 1600 lb per lin ft, and the actual external bending moment is

$$M = \frac{1}{8} \times 1600 \times 18^2 \times 12 = 778{,}000 \text{ in.-lb}$$

$$r = \frac{18{,}000}{1125} = 16.0$$

From Eq. (3.7),

$$k = \frac{12}{12 + 16.0} = 0.428$$

$$j = 1 - \frac{0.428}{3} = 0.857$$

Substituting in Eq. (3.4),

$$778{,}000 = \frac{1}{2} \times 1125 \times 0.428 \times 0.857 \times bd^2$$

from which

$$bd^2 = 3770 \text{ in.}^3$$

Let $b$ be taken as 10 in.; then $d$ (required) = 19.4 in., and

$$d \text{ (selected)} = 20 \text{ in.}$$

Adding 2 in. below the center of the steel, the total cross section is 10 by 22 in., and the weight of the beam is 230 lb per lin ft.

The actual weight of the beam does not agree with the assumed value. Hence it is necessary to check back to see whether any revision should be made in the design. The revised bending moment is

$$\begin{aligned} M &= \tfrac{1}{8} \times 1630 \times 18^2 \times 12 = 792{,}000 \text{ in.-lb} \\ 792{,}000 &= \tfrac{1}{2} \times 1125 \times 0.428 \times 0.857 \times bd^2 \\ bd^2 &= 3830 \text{ in.}^3 \end{aligned}$$

With $b = 10$ in., $d = 19.6 = 20$ in. Since these results agree with those assumed in the revision, the design is satisfactory.

With these values of $b$ and $d$, the required area of steel is obtained from Eq. (3.5) as follows:

$$\begin{aligned} 792{,}000 &= A_s \times 18{,}000 \times 0.857 \times 20 \\ A_s &= 2.57 \text{ sq in.} \end{aligned}$$

Two No. 8 bars and one No. 9 bar, area 2.57 sq in., are selected.

If the ideal steel ratio had been used in determining the quantity of steel to be placed in the beam, the procedure would have been as follows: from Eq. (3.8),

$$p = \frac{12}{2 \times 16.0(12 + 16.0)} = 0.0134$$

and

$$A_s = 0.0134 \times 10 \times 20.0 = 2.68 \text{ sq in.}$$

This, it will be noticed, is slightly in excess of the required area, as determined by the first method. This difference may be accounted for as follows: The value of $A_s$, as computed by this latter method, represents an area of steel that will develop a tensile force in the steel equal to the full allowable compressive strength of the concrete of a beam whose effective dimensions are 10 by 20 in. But in order fully to develop a moment of 792,000 in.-lb as required by the problem, an effective cross section of only 10 by 19.6 in. was needed. The greater value of $d = 20.0$ in. was selected to simplify the dimensioning of the plans. The development of the full strength of a 10- by 20-in. beam therefore furnishes an excess of steel over that required to provide for the maximum external bending moment in the beam, and hence is a waste of material. In this case the two No. 8 bars and the one No. 9 bar would not be sufficient. A greater difference between the theoretical value of $d$ required and that actually furnished would emphasize this to a greater extent.

The latter method involving the ideal steel ratio is therefore not recommended for general use.

## Problems

1. A simply supported rectangular beam has a total cross section of 10 by 16 in. and a length of 20 ft-0 in. It is reinforced with four No. 5 bars in one row. The distance from the centers of the bars to the lower surface of the beam is 2½ in. With 2500-psi concrete and intermediate-grade steel, what is the resisting moment of the beam?

2. If a concentrated load of 3500 lb were placed on the beam of Prob. 1, at a distance of 7 ft-0 in. from the support, what would be the maximum unit stress in the concrete and the maximum unit stress in the steel?

3. A simply supported rectangular beam with a span of 18 ft-0 in. supports a uniform live load of 975 lb per lin ft and a concentrated load of 3000 lb at the middle of the span. With $f'_c = 3000$ psi and with intermediate-grade reinforcing steel, determine the required cross section and steel area.

4. A simply supported rectangular beam with a span of 17 ft-0 in. supports a live load which varies in amount from zero at the left support uniformly to an amount of 1000 lb per lin ft at the right support. A 2500-psi concrete and structural-grade reinforcing bars are to be used. Design the beam

5. A rectangular beam, simply supported, has a width of 14 in. and an effective depth of 26 in. If $n = 12$ and if the allowable unit stresses are 1000 psi and 20,000 psi for the concrete and steel, respectively, what steel area must be used in order that the resisting moment with respect to the strength of the concrete may be the same as the resisting moment with respect to the steel?

6. What would be the resisting moment of the beam in Prob. 5, if the reinforcement consisted of four No. 7 bars?

**3.10. Tables for Rectangular Beams.** Many of the computations involved in the design and review of rectangular beams may be eliminated by the use of previously prepared tables. For example, in design, the value of $k = n/(n + r)$ depends only upon the ratio of the moduli of elasticity and the allowable unit stresses of the two materials. Table 5, Appendix D, gives, for the most common values of $n$ and for all practical combinations of $f_s$ and $f_c$, the values of $k$ as determined by the above equation. Corresponding values of $j = 1 - (k/3)$, $K = \frac{1}{2}f_c kj = pf_s j$ (for use in the formula $M = Kbd^2$), and $p = n/2r(n + r)$ are also given.

Similarly, Table 6, Appendix D, for use in the review of beams, gives the value of $k = \sqrt{2pn + (pn)^2} - pn$, and $j = 1 - (k/3)$ for sufficient values of the variable $p$ and $n$ to make the solution of ordinary problems possible with but a slight amount of interpolation.

**3.11. Illustrative Examples Involving the Use of Tables.** I. The use of Table 5, Appendix D, in designing a beam may be shown by its application to Example IV, Art. 3.9. From this table (for values of $n = 12$, $f_c = 1125$, and $f_s = 18{,}000$), $K = 207$ and $j = 0.857$. Therefore, from Eq. (3.6),

$$792{,}000 = 207\ bd^2$$

and

$$bd^2 = 3830 \text{ in.}^3$$

Let $b = 10$ in.; then $d = 19.6$, or 20 in. From Eq. 3.5,

$$792{,}000 = A_s \times 18{,}000 \times 0.857 \times 20$$

from which

$$A_s = 2.57 \text{ sq in.}$$

The use of the equation $M = Kbd^2$ is identical with the use of

$$M_c = \tfrac{1}{2} f_c k j b d^2$$

since $K = \frac{1}{2} f_c k j$.

II. Problem I, Art. 3.9, may be solved by reference to Tables 5 and 6. The problem might be reworded as follows: A rectangular reinforced concrete beam has a total cross section of 12 by 24 in. It is reinforced with four No. 8 bars in one row, the centers of the bars being $2\frac{1}{2}$ in. above the lower surface of the beam. As the load is increased, which will reach the

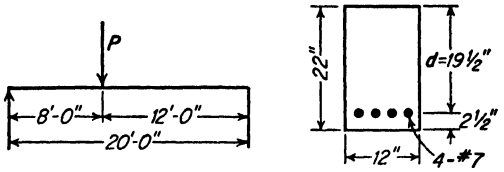



FIG. 3.4

specified stress limit first, the concrete or the steel, *i.e.*, is the beam overreinforced or underreinforced? What is the resisting moment of the beam? Use a 2500-psi concrete, structural-grade steel, and allowable unit stresses as given in the ACI Code.

The ideal percentage of steel required to give equal strengths in tension and compression, with allowable unit stresses of 18,000 and 1125, and with $n = 12$, is given in Table 5 as 0.0134. The actual value of $p$ in the beam $= 3.14/(12 \times 21.5) = 0.0122$. The beam is, therefore, underreinforced and its strength is limited by that of the steel. From Table 6, $k = 0.414$, and $j = 0.862$, and the resisting moment is

$$M = 2.14 \times 18{,}000 \times 0.862 \times 21.5 = 1{,}047{,}000 \text{ in.-lb}$$

NOTE: Table 6 has been used only to determine the relative strengths of the steel and the concrete in the beam, and not to determine the values of $k$ and $j$.

III. The concrete used in constructing the beam shown in Fig. 3.4 has an ultimate 28-day strength in compression of 2500 psi and the reinforcing bars are of intermediate-grade steel. What maximum concentrate load $P$ can be placed in the position shown without exceeding stresses of 1000 psi in the concrete and 20,000 psi in the steel?

$$p = \frac{2.41}{12 \times 19.5} = 0.0103$$

From Table 6, $k = 0.390$, and $j = 0.870$. Table 5 shows that the beam is overreinforced (ideal $p = 0.0094$), and the strength is governed by that of the concrete in compression.

$$M_c = \tfrac{1}{2} \times 1000 \times 0.390 \times 0.870 \times 12 \times (19.5)^2 = 776{,}000 \text{ in.-lb}$$

Since the principal load is the concentrated load (the only other load is the weight of the beam, which is comparatively small) and since this load is fairly close to the center of the span, it is probable that the maximum moment will occur under the load, *i.e.*, at a distance of 8 ft from the left support. At this section, the moment $M_D$ due to the weight of the beam (275 lb per ft) is

$$M_D = \left(275 \times \frac{20}{2} \times 8 - \frac{275 \times 8^2}{2}\right) 12 = 158{,}400 \text{ in.-lb}$$

The moment available for the concentrated load is then

$$776{,}000 - 158{,}400 = 617{,}600 \text{ in.-lb}$$

The moment under the concentrated load, caused by this load, is

$$P \times {}^{12}\!/_{20} \times 8 \times 12 = 57.6P \text{ in.-lb}$$

Equating this to its maximum allowable numerical value,

$$57.6P = 617{,}600$$

from which

$$P = 10{,}700 \text{ lb (maximum)}$$

With this load on the beam, in addition to the weight of the beam, the shear passes through zero under the concentrated load

$$(V_8 = 10{,}700 \times {}^{12}\!/_{20} + 275 \times {}^{20}\!/_{2} - 275 \times 8 - 10{,}700 = -3730 \text{ lb})$$

which indicates that the maximum moment occurs at this section, as assumed.

**3.12. Analysis of Rectangular Beams by the Principle of the Transformed Section.** In a homogeneous beam the neutral axis passes through the center of gravity of the cross section. A reinforced concrete beam can be treated as a homogeneous beam, if the steel is considered to be replaced by concrete, so placed, and of such an amount, as to produce the same effect as the steel in resisting the bending moment. This hypothetical concrete is assumed to be able to resist tension, in contrast to the real properties of the material. This alternate approach of replacing the real reinforced concrete section by fictitious, equally strong homogeneous sections gives the same results as those derived in Art. 3.4. For beams of standard configurations, it has no advantage over the first method. However, for beams of unusual cross-sectional shape or for

reinforcement of an unusual type or distribution, it is often useful since it allows the application of the simple, familiar formulas for the analysis of homogeneous beams. The equivalent amount of concrete and the required location of this concrete are obtained from the following analysis.

According to the theory of flexure, the unit stress on any fiber of a homogeneous beam at a given distance from the neutral axis is the same as the unit stress on any other fiber at the same distance from the neutral axis. Hence, if the moduli of elasticity of the steel and the concrete were equal to each other, the steel could be replaced by the same area of concrete in the same horizontal plane as the steel. The moduli of elasticity are not equal, however, and in order to resist the same total tensile stress the equivalent area of concrete must be modified accordingly. The deformation of the hypothetical equivalent concrete, which will be assumed in the same horizontal plane as the steel, is the same as that of the original steel, and, in Fig. 3.2, the deformation of a unit length of beam can be represented by the distance $BB'$. Since $E$ = unit stress divided by unit deformation, if the unit stress in the equivalent concrete is $f_{ec}$ and the unit stress in the steel is $f_s$, the following relations are obtained:

$$E_c = \frac{f_{ec}}{BB'} \quad \text{or} \quad BB' = \frac{f_{ec}}{E_c}$$

$$E_s = \frac{f_s}{BB'} \quad \text{or} \quad BB' = \frac{f_s}{E_s}$$

Hence

$$\frac{f_{ec}}{E_c} = \frac{f_s}{E_s} \quad \text{and} \quad f_s = \frac{E_s}{E_c} f_{ec} = n f_{ec}$$

It is thus seen that the unit stress in the steel is $n$ times the unit stress in the equivalent concrete, and in order to resist the same total force (caused by a given bending moment) every square inch of steel must be replaced by $n$ square inches of concrete, all in the same horizontal plane as the steel. The resulting equivalent homogeneous beam will then be as shown in the cross section in Fig. 3.5, all the tension being resisted by the shaded strip of concrete, of area $nA_s$, at a distance $d$ below the extreme compression fiber, and all the compression being resisted by the shaded concrete area above the neutral axis. The unshaded concrete area below the neutral axis is cracked and serves only as a means of connecting the effective tension and compression areas so as to permit the development of the stresses on these areas. The cross section shown in Fig. 3.5 is called the transformed section and represents

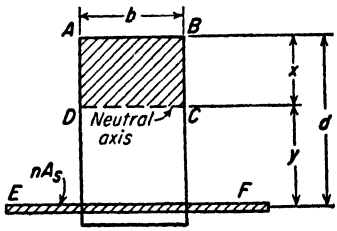



FIG. 3.5

the homogeneous section which would be equivalent, as far as resisting bending stresses is concerned, to the original composite steel-and-concrete beam.

Since the neutral axis of the homogeneous beam passes through the center of gravity of the cross section, the moment of the area $ABCD$ in the transformed section (Fig. 3.5) about the neutral axis $DC$ must equal the moment of the equivalent area $EF$ about $DC$, or

$$bx\frac{x}{2} = nA_s y = nA_s(d - x)$$

This one equation, in conjunction with others which are obvious from the fundamental flexure theories and assumptions, is all that is necessary to complete the analysis of beams with tension reinforcement.

**3.13. Illustrative Examples.** I. Solve Example I, Art. 3.9, by means of the principle of the transformed section. This problem involves the determination of the resisting moment of a beam whose dimensions and

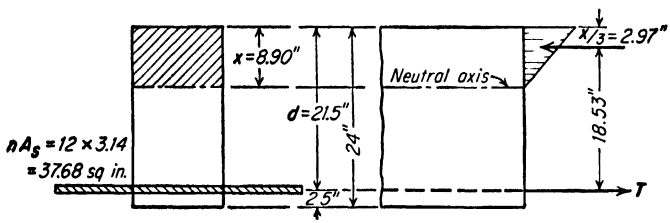



FIG. 3.6

reinforcement are known. The neutral axis is located from the equation in the preceding articles, as follows:

$$\frac{12x^2}{2} = 12 \times 3.14(21.5 - x)$$
$$6x^2 = 810.1 - 37.7x$$
$$x = 8.90 \text{ in.}$$

The arm of the resisting couple (see Fig. 3.6) is equal to

$$21.5 - (8.90/3) = 18.53 \text{ in.}$$

If the full strength of the steel is developed, the total tension will be $18{,}000 \times 3.14 = 56{,}500$ lb, and the simultaneous total compression will be the same. If the maximum allowable extreme compression fiber stress is developed, the total compressive force will be

$$\tfrac{1}{2} \times 1125 \times 8.90 \times 12 = 60{,}100 \text{ lb}$$

and the simultaneous total tensile force will also be 60,100 lb. The full strength of the concrete obviously cannot be developed without overstressing the steel in tension. The beam is therefore underreinforced,

and the resisting moment is equal to the total allowable tension multiplied by the lever arm of the internal stress couple, or

$$M = 56{,}500 \times 18.53 = 1{,}047{,}000 \text{ in.-lb}$$

This agrees with the value obtained in Example I, Art. 3.9.

II. By means of the principle of the transformed section, design the beam which is described in Example IV, Art. 3.9. Assuming that the weight of the beam is 230 lb per lin ft, the maximum bending moment is

$$M = \tfrac{1}{8} \times 1630 \times 18^2 \times 12 = 792{,}000 \text{ in.-lb}$$

In an ideal beam, the steel and the concrete are both stressed to the limit. When the beam is fully loaded, the extreme fiber stress in compression at the section of maximum moment should be 1125 psi and the simultaneous stress in the equivalent tension concrete should be 18,000/12 = 1500 psi.

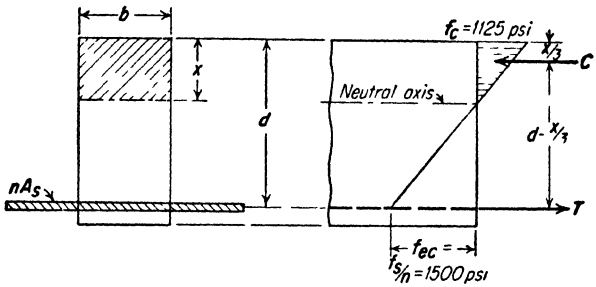



FIG. 3.7

The stress diagram is shown in Fig. 3.7. In order that these stresses may be realized, the following proportion must be true:

$$\frac{x}{d} = \frac{1125}{1125 + 1500}$$

from which

$$x = d(\tfrac{1125}{2625}) = 0.428d$$

The lever arm of the stress couple is then $d - (0.428d/3) = 0.857d$, and the resisting moment of the beam, expressed in terms of the strength of the concrete in the compression area, is

$$M = \tfrac{1}{2} \times 1125 \times 0.428d \times b \times 0.857d = 207bd^2$$

Since the resisting moment must be equal to the maximum bending moment, the following equation can be written:

$$792{,}000 = 207bd^2$$

from which

$$bd^2 = 3830 \text{ in.}^3$$

with $b = 10$ in., $d = 19.6$, or 20 in., and the weight of the beam is 230 lb per ft, as assumed.

With $d = 20$ in., the maximum compressive stress in the concrete will be slightly less than 1125 psi and the value of $x$ as computed above will no longer be the true value of $x$. The error will not be great, however, unless the discrepancy between the true value of $d$ and the actual value is of an appreciable amount. For design purposes, the neutral axis can be assumed to be in the same position relative to the top of the beam as computed above, and the lever arm of the stress couple is

$$0.857 \times 20 = 17.14 \text{ in.}$$

The total tension in the steel is then

$$T = \frac{792{,}000}{17.14} = 46{,}200 \text{ lb}$$

The required steel area is therefore

$$A_s = \frac{46{,}200}{18{,}000} = 2.57 \text{ sq in.}$$

These results agree with those obtained in Example IV, Art. 3.9.

If a theoretically correct solution were desired for the steel area, the true value of $x$, for $d = 20$ in., could be obtained by equating the resisting moment with respect to the strength of the compression concrete to the maximum bending moment. The maximum unit compressive stress in the extreme fiber in this equation would be $1500 \times x/(20 - x)$ instead of 1125 lb. The remainder of the solution is similar to that given above. The essential computations are as follows:

$$\left(\tfrac{1}{2} \times 1500 \times \frac{x}{20 - x} \times 10x\right)\left(20 - \frac{x}{3}\right) = 792{,}000 \text{ in.-lb}$$

$$x = 8.34 \text{ in.}$$

$$f_c = 1500 \times \frac{8.34}{20 - 8.34} = 1070 \text{ psi}$$

$$C = T = \tfrac{1}{2} \times 1070 \times 10 \times 8.34 = 44{,}600 \text{ lb}$$

$$A_s = \frac{44{,}600}{18{,}000} = 2.48 \text{ sq in.}$$

## SLABS

**3.14. Types of Slabs.** Slabs may be supported on two sides only, or they may rest on beams along all four edges. A slab which is supported on two sides only is essentially a rectangular beam of comparatively large ratio of width to depth. There are, however, certain modifications entering into the design and review of such slabs which it was not necessary

to consider in the solution of rectangular beams. Slabs which are supported on four sides, with reinforcement in two directions, present the additional problems of determining the proportion of the total load that is transmitted in each direction to the supporting beams, and also of allocating this proportion of the total load to the various strips into which the slab is assumed to be divided. These problems are considered in Arts. 3.19 to 3.22.

Floor slabs in buildings are usually designed for a uniform live load covering the entire slab area. The following discussions are intended to apply only to the analysis of such uniformly loaded slabs. Concentrated loads on concrete slabs are supported by a greater width of slab than the mere contact width. Methods of computing the probable distribution of concentrated loads are explained in Sec. 11. Multiples of $\frac{1}{4}$ in. may be used in selecting the total thickness of slabs.

**3.15. Slabs Supported on Two Sides Only.** The simplest form of slab is one of indefinite width, supported by only two beams, one at each edge of the slab. If a 12-in. strip of slab were cut out at right angles to the supporting beams, such as either of the shaded areas in Fig. 3.8, a rectangular beam 12 in. in width would result, with a depth equal to the thickness of the slab and a length equal to the distance between supports. This strip could then be analyzed by the same formulas which were used in problems dealing with rectangular beams, the bending moment being computed for a width of 1 ft. The load per square foot on the slab would then be the load per linear foot on the imaginary beam. Since all the load on the slab must be transmitted to the two supporting beams, it follows that all the reinforcing steel should be placed at right angles to these beams, with the exception of any bars that may be placed in the other direction to take care of shrinkage and temperature stresses. A slab which is supported on two sides only thus consists (in theory) of a series of rectangular beams side by side.

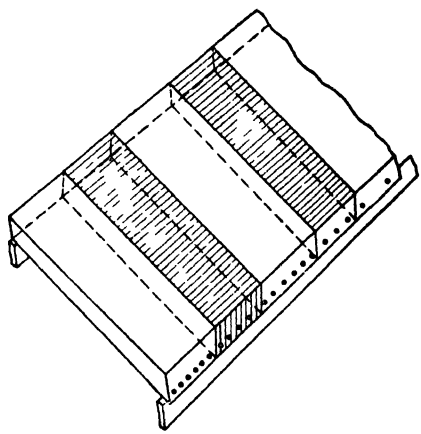


FIG. 3.8

The ratio of steel in a slab may be determined by dividing the sectional area of one bar by the area of concrete between two successive bars, the latter area being the product of the depth to the center of the bars and the distance between them, center to center. The ratio of steel may also be determined by dividing the average area of steel per foot of width

by the effective area of concrete in a 1-ft strip. The average area of steel per foot of width is equal to the area of one bar times the average number of bars in a 1-ft strip (12 divided by the spacing in inches), and the effective area of concrete in a 1-ft (or 12-in.) strip is equal to 12 times the effective depth $d$.

To illustrate the latter method of obtaining the steel ratio $p$, assume a 5-in. slab with an effective depth of 4 in., and with No. 4 bars spaced 4½ in., center to center. The average number of bars in a 12-in. strip of slab is $12/4\frac{1}{2} = 2.7$ bars, and the average steel area in a 12-in. strip is $2.7 \times 0.20 = 0.54$ sq in. Hence $p = 0.54/(12 \times 4) = 0.0112$. By the other method, $p = 0.20/(4\frac{1}{2} \times 4) = 0.0112$.

The spacing of bars which is necessary to furnish a given area of steel per foot of width is obtained by dividing the number of bars required to furnish this area into 12. For example, to furnish an average area of 0.46 sq in. per ft, with No. 4 bars, requires $0.46/0.20 = 2.3$ bars per foot; the bars must be spaced not more than $12/2.3 = 5.2$ in., center to center.

If the slab is of one span only and if it rests freely on its supports, the maximum positive moment $M$, assuming a uniform load of $w$ psf, is $M = \frac{1}{8}wl^2$. The span length $l$ is taken as the distance center to center of supports, but it need not exceed the clear span plus the depth of the slab. If a single-span slab is built monolithically with the supporting beams, the positive and negative moments are computed in accordance with the provisions of the ACI Code given in Art. 3.6. Where the width of the support is not known, it may be assumed as 1 ft in order to obtain the value of the clear span for use in computing the moments. Such an assumption usually can be made with sufficient accuracy so that no revision is required when the exact width of the support becomes known.

**3.16. Placing the Reinforcement.** The insulation at the bottom should follow the recommendation of the ACI Code unless conditions warrant some change (see Art. 3.7). In the average slab, a depth of 1 in. below the center of the steel may be used.

The lateral spacing of bars, except those which are used only to prevent shrinkage and temperature cracks, should not exceed three times the thickness of the slab; the minimum spacing is given in Art. 3.7.

**3.17. Temperature Reinforcement.** Concrete shrinks in the process of hardening of the cement paste, as was pointed out in Art. 1.26. It is advisable to minimize such shrinkage by using concretes with the smallest possible amounts of water and cement compatible with other requirements, such as strength and workability, and by thorough moist curing of sufficient duration. However, no matter what precautions are taken, a certain amount of shrinkage is usually unavoidable. If a slab of moderate dimensions rests freely on its supports, it can contract to accommodate the shortening of its length produced by shrinkage. Usually,

however, slabs and other members are joined rigidly to other parts of the structure and cannot contract freely. This results in tension stresses which are known as shrinkage stresses. A decrease in temperature relative to that at which the slab was poured, particularly in outdoor structures such as bridges, has the same effect as shrinkage. That is, the slab tends to contract and, if restrained from doing so, becomes subject to tensile stresses.

Since concrete is weak in tension, these temperature and shrinkage stresses are likely to result in cracking. Cracks of this nature are not detrimental, provided their size is limited to what is known as hairline cracks. This can be achieved by placing reinforcement in the slab to counteract and equalize the contraction. As the concrete tends to shrink, such reinforcement resists the contraction and, consequently, becomes subject to compression. The total shrinkage of a slab so reinforced is less than for one without reinforcement and, in addition, whatever cracks do occur will be of smaller width and more evenly distributed by virtue of the reinforcement.

In one-way slabs (supported on two sides only) the reinforcement provided for resisting the bending moments at the same time has the desired effect of reducing and equalizing shrinkage. However, as contraction takes place equally in all directions, it is necessary to provide special reinforcement for the shrinkage and temperature contraction in the direction perpendicular to the main reinforcement. This added steel is known as temperature or shrinkage reinforcement.

Reinforcement for shrinkage and temperature stresses normal to the principal reinforcement should be provided in slabs where the principal reinforcement extends in one direction only. The ACI Code specifies the following minimum ratios of reinforcement area to effective concrete area, but in no case shall such reinforcement bars be placed farther apart than five times the slab thickness or more than 18 in.

| | |
|---|---|
| Floor slabs where plain bars are used | 0.0025 |
| Floor slabs where deformed bars are used | 0.0020 |
| Floor slabs where wire fabric is used, having welded intersections not farther apart in the direction of stress than 12 in | 0.0018 |
| Roof slabs where plain bars are used | 0.0030 |
| Roof slabs where deformed bars are used | 0.0025 |
| Roof slabs where wire fabric is used, having welded intersections not farther apart in the direction of stress than 12 in | 0.0022 |

**3.18. Illustrative Examples.** I. Design a simply supported slab, not built integrally with its supports, to sustain a live load of 125 psf. The span of the slab center to center of supports is 11 ft-0 in. and the concrete specified is a 2000-psi concrete with the design stress in accordance with the ACI Code. The design stress for the steel is 18,000 psi. Assuming

*a 5-in. slab and considering a 12-in. strip at right angles to the support,* the maximum external bending moment on this strip, which may be considered as a rectangular beam 12 in. in width, is

$$M = \tfrac{1}{8} \times 187 \times 11^2 \times 12 = 33{,}900 \text{ in.-lb}$$

Since $M = Kbd^2$, in which $K = 165$ (Table 5, Appendix D), the required depth of the imaginary beam is

$$d = \sqrt{\frac{33{,}900}{12 \times 165}} = 4.15 \text{ in.}$$

Providing 1 in. from the center of the steel to the bottom of the slab, it is seen that the depth of the slab will exceed 5 in. and revision is necessary.

Taking a depth of slab of $5\frac{1}{2}$ in., the total load per square foot is 194 lb., $M = 35{,}200$ in.-lb, and $d = 4.21$ in. Selecting $d = 4\frac{1}{2}$ in., the total thickness of slab is $5\frac{1}{2}$ in., as assumed. From Table 5, $j = 0.857$ and the area of steel required is

$$A_s = \frac{35{,}200}{18{,}000 \times 0.857 \times 4\frac{1}{2}} = 0.507 \text{ sq in.}$$

Using No. 4 bars, $0.507/0.20 = 2.5$ bars are required per foot of width. The maximum spacing is then $12/2.5 = 4.8$ in. In order to simplify the construction, a spacing of $4\frac{1}{2}$ in. will be used throughout the width of the slab. The reinforcement required normal to the main reinforcement for shrinkage and temperature stresses is

$$0.002 \times 4\tfrac{1}{2} \times 12 = 0.108 \text{ sq in. per ft of slab}$$

This will be provided by using No. 3 bars placed on 12-in. centers which furnish an area of 0.11 sq in. per ft of slab.

II. Review the slab designed in Example I, to determine the maximum unit stresses in the steel and in the concrete. The maximum bending moment on a 12-in. strip of slab, as computed in Example I, is 35,200 in.-lb. The area of steel provided in a 12-in. strip is

$$12/11\tfrac{1}{2} \times 0.20 = 0.53 \text{ sq in.}$$

From Table 6, $k = 0.414$ and $j = 0.862$, and

$$f_s = 35{,}200/(0.53 \times 0.862 \times 4\tfrac{1}{2}) = 17{,}100 \text{ psi}$$

The unit stress in the concrete is

$$f_c = \frac{17{,}100 \times 0.414}{15(1 - 0.414)} = 810 \text{ psi}$$

III. A reinforced concrete slab is built integrally with its supports and consists of two equal spans, each with a clear span of 15 ft-0 in. The

live load is 100 psf and 2500-psi concrete and intermediate-grade steel is to be used. Design the slab following the provisions of the ACI Code.

It is evident that the negative moment at the interior support will govern the design. Assuming a 6-in. slab, the total load per foot is 175 lb and the negative bending moment is

$$\tfrac{1}{9} \times 175 \times 15^2 \times 12 = 52{,}500 \text{ in-lb}$$

From Table 5, $K = 196$ and $j = 0.866$ and

$$d = \sqrt{\frac{52{,}500}{196 \times 12}} = 4.72 \text{ in}$$

With $d$ taken as 5 in. and allowing 1 in. from the center of the reinforcement to the top or bottom of the slab, the total thickness of the slab is 6 in., as assumed.

The area of reinforcement required per foot of width in the top of the slab at the interior support to resist the negative bending moment is $52{,}500 \div (20{,}000 \times 0.866 \times 5) = 0.61$ sq in. The positive bending moment is $\tfrac{1}{14} \times 175 \times 15^2 \times 12 = 33{,}700$ in.-lb, and the area of reinforcement required per foot of width in the bottom of the slab to resist this moment is 0.39 sq in. The negative moment at the exterior support is $\tfrac{1}{24} \times 175 \times 15^2 \times 12 = 19{,}700$ in.-lb, and the area of reinforcement required per foot of width in the top of the slab is 0.23 sq in. These steel areas may be provided in a variety of different ways, but whatever the selection is, due consideration must be given to actual placing of the steel during construction. The arrangement should be such that the steel may be placed rapidly with the minimum of labor costs even though excess steel is necessary to achieve this end.

Two possible arrangements of reinforcement are shown in Fig. 3.9. In (*a*) bent bars are used, while in (*b*) all bars are straight. The arrangement of (*a*) requires more steel and the additional cost of bending, but on the other hand requires less supporting chairs and is somewhat easier of placement.

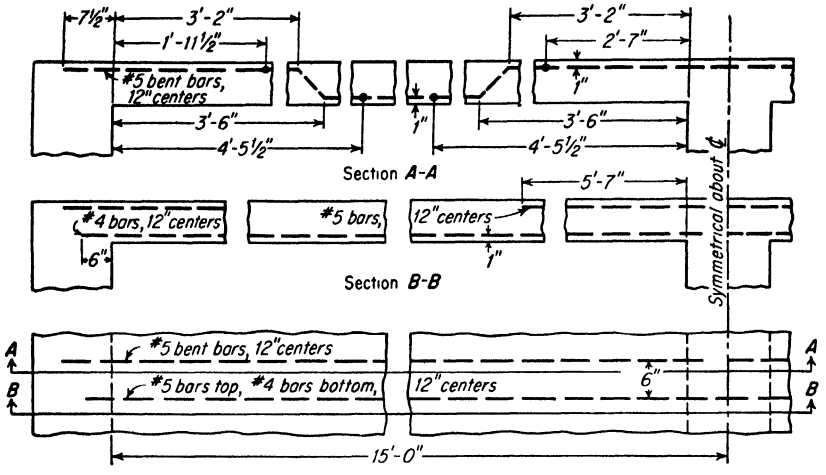
In the arrangement shown in Fig. 3.9*a*, the bent No. 5 bars furnish 50 per cent of the reinforcement required over the interior support. From Diagram 2, Appendix D,[1] it is determined that 50 per cent of the reinforcement may be bent downward at a distance of

$$0.13 \times 15 = 1.95 \text{ ft} = 1 \text{ ft-11}\tfrac{1}{2} \text{ in.}$$

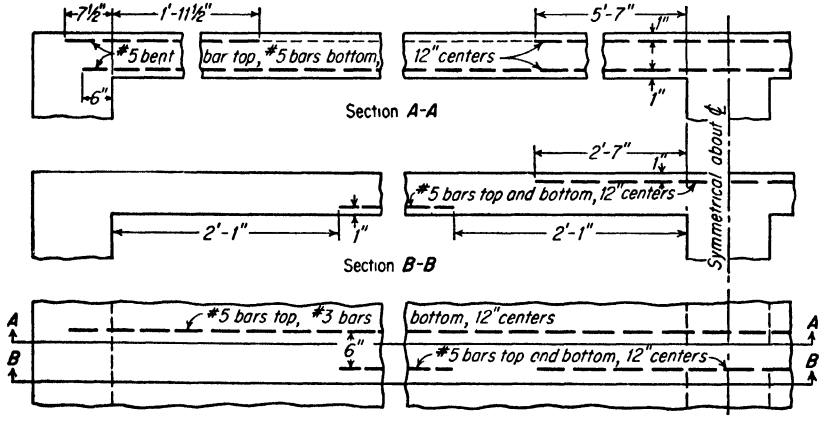
from the face of the support. However, to comply with the ACI Code (902), the point of downward bend must be 12 diameters of the bar

[1] Diagram 2 has been constructed using the moment coefficients specified in the ACI Code with the addition of curves for $-\tfrac{1}{16}\, wl'^2$ and $-\tfrac{1}{24}\, wl'^2$, as given in Art. 3.4. It may be used for uniformly loaded beams and slabs wherever the moment coefficients of Art. 3.6 are used for design.

farther from the support, or $\frac{5}{8} \times 12 = 7\frac{1}{2}$ in., so that the actual point at which these bars may be bent downward must be at least 2 ft-7 in. from the face of the support. The No. 5 straight bars over the interior support are required for a distance of 12 diameters beyond the point




PARTIAL PLAN
(a)




PARTIAL PLAN
(b)

FIG. 3.9

where the negative moment becomes zero, which, from Diagram 2, is $0.33 \times 15$ ft $+ 7\frac{1}{2}$ in. $= 5$ ft-7 in. from the face of the support. At the exterior support, the negative moment becomes zero at a distance of $0.09 \times 15 = 1.35$ ft $= 1$ ft-4 in. from the face of the support. There-

fore, the No. 5 bent bars may be bent downward a distance of 1 ft-11½ in. from the face of the support. The No. 4 bars in the bottom of the slab furnish 0.20 ÷ 0.39 = 50 per cent of the positive reinforcement required. Therefore, the remainder of the positive reinforcement (the No. 5 bent bars) may be bent upward at the point where 50 per cent of the reinforcement is no longer needed. This distance is (from Diagram 2)

$$0.33 \times 15 = 4.95 \text{ ft}$$

from the face of the interior support, or, in accordance with the provisions of the ACI Code, these bars may be bent upward at a distance of 4 ft-11½ in. − 6 in. = 4 ft-5½ in. from the face of the support. (While No. 3 bars rather than No. 4 bars would furnish sufficient additional positive reinforcement, they would not allow the bending up of the No. 5 bars.)

In the arrangement shown in Fig. 3.9*b*, the short No. 5 bars in the top of the slab over the center support may be discontinued at the theoretical point of downward bend of the bent bars of Fig. 3.9*a*, or 2 ft-7 in. from the face of the support, while the longer bars are discontinued at a distance of 5 ft-7 in. from the face of the interior support and 1 ft-11½ in. from the face of the exterior support. The No. 3 bars in the bottom of the slab furnish 0.11 ÷ 0.39 = 28 per cent of the positive reinforcement required, indicating that the remainder of the reinforcement (72 per cent) must be continued (Diagram 2) to a distance of 0.18 × 15 − 12 bar diameters, or 2 ft-8½ in. − 7½ in. = 2 ft-1 in. from the face of the supports. In order to avoid the placement of so many short bars, it is often preferable to run the ⅝-in. bars required at both supports continuously across the slab, as shown by the dotted lines in Fig. 3.9*b*.

### Problems

1. Design a slab freely supported on two sides only, to support a uniform live load of 200 psf. The span of the slab is 10 ft-0 in., and a 2500-psi concrete is to be used, with structural-grade reinforcing steel.

2. A slab of one span, freely supported on two sides only, has a span of 11 ft-0 in. and a total thickness of 5 in. and is reinforced with No. 4 bars 7 in. on centers, the centers of the bars being 1 in. above the lower surface of the slab. If $f'_c$ = 2500 psi and $f_s$ = 20,000 psi, what is the safe uniform live load, in pounds per square foot, that can be placed upon the slab?

## SLABS SUPPORTED ON FOUR SIDES

**3.19. General Considerations.** When a slab panel is square, or nearly so, and there are beams or walls at the four edges of the panel, the slab should be reinforced in two directions so as to transmit the total load to all four supports. The amount of load that is transmitted in each direc-

tion will depend upon the relative lengths of the sides of the panel and the conditions of continuity that exist at the four edges.

If the panel is square and if the construction is such that the same degree of restraint exists at each edge, one-half of the total load will be carried in each direction. If the panel is longer in one direction than in the other, more than one-half of the load will be transmitted in the shorter direction, and the remainder will be transmitted in the longer direction. If, however, one side of the panel is very much longer than the other, such a large proportion of the total load will be transmitted in the shorter direction that reinforcement parallel to the longer side would be of little practical value.

When a slab is reinforced in two directions, the value of $d$ that is established for one set of bars definitely fixes the corresponding value to be used in the computations of the other set. The two sets of bars are placed one above the other, the upper resting directly on the lower. In a square slab it is customary to use the effective depth for the upper row in all computations and to place the same reinforcement, similarly spaced, in the lower row. On account of the larger value of $d$, this provides a slight excess of steel in the lower row, but it simplifies the details of construction. In rectangular slabs, it will generally be found economical to place the shorter bars, which carry the larger part of the load, underneath the longer bars, thus making the $d$ for the short bars as great as possible with a given slab thickness.

**3.20. Distribution of Load Based on Deflections.** A theoretical analysis of a slab supported on four sides is complicated by many factors, such as the condition of restraint or continuity at each edge of the panel and the effect of the stiffening action of the beams on the portions of the slab adjacent to these beams.

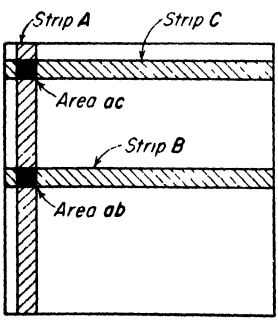


FIG. 3.10

A strip of slab such as strip $A$, Fig. 3.10, parallel and adjacent to a supporting beam supports practically none of the load in the middle portion of the span of the strip (for example, the load on area $ab$); the strip $B$ at right angles to $A$ supports the entire load in that region. The load on an area such as $ac$, near the corner of the panel, would be supported more or less equally by the strips $A$ and $C$. In a square slab, the loading curve on any strip actually approximates a parabola with ordinates varying from a minimum at the center to $w$ psf at the ends, $w$ being the total load per square foot on the slab. The center minimum depends upon the location of the strip in the slab, decreasing from $w/2$

for a strip at the center of the panel to zero for a strip close to one of the supporting beams.

In a rectangular slab of length $l$ and width $l'$, the amount of load carried by each system of reinforcement is found analytically by equating the deflections of two strips, parallel to and midway between each pair of beams. The deflection of each strip is proportional to the load per foot multiplied by the fourth power of the span. Hence, since the deflections of the two strips must be equal at the center, if $w_1$ represents that part of the load $w$ which is transmitted in the short direction and $w_2$ that part which is transmitted in the long direction, the following equation can be written

$$w_1 l'^4 = w_2 l^4$$

from which

$$\frac{w_1}{w_2} = \frac{l^4}{l'^4}$$

The amount of load per square foot that is carried by each of the two sets of bars is thus inversely proportional to the fourth power of the span of the strip in which the bars are placed. Moment computations based on this theory, which assumes a uniform distribution of the load on each strip, are not theoretically correct, since for points near the short edges of the panels the proportion of the load carried by the longitudinal strips will be greater than indicated above, and for points near the long edges of the panels it will be smaller. The theory does not recognize any effect of conditions of continuity on distribution, and another error is thus introduced in practical application.

**3.21. Design of Two-way Slabs According to the ACI Code.** An exact analysis of a concrete slab panel which is supported on four sides should take into consideration the conditions of continuity along each edge of the panel, the relative stiffness of the beams compared with that of the slab, and the relative stiffness of each adjoining panel. Several conditions of continuity are represented in the following types of panels: (1) a single panel, (2) the end panel in a single row of panels, (3) an intermediate panel in a single row of panels, (4) a corner panel, (5) a wall panel in a rectangular group of panels, and (6) an interior panel in a rectangular group.

Elaborate mathematical analyses have been made of the magnitude of moments in slabs in the six enumerated conditions. Load tests made on slabs indicate that their actual strength is in excess of that which should obtain from the theoretical computations. This seems to be due not to the inadequacy of the theory but to the fact that in shallow slabs the tension in the concrete in the two directions contributes significantly to

the ultimate strength of such slabs. For this reason the moment coefficients indicated by the mathematical theory are usually reduced by a percentage to conform with the results of the load tests. These factors were considered in the development of the reasonably simple method of two-way slab design that appears in the ACI Code.

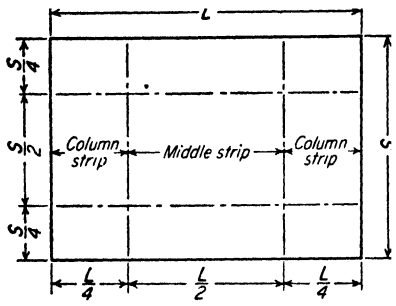


Fig. 3.11

In this method, the slab is considered to be built monolithically with the supporting walls or beams, so that a negative moment is developed on all exterior edges. The panel is divided into middle strips and column strips (see Fig. 3.11). If the panel is nearly square, the middle strip has a width of one-half panel, and each column strip has a width of one-quarter panel. In panels where the ratio of the short span to the long span is less than 0.5, the width of the middle strip that extends in the short direction is equal to the difference between the long and short spans, the remaining area being divided equally between the two column strips.[1]

The span lengths are taken as the distance between the centers of supports or the clear span plus twice the thickness of the slab, whichever is the smaller. The critical sections for bending moment are along the center lines of the panel for positive moment and along the edges of the panel at the faces of the supporting beams or walls. The bending-moment coefficients for several conditions of continuity are given in the table on page 73. These coefficients are for a strip of slab 1 ft wide in the middle strip and are in terms of $wS^2$, where $w$ is the uniform load per square foot on the panel and $S$ is the short-span length. The bending moment in a strip 1 ft wide in the parallel column strip is equal to two-thirds of the bending moment in the 1-ft strip of the middle strip. The bending moments in both the short- and long-span directions are computed in terms of the short-span length, that is, $M = CwS^2$ for both directions. In determining the spacing of the reinforcement in the column strip, the bending moment in the column strip may be assumed to vary from a maximum at the edge of the middle strip to a minimum at the edge of the support. However, the principal reinforcement shall not be spaced farther apart than three times the slab thickness and shall be

[1] Where the ratio of the short span to the long span is less than 0.5, it is doubtful if any appreciable amount of the load is carried in the long direction, and conservative design would consider all load carried in the short direction with moment coefficients as for continuous beams, as given in Art. 3.6.

**TABLE 3.1. Moment Coefficients in Middle Strip for Two-way Slabs**

| Moments | Short span: Ratio of short span to long span | | | | | | Long span, all values of $m$ |
|---|---|---|---|---|---|---|---|
| | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 and less | |
| Case 1. Interior panels | | | | | | | |
| Negative moment at | | | | | | | |
| Continuous edge | 0.033 | 0.040 | 0.048 | 0.055 | 0.063 | 0.083 | 0.033 |
| Discontinuous edge | | | | | | | |
| Positive moment at mid-span | 0.025 | 0.030 | 0.036 | 0.041 | 0.047 | 0.062 | 0.025 |
| Case 2. One edge discontinuous | | | | | | | |
| Negative moment at | | | | | | | |
| Continuous edge | 0.041 | 0.048 | 0.055 | 0.062 | 0.069 | 0.085 | 0.041 |
| Discontinuous edge | 0.021 | 0.024 | 0.027 | 0.031 | 0.035 | 0.042 | 0.021 |
| Positive moment at mid-span | 0.031 | 0.036 | 0.041 | 0.047 | 0.052 | 0.064 | 0.031 |
| Case 3. Two edges discontinuous | | | | | | | |
| Negative moment at | | | | | | | |
| Continuous edge | 0.049 | 0.057 | 0.064 | 0.071 | 0.078 | 0.090 | 0.049 |
| Discontinuous edge | 0.025 | 0.028 | 0.032 | 0.036 | 0.039 | 0.045 | 0.025 |
| Positive moment at mid-span | 0.037 | 0.043 | 0.048 | 0.054 | 0.059 | 0.068 | 0.037 |
| Case 4. Three edges discontinuous | | | | | | | |
| Negative moment at | | | | | | | |
| Continuous edge | 0.058 | 0.066 | 0.074 | 0.082 | 0.090 | 0.098 | 0.058 |
| Discontinuous edge | 0.029 | 0.033 | 0.037 | 0.041 | 0.045 | 0.049 | 0.029 |
| Positive moment at mid-span | 0.044 | 0.050 | 0.056 | 0.062 | 0.068 | 0.074 | 0.044 |
| Case 5. Four edges discontinuous | | | | | | | |
| Negative moment at | | | | | | | |
| Continuous edge | | | | | | | |
| Discontinuous edge | 0.033 | 0.038 | 0.043 | 0.047 | 0.053 | 0.055 | 0.033 |
| Positive moment at mid-span | 0.050 | 0.057 | 0.064 | 0.072 | 0.080 | 0.083 | 0.050 |

sufficient in amount to satisfy the requirements for shrinkage and temperature reinforcement

The loads on the supporting beams for two-way rectangular panels may be assumed to be uniformly distributed throughout the span of the beams. The amount of load supported by each beam is considered to be all the load on an area bounded by the intersection of 45° lines from the corners of the panel with the center line of the panel parallel to the long side, as shown in Fig. 3.12. The equivalent uniform load per lineal foot of beam for each panel supported may be taken as

$$\frac{wS}{3} \quad \text{for the short span}$$

and

$$\frac{wS}{3}\frac{(3 - m^2)}{2} \quad \text{for the long span}$$

where $w$ is the load per square feet on the panel, $S$ is the short-span length, and $m$ is the ratio of short span to long span.

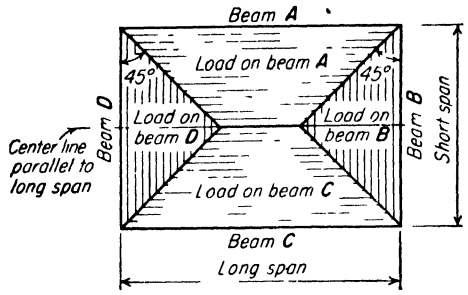



Fig. 3.12

**3.22. Illustrative Example.** Design a slab panel 20 ft-0 in. by 25 ft-0 in. to support a uniform live load of 125 psf. The panel is the corner panel of a group. A 2500-psi concrete and structural-grade steel is to be used and the provisions of the ACI Code are to be followed.

Assuming a 6-in. slab, the total load is 200 psf. The ratio of the short span to the long span is 0.8. The moments for the 1-ft strips are:
Short span, middle strip, positive moment:

$$M = 0.048 \times 200 \times 20^2 \times 12 = 46{,}100 \text{ in.-lb}$$

Middle strip, negative moment at discontinuous edge:

$$M = 0.032 \times 200 \times 20^2 \times 12 = 30{,}700 \text{ in.-lb}$$

Middle strip, negative moment at continuous edge:

$$M = 0.064 \times 200 \times 20^2 \times 12 = 61{,}400 \text{ in.-lb}$$

Long span, middle strip, positive moment:

$$M = 0.037 \times 200 \times 20^2 \times 12 = 35{,}500 \text{ in.-lb}$$

Middle strip, negative moment at discontinuous edge:

$$M = 0.025 \times 200 \times 20^2 \times 12 = 24{,}000 \text{ in.-lb}$$

Middle strip, negative moment at continuous edge:

$$M = 0.049 \times 200 \times 20^2 \times 12 = 47{,}000 \text{ in.-lb}$$

The critical moment to be used in determining the slab thickness is 61,400 in.-lb, and the depth to the steel in the short span is

$$d = \sqrt{\frac{61{,}400}{207 \times 12}} = 4.96, \text{ or } 5 \text{ in.}$$

Using a $d$ of 5 in. for the bars in the short direction and allowing 1 in. from the center of the steel to the bottom of the slab, the total thickness of the slab is 6 in., as assumed.

The area of steel required in the short span, middle strip, at the center of the span is

$$A_s = 46{,}100 \div (18{,}000 \times 0.857 \times 5) = 0.60 \text{ sq in.}$$

This amount can be provided with No. 5 bars[1] 6 in. on centers. The area of steel required at the continuous edge is

$$A_s = 61{,}400 \div (18{,}000 \times 0.857 \times 5) = 0.80 \text{ sq in.}$$

This amount can be provided by bending up two out of every three bars from this panel and two out of three bars from the adjacent panel. This will result in having four bars in every 18 in. over the support, giving an average spacing of 4½ in. which provides a steel area of 0.82 sq in. The other bars (one out of three) from each panel are straight bars and may be carried straight through the support or may be cut off short of the support where not required to resist the positive moment.

The area of steel required at the discontinuous edge is 0.40 sq in., which can be furnished by bending up two out of every three bars of the positive moment reinforcement.

The point of inflection in panels of this type may, in accordance with the ACI Code, be assumed to be one-eighth of the clear span length from the face of the support. Reference to Diagram 2 shows that two-thirds of the positive reinforcement may be bent up at a distance of 0.09 of the distance between the points of inflection, or $0.09 \times \frac{3}{4} \times S = 0.07S$.[2] This indicates that the two bars may be bent upward a distance of 0.125 + 0.07, or $0.195 \times 20 = 3.90$ ft from the face of the support.

In the column strip the amount of steel required being two-thirds of that required in the middle strip, the spacing of the No. 5 bars may be increased to 9 in. and the same arrangement of bent-up and straight bars used as has been indicated for the middle strip.

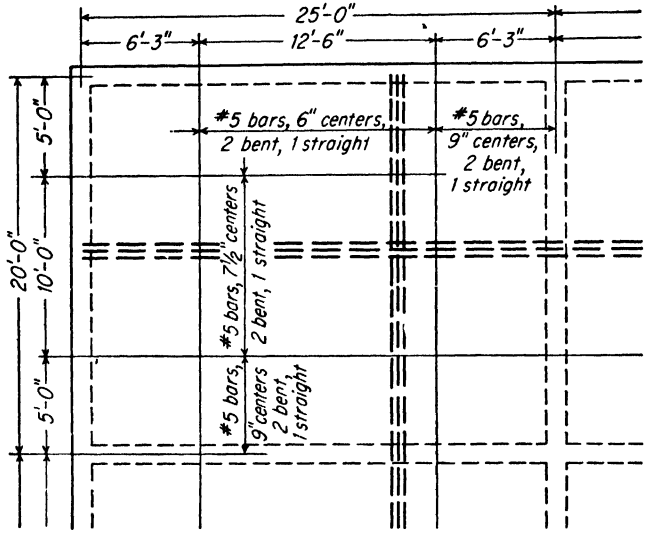
[1] This actually gives protection to the steel of $\frac{1}{16}$ in. less than the $\frac{3}{4}$ in. usually required. However, since reinforcing bars cannot be placed with this degree of accuracy, this slight infringement on the requirements is allowable.

[2] This distance could be computed as follows: In the parabola representing the moment curve between points of inflection the ordinate $y$ from the vertex is taken as two-thirds the maximum $y$. Then $x^2 = 2/3y$ (where $x$ is the horizontal distance measured from an axis through the vertex) or $x = 0.82$ of one-half the distance between points of inflection.

The depth of slab required in the long direction for the positive reinforcement is

$$d = \frac{35{,}500}{207 \times 12} = 3.78 \text{ in.}$$

With the longitudinal bars laid directly on and in contact with the bars in the short direction, the $d$ furnished is 4.7 in. For negative moment, the steel can be placed within 1 in. of the top of the slab so that $d = 5$ in.




Plan of Slab

| Direction of bars | Discontinuous edge | | | | Continuous edges | | | |
|---|---|---|---|---|---|---|---|---|
| Short | 7½" | 3'-6½" | 4" | * | 4" | 3'-6½" | * | 2'-1" |
| Long | 7½" | 4'-7" | 3⅜" | * | 3⅜" | 4'-7" | * | 2'-4" |

*These dimensions depend on width of support

Dimensions of Bent Bars

FIG. 3.13

The area of steel required in the long-span middle strip at the center of the span is

$$A_s = 35{,}500 \div (18{,}000 \times 0.857 \times 4.7) = 0.49 \text{ sq in.}$$

This amount can be provided with No. 5 bars 7½ in. on centers.

The area of steel required for negative moment at the continuous edge of the panel is 0.61 sq in. and at the discontinuous edge 0.31 sq in. At

the continuous edge, by bending up two out of every three bars from this panel and two out of every three from the adjacent panel, an average spacing slightly in excess of 5½ in. will be realized and an area of about 0.67 sq in. per ft strip furnished. By bending up two out of every three bars at the discontinuous edge, 0.34 sq in. per ft strip is furnished.

In the column strip, the spacing of the reinforcement could be increased to $7\frac{1}{2} \div \frac{2}{3} = 11$ in., but in order to simplify the construction, the same spacing as in the short direction (9 in.) will be used. The arrangement of steel shown in Fig. 3.13 satisfies the requirements for both positive and negative moments and also the requirements for bond[1] (see Art. 3.35).

Where a large number of interior panels are involved in the structure of which this panel is a part, since the reinforcement required in those panels is only three-fourths of that required in the corner panel (see table on page 73), it may be desirable to change either the spacing of the No. 5 bars to 8 in. or to use No. 4 bars at 5 in. In order to simplify the form construction, the total thickness of the slab will be kept constant at 6 in.

## DIAGONAL TENSION, SHEAR, AND BOND

**3.23. Stresses in a Concrete Beam.** The preceding articles contain an outline of the methods of calculating the maximum fiber stresses in the concrete and steel of a reinforced concrete beam and of so proportioning the amounts of steel and concrete that the working strength of any part of the beam in flexure is not exceeded.

There are other internal stresses existing in a concrete beam which, if not properly cared for, may in themselves cause failure of the beam. These stresses are (1) shearing stresses, or those tending to make one plane of concrete, either vertical or horizontal, slide along an adjacent

[1] Referring to Fig. 3.13, the bent bars in each direction must be carried 12 diameters (see Art. 3.35), or 7½ in., beyond the face of the outer support. These bars must remain in the top of the slab to resist negative moment until the point of inflection is reached, which is 2 ft-6 in. and 3 ft-1½ in. for the short and long span, respectively. They must reach the bottom of the slab at a distance of 3.90 ft and 4.88 ft, respectively. The dimensions shown in the figure comply with these requirements. At the continuous edge, the bent bars from the adjacent panel reach the top of the slab 3 ft-6½ in. and 4 ft-7 in. from the edge of the supports. The bars from the corner panel furnish one-half of the negative reinforcement required and may be discontinued where not more than one-half of such reinforcement is needed. Reference to Diagram 2, applying the curve for $-\frac{1}{16}\,wl'^2$, shows that these distances are

$$0.07 \times 20 = 1.40 \text{ ft and } 0.07 \times 25 = 1.75 \text{ ft}$$

respectively. However, the ACI Code requires that they be extended 12 diameters further; hence, the lengths required beyond the edge of the support are 2 ft-1 in. and 2 ft-4 in., respectively. The straight bars must be carried 6 in. (see Art. 3.35) beyond the face of the supports.

plane; (2) diagonal tension stresses, or those which cause cracks in the concrete along inclined planes near points of high shear; and (3) in reinforced beams, bond stresses, or those tending to cause the steel to pull away from the concrete when under stress and thus destroy the unity of the beam.

**3.24. Shearing Stresses.** If a pile of boards is used to support a load, the boards being free to slip on each other, it is noticeable that the ends overlap even when the boards are of equal length (see Fig. 3.14). Slipping has occurred along the surfaces of contact. If, however, they are glued together, piled as before, the slipping is prevented, but the *tendency to slip* still exists and is known as the shearing stress in surfaces parallel to the neutral axis. These shearing stresses exist in beams of any material as long as the two sides of the surface considered form a continuous substance.

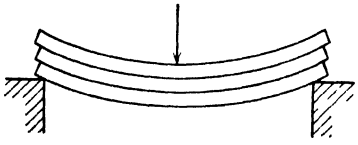


FIG. 3.14

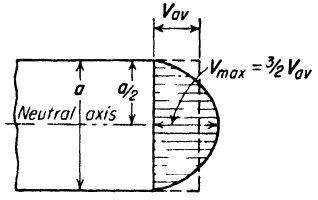



FIG. 3.15

In addition to the horizontal shearing stresses described above, there are vertical stresses of the same nature, *i.e.*, a tendency for the beam on one side of a vertical section to slide upward past the other side. These two kinds of shearing stresses are of the same intensity per unit of area at any point in the beam.

**3.25. Intensity of Shearing Stress in a Plain Concrete Beam.** As stated in Art. 3.1, the value of the unit shear $v$ in a plain concrete beam (or any homogeneous beam) is represented by the equation

$$v = \frac{VQ}{Ib}$$

The value of the shear stress, as represented by this equation, becomes zero at the top and bottom surfaces of the beam and a maximum at the neutral axis. The distribution for a rectangular section is shown in Fig. 3.15. The maximum stress equals $1\frac{1}{2}$ times the average, or $\frac{3}{2}\frac{V}{ba}$ (see Art. 3.1). Between the neutral axis and the extreme surfaces the value of the unit shear varies as the ordinates of a parabola. The absolute maximum value of $v$ occurs in the section at which the total external vertical shear is a maximum.

**3.26. Intensity of Shearing Stress in a Reinforced Concrete Beam.** In a nonhomogeneous beam the unit shearing stresses vary in a very dif-

ferent manner from that described above. To derive an equation which expresses the variation of the unit shear at any section of a reinforced concrete beam, consider a short length of the beam as a free body (Fig. 3.16*a*). The forces acting on this element are those of compression $C$ and $C'$, of tension $T$ and $T'$, and of total vertical shear $V$. Hence in Fig. 3.16*a*, which represents a length so short that no part of the external vertical load need be considered (the total vertical shear on the left equals that on the right of the element), $T - T'$ is equal to the total shearing force, which tends to slide the lower portion along the upper, on any plane between the steel and the neutral axis, assuming that the concrete will take none of the tension.

This is illustrated in Fig. 3.16*a*. If the element is assumed on the left half of the span, and the beam is subjected to a uniform load, the fiber

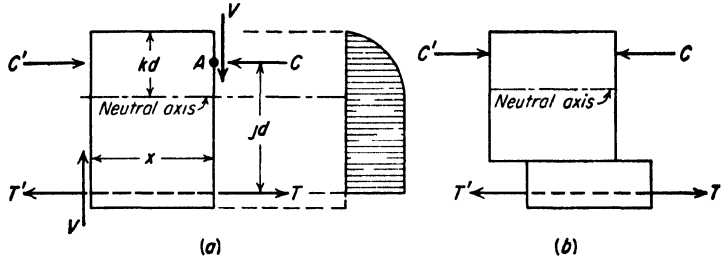



FIG. 3.16

stresses on the right side of the element are greater than those on the left. The lower portion tends to slide toward the right and the upper portion toward the left, as shown in Fig. 3.16*b*. The force producing this horizontal sliding is equal to the difference in the forces acting on each part of the section, that is, $T - T'$ for the lower part and $C - C'$ for the upper. $C - C'$ must equal $T - T'$, and the shearing strength of any two consecutive horizontal planes between the neutral axis and the tension steel must be sufficient to transmit the effect of one set of these forces to the other, so as to prevent the movement indicated in Fig. 3.16*b*.

The horizontal shear stress on any such plane as described above is

$$v = \frac{T - T'}{bx}$$

$bx$ being the area of the surface under consideration. The external forces acting on this portion of the beam must be in equilibrium: hence the summation of moments about any point such as $A$ on the line of action of the compressive forces must equal zero, or

$$(T - T')jd = Vx$$

Therefore

$$T - T' = \frac{Vx}{jd}$$

Substituting this value of $T - T'$ in the above equation for $v$,

$$v = \frac{V}{bjd} \tag{3.9}$$

This represents the value of the unit horizontal shearing stress along any plane between the steel and the neutral axis, and, as shown in Fig. 3.16*a*, it is also the maximum unit shear in the section. The amount of this shear per linear inch of beam for the full width $b$ is

$$vb = \frac{V}{jd} \tag{3.10}$$

The value of $j$ for working loads varies between narrow limits, and this variation causes but insignificant differences in values of $v$. For this reason it is satisfactory to use the value of $j = \frac{7}{8}$ in all computations involving shear and bond. This is an average value for beams in ordinary construction. Above the neutral axis the shear follows the parabolic law, as in the plain concrete beam (see Fig. 3.16*a*).

*Illustrative Problem.* A reinforced concrete beam (Fig. 3.17) has a span of 15 ft-0 in. and supports a concentrated load of 15,000 lb placed 5 ft-0 in. from the left support, in addition to a uniform live load of 600 lb per lin ft over the whole span. The over-all cross section of the beam is 12 by 20 in., and the reinforcement consists of four No. 7 bars in one row, the center of which is $2\frac{1}{2}$ in. above the lower surface of the beam. Determine the maximum unit shearing stress in sections at the left and right supports, and in sections just to the left and right of the concentrated load. The dead weight of the beam is 250 lb per ft, and the total uniform load is 600 + 250 = 850 lb per ft. At the left support,

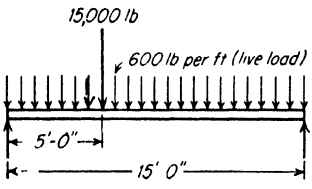


FIG. 3.17

$$V = \tfrac{2}{3} \times 15{,}000 + \tfrac{15}{2} \times 850 = 16{,}400 \text{ lb}$$

$$v = \frac{16{,}400}{12 \times \frac{7}{8} \times 17.5} = 89 \text{ psi}$$

At the right support,

$$V = \tfrac{1}{3} \times 15{,}000 + \tfrac{15}{2} \times 850 = 11{,}400 \text{ lb}$$

$$v = \frac{11{,}400}{12 \times \frac{7}{8} \times 17.5} = 62 \text{ psi}$$

Just to the left of the concentrated load,

$$V = 16{,}400 - 5 \times 850 = 12{,}100 \text{ lb}$$
$$v = \frac{12{,}100}{12 \times \frac{7}{8} \times 17.5} = 66 \text{ psi}$$

Just to the right of the concentrated load,

$$V = 16{,}400 - 5 \times 850 - 15{,}000 = -2900 \text{ lb}$$
$$v = \frac{2900}{12 \times \frac{7}{8} \times 17.5} = 16 \text{ psi}$$

The practical importance of the above values is indicated in Example I, Art. 3.31.

**3.27. Inclined Tension Stresses (Diagonal Tension).** Assume an infinitely small cube to be removed from a beam at any section along the neutral axis. Two pairs of shearing forces, horizontal and vertical, must be considered. These forces form two couples acting as shown in Fig. 3.18. Since the prism has been assumed at the neutral axis, the flexural stresses of tension and compression are zero. Therefore, the shearing forces $vbx$ develop inclined stresses of tension in the direction $MN$, and compression along the line $PQ$, each equal to $2vbx/\sqrt{2}$. Since the length of each diagonal of the prism is $x\sqrt{2}$, the intensity of these inclined stresses, *i.e.*, the amount per unit area, equals $2vbx/\sqrt{2}$ divided by $bx\sqrt{2}$, or $v$. It follows that at any point along the neutral plane there exist tensile and compressive stresses inclined at 45° with the horizontal and that the value of these stresses per unit of area equals the unit shear at that point.

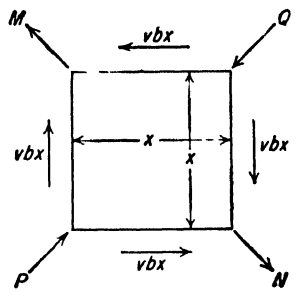


FIG. 3.18

Since at all points above and below the neutral axis there exist, in addition to the horizontal and vertical shearing forces, horizontal fiber stresses of tension and compression, the values of the inclined tensile and compressive stresses at such points are found by combining the fiber stresses with the shearing stresses.

Treatises on mechanics prove that the intensity of the inclined stress $t$ at any point in a beam is represented by the equation

$$t = \tfrac{1}{2}f + \sqrt{\tfrac{1}{4}f^2 + v^2} \tag{3.11}$$

and the direction of this stress by the equation

$$\tan 2\alpha = \frac{2v}{f}$$

where $f$ = fiber stress per unit of area
$v$ = intensity of vertical or horizontal shearing stress at point
$\alpha$ = angle made by stress $t$ with the horizontal

In the equation for the angle of inclination of the inclined stress, two values of $\alpha$, differing by 90°, will satisfy, showing that the maximum compressive stress and the maximum tensile stress at any point make an angle of 90° with each other.

On account of the comparatively large compressive strength of concrete, the inclined compressive stresses as determined above may be neglected; failure, if any, occurs because of the cracking of the concrete due to tensile stresses in excess of its strength.

For ordinary beams of homogeneous materials, such as beams of steel or timber, a determination of the normal, or flexural, stresses and the shearing stresses, described above, gives sufficient information for purposes of design. In concrete beams, both plain and reinforced, the inclined stresses of maximum tension induced by the shearing and bending stresses are often fully as important as the maximum fiber stresses, and it is necessary to make some provision for them. This is because of the very low strength of concrete in tension. Hence the necessity for the further investigation of such stresses.

**3.28. Diagonal Tension in Plain Concrete Beams.** Examination of the above equations shows that at the center of a homogeneous beam, where the moment is a maximum, the direction of the lines of maximum tension is horizontal for the entire depth of the beam. As the end of the beam is approached, the shear becomes large and the bending moment small (assuming a simply supported beam). Here the effect of the shear on the diagonal tension is great, while the horizontal fiber stress has little weight in determining the inclination and amount of the inclined tensile stress near the support. At the end of the beam, where the horizontal tension is zero, the diagonal tension stresses are inclined at practically 45° throughout the entire depth of the beam.

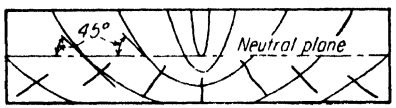


FIG. 3.19

Figure 3.19 illustrates the variation in direction of the maximum inclined tension stresses in a homogeneous rectangular beam. As seen above, the exact direction at any point depends upon the relation between shear and bending moment at the point. The short wavy lines represent the probable cracks which may form with large shearing stresses. Lines of maximum compression run at right angles to those of maximum tension.

**3.29. Diagonal Tension in Reinforced Concrete Beams.** In reinforced beams, owing to the concentration of the tension in the steel, the direction

of maximum tension at various depths is somewhat different from that in plain or homogeneous beams. Large shearing stresses exist immediately above the steel, and the maximum tensile stresses become considerably inclined at that plane, the exact direction depending upon the relation between the shear and horizontal fiber stress.

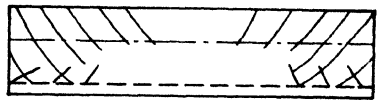


FIG. 3.20

Figure 3.20 represents the general direction of the inclined tensile stresses in a uniformly loaded beam, the wavy lines representing the probable planes of rupture. The diagonal cracks near the bottom are approximately vertical at the center and become more and more inclined as the end of the beam is approached.

In beams with tension reinforcement, the value of $t$ as expressed by Eq. (3.11) is indeterminate, since the value of $f$, the horizontal fiber stress in the concrete, is variable. This is due to the fact that as the loading increases, the concrete cracks more and more, and the amount of tensile stress carried by it therefore decreases. The excess is picked up by the steel, and immediate failure prevented, whereas in a plain beam, increasing the loading after the concrete commences to crack results in immediate failure of the beam. Therefore the exact amounts of the inclined tension stresses are unknown. It is seen, however, from a study of Eq. (3.11), that the vertical shearing stresses furnish a means of comparing or measuring the diagonal tension stresses. It must be remembered that *the vertical shearing stress is not the numerical equivalent of the diagonal tension stress*, nor is there any definite relation between them.

By limiting the allowable unit shearing stress to a value which has been found by actual tests to be low enough to ensure against failure by diagonal tension, it may be considered that the danger of such failure has been eliminated. This limit of the allowable shearing stress is considerably below the safe working stress of concrete in direct shear because of the fact that, when the shear in a beam is still low, the diagonal tension may be excessive. Failure occurs not by direct shear, but by the cracking of the concrete along inclined planes.

An examination of Eq. (3.11) shows that diagonal tension at any point varies with both the shear and horizontal tension in the concrete. In order to reduce the danger of failure by diagonal tension, large shearing stresses should be avoided and the horizontal tension in the concrete kept as small as possible. This latter may be accomplished by furnishing an excess of longitudinal steel at points of heavy shear, thus reducing the horizontal deformation and consequently the tension in the concrete.

**3.30. Reinforcement for Diagonal Tension Stresses.** Where in the interest of economy of design it becomes necessary to allow relatively

large shearing stresses in a beam, additional reinforcement must be provided to care for the diagonal tension stresses (the shear being a measure of the diagonal tension stresses). A study of Figs. 3.19 and 3.20 shows that the most efficient type of such reinforcement, usually referred to as "web reinforcement," consists of an arrangement of steel as shown in Fig. 3.21, the inclined bars being either a continuation of horizontal bars or additional bars rigidly connected to the horizontal bars. Such an arrangement is, however, not practical. Slight variations between the inclination of the web reinforcing bars and the lines of maximum tension have but little effect on the efficiency of the system; hence in practice the most commonly used methods of arranging diagonal tension reinforcement are divided into three groups:

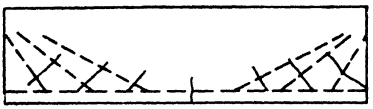


FIG. 3.21

1. Bars or stirrups perpendicular to and looped around or welded to the longitudinal steel.

2. Bars or stirrups welded or otherwise rigidly attached to the longitudinal steel and making an angle of 30° or more thereto.

3. Longitudinal bars bent so that the axis of the inclined portion of the bar makes an angle of 15° or more with the axis of the longitudinal portion of the bar.

All inclined web reinforcement bars in a given beam are placed usually at the same angle with the horizontal, in the majority of cases a 45° angle is used. A combination of vertical stirrups and bent bars properly arranged gives the most practical and effective protection against diagonal tension failure. Separate inclined bars are seldom used.

When separate members, either vertical or inclined, are used as diagonal tension reinforcement, care must be taken to see that they are properly connected to the longitudinal steel so that slipping is prevented. When web reinforcement comes into action as the principal tension web resistance, the bond stresses between the longitudinal bars and the concrete are not distributed so uniformly along the bars as they otherwise would be, but tend to be concentrated at and near stirrups and at and near the points where bars are bent up. When stirrups are not rigidly attached to the longitudinal bars and the proportioning of bars and stirrups is such that local slip of the bars occurs at stirrups, the effectiveness of the stirrups is impaired, though their presence still gives an element of resistance to diagonal tension failure. It is on the tension side of a beam that diagonal tension develops in a critical way; therefore proper connection should always be made between stirrups or other web reinforcement and the longitudinal tension reinforcement. Where negative moment exists, as is the case near the supports in a continuous beam,

web reinforcement to be effective must be looped over, or wrapped around, or be connected with the longitudinal tension reinforcing bars at the top of the beam in the same way as is necessary at the bottom of the beam at sections where the bending moment is positive. Requirements of the ACI Code with regard to the anchorage of web reinforcement bars are given in Art. 3.38. In all cases stirrups should be carried as close to the upper and lower surfaces as fireproofing requirements will permit.

**3.31. Working Unit Shearing Stresses.** The ACI Code specifies $0.03f'_c$ as the safe limit of the unit shearing stress for beams without web reinforcement. Where the web reinforcement consists only of stirrups placed perpendicular to the longitudinal reinforcement, the unit shearing stress should not exceed $0.08f'_c$. Where bent bars, having a radius of bend of not more than three times the diameter of the bar, are used alone as web reinforcement, the allowable unit shearing stress shall not exceed $0.06f'_c$. This unit shearing stress may be increased at the rate of $0.01f'_c$ for each increase of four bar diameters in the radius of bend until the maximum allowable unit shearing stress is reached The maximum allowable unit shearing stress, where a combination of stirrups perpendicular to the longitudinal reinforcement and bent bars, inclined stirrups, or bent bars with a sufficient radius of bend are used for web reinforcement, shall not exceed $0.12f'_c$.

Tests show that, in beams with web reinforcement, both the steel and the concrete resist the diagonal tension. Therefore, the web reinforcement is designed to carry only the unit shear over and above the $0.03f'_c$ which the concrete itself can resist.

**3.32. Region Where No Web Reinforcement Is Required.** As mentioned in the previous article, web reinforcement is not required in regions where the unit shear is less than a given percentage of the ultimate compressive strength of the concrete; that is to say, the concrete is assumed capable of withstanding all the diagonal tension as measured by a unit shear of that amount.

*In a uniformly loaded beam,* the distance from the support beyond which stirrups are not required is determined as follows: Let $x_1$ equal the distance to be found, $v_c$ the unit shear $x_1$ ft from the support, and $V_c$ the total shear at that point.

$$V_c = \frac{wl}{2} - wx_1 \qquad \text{and} \qquad v_c = \frac{V_c}{bjd}$$

By substituting the value of $V_c$ from the former in the latter equation, and solving for $x_1$, this becomes

$$x_1 = \frac{1}{2} - \frac{v_c bjd}{w} \tag{3.12}$$

For beams with unsymmetrical or concentrated loads, the points where web reinforcement may be discontinued may be located by constructing the shear diagram for the beam and noting the point or points at which the unit shear is less than the given proportion (see Art. 3.26) of the compressive strength of the concrete.

*Illustrative Examples.* I. Consider the beam of the problem in Art. 3.26. Since for that beam $f'_c = 2500$ psi, the allowable unit shear without web reinforcement is $0.03 \times 2500 = 75$ psi. Determine the regions over which web reinforcement is required.

Since the unit shear at any section between the right support and the concentrated load (Fig. 3.17) is less than 75 psi, no web reinforcement is required throughout this distance. The unit shear at the left support is 89 psi, and this shear decreases uniformly to 66 psi at a section just to the left of the concentrated load. Hence, web reinforcement is required from the left support to the section at which the unit shear is equal to 75 psi. This section can be located by a simple proportion, as follows: Let $x_1$ be the required distance from the left support. Then

$$\frac{x_1}{5} = \frac{89 - 75}{89 - 66}$$

and

$$x_1 = 3.0 \text{ ft}$$

Another method of computing $x_1$ is as follows. The unit shear at any section $x_1$ ft from the left support is

$$v = \frac{V}{bjd} = \frac{16{,}100 - 850x_1}{12 \times \frac{7}{8} \times 17.5} = 75$$

from which $x_1 = 3.0$ ft as above. Web reinforcement is required therefore for a distance of 3.0 ft from the left support.

II. A reinforced concrete beam has a span of 14 ft-0 in. and is to sustain a uniform live load of 1000 lb per lin ft. The reinforcement consists of two No. 9 bars, placed $11\frac{1}{2}$ in. below the upper surface and $2\frac{1}{2}$ in. above the lower surface of the beam. The width of the beam is 8 in. and a 2500-psi concrete is to be used in the construction. Determine the regions over which web reinforcement is required.

The weight of the beam is 120 lb per ft, and the total load is therefore $1000 + 120 = 1120$ lb per ft. The unit shear at the supports, Eq. (10), is 97 psi; since this exceeds the allowable value of 75 psi for beams without web reinforcement, such reinforcement is required at the supports and for a distance $x_1$ ft from the supports. Since the beam supports only a uniform load, Eq. (3.12) can be used in obtaining the value of $x_1$. Hence,

$$x_1 = \frac{14}{2} - \frac{75 \times 8 \times \frac{7}{8} \times 11\frac{1}{2}}{1120} = 1.6 \text{ ft}$$

**3.33. Design for Web Reinforcement.** Wherever the unit shear exceeds the value that can be taken by the concrete, web reinforcement must be provided. This reinforcement must be sufficient to take care of the stresses over and above that taken by the concrete.

Reference to Figs. 3.19 and 3.20 shows that near the support of a beam where the shear is the greatest the diagonal tensile stresses are inclined at an angle of approximately 45° with the horizontal. Therefore, in determining the amount of diagonal tensile reinforcement required in a beam, it will be assumed that the forces to be resisted make an angle of 45° with the horizontal.

In Fig. 3.22*a* let $P$ represent the diagonal tension force inclined at 45° with the horizontal, and $a$, $c$, and $d$ bars spaced a distance $s$ along the

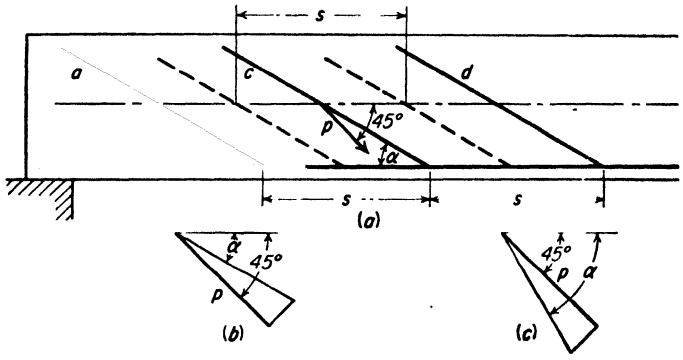



FIG. 3.22

beam making an angle $\alpha$ with the horizontal. The bar $c$ must resist the diagonal tensile forces not taken by the concrete over an area $bs$, and since the unit shear at any point in the beam is taken as the measure of the diagonal tension,

$$P = v'bs \sin 45°$$

where $v' = v - v_c$ with $v_c$ being the allowable unit shear taken by the concrete.

Since the angle of inclination of the bar is $\alpha$, reference to the force triangle (Fig. 3.22*b*) shows that the force to be resisted by the bar is

$$P \div \cos(45° - \alpha) = \frac{v'bs \sin 45°}{\cos 45° \cos \alpha + \sin 45° \sin \alpha} = \frac{v'bs}{\cos \alpha + \sin \alpha}$$

Since $v' = V'/bjd$,

$$P = \frac{V's}{jd(\cos \alpha + \sin \alpha)}$$

Since $A_v f_v$ must equal $P$, where $A_v$ is equal to the cross-sectional area of

the web reinforcement and $f_v$ is the allowable unit tensile stress in that reinforcement,

$$s = \frac{A_v f_v jd(\cos \alpha + \sin \alpha)}{V'} \tag{3.13}$$

or

$$s = \frac{A_v f_v(\cos \alpha + \sin \alpha)}{v'b} \tag{3.13a}$$

Reference to Fig. 3.22*c* indicates that identical equations to those of (3.13) and (3.13*a*) will result when $\alpha$ is greater than 45°.

From the above it is seen that, since the sum of cos $\alpha$ and sin $\alpha$ is a maximum with $\alpha = 45°$ the maximum efficiency of inclined bars or stirrups is obtained where $\alpha = 45°$. In this case Eq. (3.13) becomes

$$s = \frac{A_v f_v jd}{0.7V'} \tag{3.14}$$

## SPACING OF VERTICAL STIRRUPS

Where vertical stirrups are used as web reinforcement, $\alpha = 90°$ and Eq. (3.13) becomes

$$s = \frac{A_v f_v jd}{V'} \tag{3.15}$$

**3.34. Actual Spacing of Web Reinforcement.** The theoretical spacing of the various types of reinforcement has been shown in the previous article. However, certain ACI Code limitations and practical considerations often require that the spacing be somewhat different from the theoretical.

The ACI Code specifies that "where web reinforcement is required it shall be so placed that every 45 degree line (representing a potential crack) extending from the mid depth of the beam to the longitudinal tension bars shall be crossed by at least one line of web reinforcement. If a shearing unit stress in excess of $0.06f'_c$ is used, every such line shall be crossed by at least two such lines of reinforcement." The ACI Code also specifies that when bent bars are used as web reinforcement "only the center three-fourths of the inclined portion of such bar, or group of bars, shall be considered effective as web reinforcement." This limitation is illustrated graphically in Fig. 3.23 for both vertical stirrups and inclined bars as web reinforcement where the shearing unit stress does not exceed $0.06f'_c$. For higher shearing unit stresses, the distances $s$ are one-half those shown in Fig. 3.23. For inclined stirrups, the maximum spacing is

$$\tfrac{1}{2}d + \tfrac{1}{2}d \cot \alpha$$

Where vertical stirrups are used, it is undesirable to space them closer than 4 in., and the size of the stirrups should be chosen so as to avoid a closer spacing. Where vertical stirrups are required over a comparatively short distance, it is good practice to space the stirrups uniformly over the entire distance, the spacing being calculated for the point of greatest shear (minimum spacing). If the web reinforcement is required over a long distance and the shear varies materially throughout this distance, it is more economical to compute the spacings required at several sections and to place the stirrups accordingly, in groups of varying spacings.

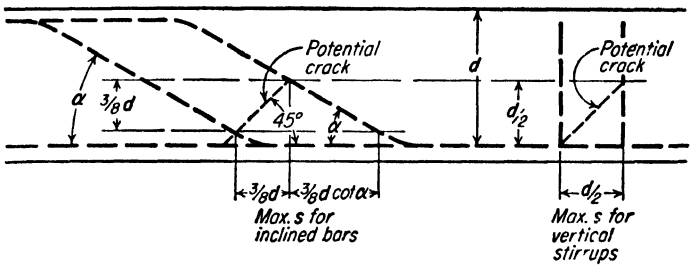



FIG. 3.23

**3.35. Arrangement of Bent-up Bars.** Main reinforcing bars may be bent up subject to limitations of the last two articles to resist the diagonal tension forces. However, Eq. (3.13) may be used only where bars are bent in two or more groups. Where a single bar or a single group of bars are bent, the total amount of shearing forces that can be resisted by such type of reinforcement is limited to $0.04f'_c bjd/\sin \alpha$ and it can be effective as web reinforcement only over the center three-fourths of the inclined portion of such bar, or group of bars.

The points where horizontal bars may be bent up are governed by the amount of reinforcement required to resist the horizontal fiber stresses caused by the bending moment at different sections along the beam. In a uniformly loaded beam the bending moment is a maximum at the center (or near the center) of the span and decreases toward the ends or points of inflection. The total tension in the reinforcement decreases in the same ratio. Sufficient reinforcement must always remain in the bottom of the beam to care for the bending stresses and the bond stresses, and the ACI Code specifies that in a simply supported beam not more than two-thirds of the reinforcement may be bent up to resist diagonal tension stresses. This Code further requires that the bars which are not bent up extend into the support a distance of 6 in.

In beams continuous over one or more supports, the bending of the bars must be made at such points that both the positive and negative

bending-moment requirements are satisfied. In such cases the ACI Code requires that at least one-quarter of the positive-moment reinforcement extend along the same face of the beam into the support a distance of 6 in., while at the outer ends of freely supported end spans, the requirements for simply supported spans given in the preceding paragraph apply.

The location of the points of bending may be determined graphically as follows: Plot the bending-moment diagram for the given loads. Since the amount of tensile reinforcement is proportional to the bending moment, the maximum ordinates of the bending-moment diagram also represent the total area of the steel reinforcement. Therefore, these maximum ordinates may be divided into distances proportional to the areas of the several bars comprising the reinforcement. Draw a horizontal line through each point of division. The intersection of these lines with the bending-moment diagram locates the points at which the several bars may be bent. However, the ACI Code requires that, except for simply supported beams and the outer freely supported ends of end spans of continuous beams, the theoretical points must be exceeded by at least 12 bar diameters. The moment diagrams for uniformly loaded beams are given in Diagrams 1 and 2, Appendix D, and may be used for such conditions of loading without the actual plotting of the moment diagram for the particular beam being designed. For other loadings it it usually advantageous actually to plot the moment diagrams, as described above.

The bars which are bent should be selected so that the symmetry of the remaining bars about the vertical axis of the cross section is not destroyed. As a rule, at least two bars are bent up together from corresponding points on either side of the beam. Sometimes, however, it becomes necessary to depart from this procedure, as when three bars are to be bent. Then either first one is bent, and then two, or all three are bent at the same point. In either case the odd bar is bent from the middle of the section of the beam if possible.

When the bending is done at two or more points, the distance between the points of bending, and if possible between the point of bending of the bar nearest the support and the edge of the support, should not exceed the limiting values given in Art. 3.34. The bent bars can then be assumed to take care of all the diagonal tension between the point at which the first group of bars is bent up and the support in accordance with the discussion in Art. 3.34, provided that the individual distances between points of bending do not exceed that computed from Eq. (3.13) If the distance from the point at which the bars nearest to the support are bent, and the support, exceeds either of these limitations, vertical or inclined stirrups will be required adjacent to the support, to provide

for the diagonal tension over the excess distance. The spacing of vertical stirrups is computed as explained in Art. 3.33.

**3.36. Bond Stresses.** When steel bars are placed in a beam, there must be sufficient bond between the steel and the concrete to prevent the bars from pulling out when stressed. This bond stress, or tendency for the steel to slide out of the concrete, per square inch of bar surface, may be determined as follows: If, in the short section of beam discussed in Art. 3.26, moments about the point $A$ (Fig. 3.24) are taken,

$$(T - T')jd = Vx$$

and

$$\frac{T - T'}{x} = \frac{V}{dj}$$

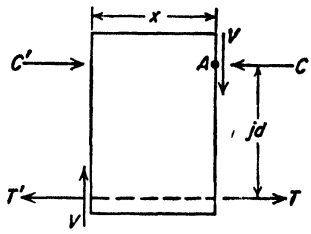



Fig. 3.24

But $T - T'$ represents the force which tends to pull the bars out of the concrete in a length $x$. Hence $(T - T')/x$ represents this force per unit length of beam. Therefore $V/jd$ is the bond force per unit of length between the two materials. If $u$ is the bond stress per unit area of exposed steel surface, and $\Sigma_0$ the total perimeter of steel,

$$u = \frac{V}{\Sigma_0 jd} \tag{3.16}$$

This equation applies to the steel in tension only.

In order to prevent the pulling out of the bars when stressed, the value of the unit bond stress as computed by Eq. (3.16) should not exceed a safe working limit which has been established from a study of the results of tests in which bond failure has occurred.

*Illustrative Example.* In the beam described in the problem in Art. 3.25, determine the maximum unit bond stress, assuming that all the bars continue in the bottom of the beam for the full length of the beam.

The maximum shear occurs at the left support (16,600 lb); hence the maximum unit bond stress exists at that support. From Eq. (3.16),

$$u = \frac{16{,}400}{4 \times 2.75 \times \frac{7}{8} \times 17.5} = 97 \text{ psi}$$

**3.37. Working Unit Bond Stresses.** The ACI Code specifies that the unit bond stress between concrete and plain reinforcing bars in beams and slabs shall not exceed $0.045f'_c$ and that the unit bond stress on approved deformed bars (see Art. 2.6) shall not exceed $0.10f'_c$. Where bars are so placed in a beam that more than 12 in. of concrete is cast in the member below the bar, the bars are classified as "top bars," and the maximum unit bond stress shall not exceed $0.03f'_c$ for plain bars or $0.07f'_c$

for approved deformed bars. All plain bars in tension except at the interior supports of continuous members shall terminate in standard hooks.

A "standard hook" is defined as (see Fig. 3.25):

1. A complete semicircular turn with a radius of bend on the axis of the bar of not less than three and not more than six bar diameters, plus an extension of at least four bar diameters at the free end of the bar.

2. A 90° bend having a radius of not less than four bar diameters and not more than six bar diameters plus an extension of twelve bar diameters.

A standard hook shall be considered as developing 10,000 psi in addition to the bond stress developed by the embedment of the bar. Standard hooks are used where required to develop the necessary anchorage in the tensile reinforcement at the ends of simply supported or cantilever beams or the freely supported ends of continuous beams.

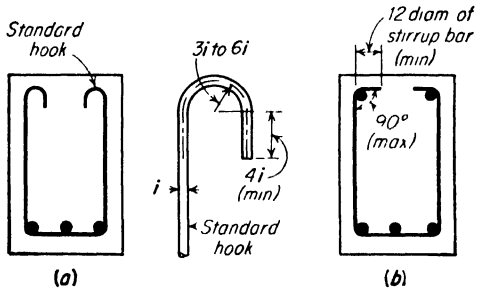



FIG. 3.25

Standard hooks, Type 1, are also used at the free ends of stirrups used for web reinforcement (see Fig. 3.25) where no compressive reinforcement is available for rigid attachment.

**3.38. Anchorage of Web Reinforcement.** Vertical U stirrups, or vertical multiple-loop stirrups, are anchored at the loop ends by bending around the longitudinal reinforcement, as shown in Fig. 3.25. At the cut ends, anchorage is provided by a standard hook, as shown in Fig. 3.25*a*, or by bending the end of the stirrup bar through an angle of at least 90° around a longitudinal reinforcing bar not less in diameter than the stirrup bar and projecting the stirrup bar at least 12 diameters past the bend, as shown in Fig. 3.25*b*. In case there is sufficient depth in the compression area of the beam to permit sufficient length of embedment to develop in bond the full stress in the stirrup bar, the hooks or bends at the cut ends of the stirrup bars may be omitted. The required length of embedment is computed from the equation in Art. 2.7. In computing the length available for embedment in the compression area, the middle of the effective depth $d$ of the beam may be considered to be the dividing line between the tension and compression areas. Inclined stirrups are

anchored in the same manner as for vertical stirrups; but, in addition, they must be welded or otherwise rigidly attached to the longitudinal reinforcing bars.

## T BEAMS

**3.39. Types of T Beams.** When a reinforced concrete floor slab is constructed as a monolith with the supporting beams, and the slab and the beam are thoroughly tied together by means of stirrups and bent-up bars, part of the slab may be assumed to assist the upper part of the beam in resisting compressive stresses. These two acting together constitute what is known as a T beam (Fig. 3.26). The slab is called the flange, and the portion of the beam beneath the slab is called the web or stem.

The exact width of slab that can be assumed as resisting the compressive forces is a variable. Tests have shown that it is dependent principally upon the relative thickness of the slab and upon the span of the

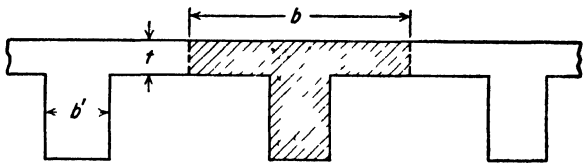



FIG. 3.26

beam. The ACI Code recommends that the effective width[1] of slab shall be determined as follows:

1. It shall not exceed one-fourth of the span length of the beam.
2. Its overhanging width on either side of the web shall not exceed eight times the thickness of the slab.
3. In any case the flange width must not be greater than the distance center to center of adjacent beams.

Another form of T beam, which is of infrequent occurrence, is one which does not form a part of a floor system, the flange being provided merely to furnish sufficient area in compression. Since the concrete in the lower part of a beam is assumed as taking no tension, its only purpose is to bind the tensile steel and the compressive concrete together. This involves mainly shearing stresses; all the rectangular section is not required in large beams, and thus a saving in concrete results when the T form is used. It is, however, usually more satisfactory to use a rectangular beam with compressive reinforcement to care for cases requiring an excessive amount of concrete rather than to resort to the T section. A saving in cost of forms, and certain evident structural advantages of

[1] For beams having a flange on one side only, the effective overhanging width shall not exceed one-twelfth of the span length of the beam, or six times the thickness of the slab, or one-half the clear distance to the next beam.

the rectangular beam, will, in general, counteract the saving in concrete in the T beam.

The neutral axis of a T beam may lie either in the flange or in the web, depending upon the relation between the thickness of the flange, the depth of the beam, and the amount of steel. When the neutral axis is in the flange, *i.e.*, when $kd$ is less than $t$, the equations derived in Art. 3.4 for rectangular beams must be used, the width of the beam being equal to the effective width $b$ of the flange. The reason for this is shown in Fig. 3.27*a*, which represents a beam T-shaped in cross section. The neutral axis is assumed above the bottom of the flange. The compressive area is represented by the shaded portion of the figure. If the additional concrete, indicated by the areas (1) and (2), had been added when the beam was poured, the physical cross section would be rectangular in shape, with a width equal to $b$. No bending strength would be added by

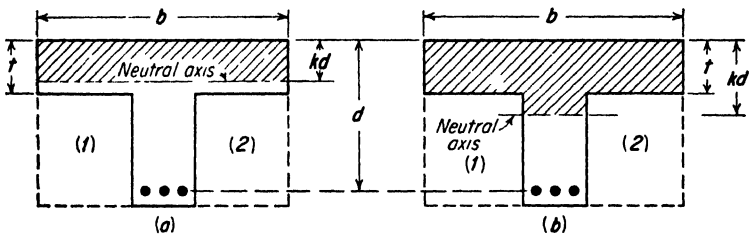



FIG. 3.27

the addition of this extra concrete, because the areas (1) and (2) are in the tension portion of the cross section, and, as stated before, the tensile strength of the concrete is disregarded in all flexure formulas. The original T-shaped beam and the revised rectangular-shaped beam are equal in flexural strength, and the rectangular-beam equations for flexure apply.

When the neutral axis is in the web, *i.e.*, when $kd$ is greater than $t$, the rectangular-beam equations no longer apply. For, in Fig. 3.27*b*, if the extra concrete represented by the areas (1) and (2) were added to the original T-shaped beam, the resulting rectangular-shaped beam would actually be stronger in flexure than the original beam, because some of the added concrete [those portions of areas (1) and (2) which are above the neutral axis] would be in compression. The application of the rectangular-beam equations to this condition would therefore be incorrect in theory. The proper equations for use in this case are derived in the following article in a manner similar to that used in the derivation of the rectangular-beam equations.

When the neutral axis is at the bottom of the flange, *i.e.*, when $kd = t$, then by comparison of Figs. 3.27*a* and *b* it is obvious that both the rec-

tangular-beam equations and the T-beam equations will give the same results; *i.e.*, the values of $k$, $j$, $M_c$, $M_s$, etc., obtained by the one set of equations, will be the same as the corresponding values obtained by the other set of equations.

**3.40. Flexure Analysis (Neutral Axis in the Web).** Figure 3.28 represents an element of a T beam. The compression force in the web, represented by the area *qrst* in the cross section, is usually small in comparison with that in the flange, and hence it is neglected in the derivation of equations for ordinary design.

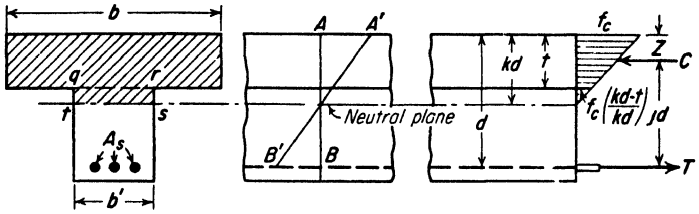



FIG. 3.28

From the assumption that deformations vary as the distance from the neutral axis,

$$\frac{AA'}{BB'} = \frac{kd}{d - kd} = \frac{k}{1 - k} \tag{a}$$

Since $E = \dfrac{\text{unit stress}}{\text{unit deformation}}$, assuming that $AA'$ and $BB'$ represent the deformations of a unit length of the beam at the extreme compression surface and the plane of the reinforcement, respectively,

$$AA' = \frac{f_c}{E_c} \qquad \text{and} \qquad BB' = \frac{f_s}{E_s}$$

Hence,

$$\frac{AA'}{BB'} = \frac{nf_c}{f_s} = \frac{n}{r} \tag{b}$$

Equating (*a*) and (*b*) and solving for $k$,

$$k = \frac{n}{n + r} \tag{3.7}$$

This gives an expression for the value of $k$ when $n$ and $r$ are known. This equation can be used only in the design of an isolated T beam, *i.e.*, one not a part of a floor system, since in such a problem just enough flange width will be provided to bring the unit stress in the concrete to its maximum allowable value simultaneously with that in the steel—the ratio $r$ is known. In a T beam which is part of a floor system, the compressive area is so large ($b$ is taken as one-fourth the span or $16t + b'$)

that when $f_s$ is a maximum, $f_c$ is only a relatively small value—the ratio $r$ is not known The total tension $= A_s f_s$. The total compression per unit of width is represented by a trapezoid whose parallel sides are $f_c$ and $f_c\ (kd - t)/kd$, the amount of compression being, therefore,

$$f_c + f_c \frac{(kd - t)/kd}{2} tb = f_c \frac{2kd - t}{2kd} bt$$

For equilibrium, the total tension must equal the total compression; hence,

$$A_s f_s = pbdf_s = f_c \frac{2kd - t}{2kd} bt \qquad (c)$$

As in the derivation of rectangular-beam equations, the relation between the unit stresses in steel and concrete is given by the equation

$$f_c = \frac{f_s k}{n(1 - k)} \qquad (3.1a)$$

Substituting from Eq (3.1$a$) in Eq. ($c$) to eliminate unit stresses,

$$k = \frac{np + \frac{1}{2}(t/d)^2}{np + (t/d)} \qquad (3.17)$$

The distance of the center of compression (center of gravity of the trapezoid) from the upper face of the beam is

$$z = \frac{3kd - 2t}{2kd - t} \frac{t}{3} \qquad (3.18)$$

and the lever arm of the couple formed by the tensile and compressive forces is

$$jd = d - z \qquad (3.19)$$

From the above analysis

$$j = \frac{6 - 6(t/d) + 2(t/d)^2 + (t/d)^3 (1/2pn)}{6 - 3(t/d)} \qquad (3.20)$$

The resisting moments of the steel and concrete are equal to the product of the lever arm $jd$ of the internal stress couple and the total tension and compression, respectively; hence,

$$M_s = A_s f_s jd \qquad (3.21)$$

and

$$M_c = f_c \left(1 - \frac{t}{2kd}\right) btjd \qquad (3.22)$$

Approximate equations for resisting moments can be developed as follows: Since the center of gravity of the compression stress trapezoid is

above the middle of the slab, the lever arm $jd$ of the resisting couple is never less than $d - \frac{1}{2}t$. The average unit compressive stress

$$\frac{f_c + f_c\,(kd - t)/kd}{2} = f_c\left(1 - \frac{t}{2kd}\right)$$

is never so small as $\frac{1}{2}f_c$ except when the neutral axis is at the bottom of the slab, in which case rectangular-beam equations apply. Equations (3.21) and (3.22) may then be approximated by substituting these limiting values for $jd$ and $fc\,[1 - (t/2kd)]$, respectively. Then

$$M_s = A_s f_s(d - \tfrac{1}{2}t) \text{ (approximate)} \tag{3.23}$$

and

$$M_c = \tfrac{1}{2}f_c bt(d - \tfrac{1}{2}t) \text{ (approximate)} \tag{3.24}$$

In the design of a continuous T beam at the support, a slightly larger amount of steel will be required there than at the center. This is due to the fact that the value of $j$ at the support will in all cases be less than at the center. Since the tension steel at the support of such beams is often provided by bending up from each side one-half of the steel furnished at the center, a slight excess at this latter point is often of advantage. The use of Eqs. (3.23) and (3.24) *in design* is therefore justified for all practical purposes. *They must not be used in review problems.*

**3.41. Shearing Strength of T Beams.** Owing to the relatively large width of flange in beams which are part of a floor system, it is safe to say that the compressive strength of the beam will seldom govern the design. Since the only stress that will be imposed upon the concrete below the neutral axis is that of shear (the concrete is assumed incapable of resisting tensile stresses), it follows that the effective cross section of the stem of the beam $b'd$ need merely be large enough to keep the horizontal shear below its allowable value. Since

$$v = \frac{V}{b'jd}$$

the amount of web area required equals

$$b'd = \frac{V}{vj} \tag{3.25}$$

As previously stated, a value of $j = \frac{7}{8}$ may be used in Eq. (3.25). Figure 3.16*b* explains the use of $b'$ instead of $b$ in Eq. (3.25).

In closely spaced long beams with light loads, it is possible that the compressive strength of the beam may govern. In such cases Eq. (3.24) may be used to get an approximate value of $d$; the review of the assumed section will then determine if any revision is necessary.

**3.42. Ratio of Depth of Beam to Breadth of Stem.** Numerous formulas have been devised to determine the economical depth of a T beam; but very often the available head room is limited, and the results of these formulas would exceed the limitation. The use of economical-depth formulas for shallow beams such as are encountered in ordinary building construction involves, for this reason, an unnecessary computation, practical considerations in most cases governing the design.

A study of numerous successful designs shows that for ordinary beams a ratio between $b'$ and $d$ of one-half to one-third gives satisfactory results. For very large and deep beams, a ratio as small as one-fourth is occasionally used. In modern building construction, the shallower and wider beam is to be preferred in order to obtain the maximum head room and minimum light obstruction.

The ACI Code recommends that beams in which the T form is used only for the purpose of providing additional compressive area of concrete shall have a flange thickness not less than one-half the width of web and a total flange width not more than four times the web thickness. This applies only to isolated T beams.

**3.43. Diagrams for Review of T Beams.** Since values of $k$ and $j$, as expressed in Eqs. (3.17) and (3.20), are dependent only upon the ratio $t/d$ and the product $pn$, both of which would be known in reviewing a T beam, values of $k$ and $j$ may be taken from Diagram 3, Appendix D, the curves of which are based on these equations and drawn with $t/d$ and $pn$ as variables. Points which fall below the broken inclined line terminating the curves indicate that the neutral axis is in the flange ($kd$ is less than $t$) and the formulas for rectangular beams (or Table 6, Appendix D) must be used. Since the value of $p$ is required in order to enter Diagram 3, this diagram cannot be used in design.

**3.44. Types of T-beam Problems.** There are three main types of problems that may be encountered in practice.

1. To find the moment of resistance or fiber stresses. The values of $k$ and $j$ may be obtained from Eqs. (3.17) and (3.20) or from Diagram 3, the values of the fiber stresses from Eqs. (3.21) and (3.1$a$), or the resisting moments from Eqs. (3.21) and (3.22), the smaller of the latter being the resisting moment of the beam. Since the resisting moment of the steel will usually govern in T beams whose flange is a part of a floor system, it is generally quicker to compute the value of $M_s$, from Eq. (3.21), and then substitute the limiting value of $f_s$ in Eq. (3.1$a$) to determine the simultaneous value of $f_c$. If this is less than the allowable value, the assumption that the steel governs is correct for the case in question. If it is greater than the allowable value, determine from Eq. (3.1$a$) the value of $f_s$ that corresponds to the maximum permissible value of $f_c$, and use this value of $f_s$ in Eq. (3.21). The result will be the moment in the steel

when the concrete is stressed just to its limit, and hence it is the true resisting moment of the beam. (If $kd$ is less than $t$, the neutral axis is in the flange; formulas for rectangular beams should then be used, the width being equal to the flange width $b$ of the T beam.)

2. To design a T beam in which the flange is a portion of a floor slab already designed. Compute, from Eq. (3.25), Art. 3.41, the cross section $b'd$ required, and select the width of stem and depth of beam with reference to the most satisfactory shape of beam, spacing of bars, etc. Usually $d$ should be taken as from two to three times $b'$. The amount of steel may then be determined from Eq. (3.23).

In order to compute the steel area more accurately, determine the value of $j$ from Eq. (3.20) or from Diagram 3, using for $p$ the value corresponding to the approximate steel area. Equation (3.21) will then give the true steel area that is required. A slight variation between the

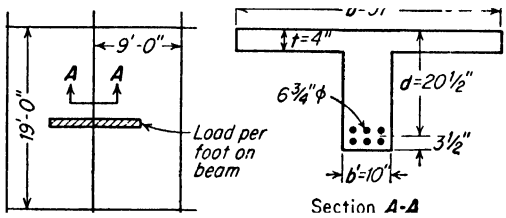



FIG. 3.29

values of $p$ as determined by the approximate method and the true method will cause only a slight difference in Eq. (3.20); thus further substituting is unnecessary. The value of $k$ should be computed from Eq. (3.17) or taken from Diagram 3 to ascertain whether the neutral axis is in the stem or flange. Equation (3.2) for rectangular beams would give the same information.

3. To design a T beam whose flange is not a part of a floor system. In designing a beam of this type, determine from Eq. (3.25), Art. 3.41, the shearing area required, and select the values of $b'$ and $d$. From Eqs. (3.7), (3.18), and (3.19), the values of $k$ and $j$ may be determined. Equations (3.21) and (3.22) will then give the area of steel and breadth of flange required.

In all work on T beams where flexural stresses are concerned, that is, in the determination of $k$, $j$, fiber stresses, resisting moments, and area of steel, the value of $b$ is the width of flange. In the determination of the shearing area required, the value of $b'$ is the width of stem. The reasons for each should be obvious from a study of the foregoing articles.

**3.45. Illustrative Example.** I. A floor slab 4 in. thick is supported by reinforced concrete beams 9 ft-0 in. center to center (Fig. 3.29) which together with the slab act as T beams. The beams are simply supported,

and their span is 19 ft-0 in. The slab supports a live load of 100 psf. The cross section of each beam below the slab is 10 by 20 in.; the reinforcement consists of six No. 6 bars in two rows, 2 in. center to center vertically, the center of the lower row being $2\frac{1}{2}$ in. above the lower surface of the beam. Assume $n = 15$. Determine $f_s$ and $f_c$.

Weight of slab $= \frac{4}{12} \times 150 = 50$ psf
Total load on slab $= 150$ psf
Load from slab on each beam $= 9 \times 150 = 1350$ lb per ft
Weight of beam below slab $= \frac{10 \times 20}{144} \times 150 = 210$ lb per ft
Total load on beam $= 1560$ lb per ft
$M = \frac{1}{8} \times 1560 \times 19^2 \times 12 = 845{,}000$ in.-lb

The breadth of flange cannot exceed $\frac{1}{4} \times 19 \times 12 = 57$ in., or $(16 \times 4) + 10 = 74$ in. Hence, $b = 57$ in.

$$p = \frac{2.65}{57 \times 20.5} = 0.0023 \qquad pn = 0.0345 \qquad \frac{t}{d} = \frac{4}{20.5} = 0.195$$

From Diagram 3, Appendix D, $k = 0.233$ and $j = 0.926$. Hence,

$$f_s = \frac{845{,}000}{2.65 \times 0.926 \times 20.5} = 16{,}800 \text{ psi}$$

$$f_c = \frac{16{,}800 \times 0.233}{(1 - 0.233) \times 15} = 340 \text{ psi}$$

II. Determine the resisting moment of the beam whose dimensions and reinforcement are given in the preceding example, assuming a 2000-psi concrete and an allowable steel stress of 18,000 psi.

If the strength of the beam were governed by that of the steel, the resisting moment would be as follows:

$$M_s = 2.65 \times 18{,}000 \times 0.926 \times 20.5 = 908{,}000 \text{ in.-lb}$$

The corresponding stress in the concrete is

$$f_c = \frac{18{,}000 \times 0.233}{15(1 - 0.233)} = 365 \text{ psi}$$

The steel; therefore, governs as assumed, since, when it is stressed to its limit, 18,000 psi, the concrete unit stress is much less than the allowable stress of $0.45 \times 2000 = 900$ psi, specified in the ACI Code.

III. Using the specifications of the ACI Code for a 2500-psi concrete and intermediate-grade steel, determine the cross section of the web below the slab and the sectional area of steel required for a simply sup-

ported T beam constructed monolithically with a 5-in. floor slab which sustains a live load of 150 psf. The distance center to center of adjacent supports is 21 ft-0 in. (Fig. 3.30).

Weight of slab = $\frac{5}{12} \times 150 = 62$ psf
Total load on slab = 212 psf
Load on beam from slab = $11 \times 212 = 2330$ lb per ft
Assumed weight of stem = 140 lb per ft
Total load on beam = 2470 lb per ft
Maximum shear $V = 2470 \times \frac{21}{2} = 26{,}000$ lb

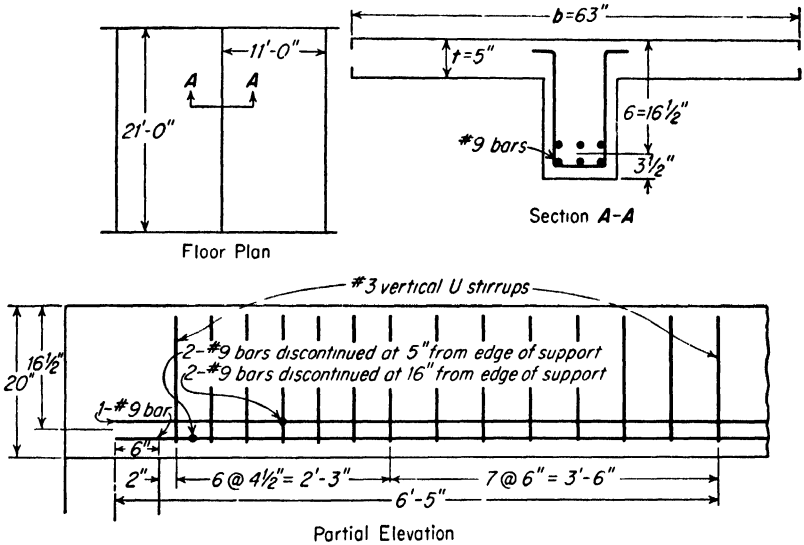



FIG. 3.30

Assuming that web reinforcement is to be provided in the form of vertical stirrups only, the allowable unit shearing stress is

$$0.08 \times 2500 = 200 \text{ psi}$$

$$b'd = \frac{26{,}000}{\frac{7}{8} \times 200} = 148 \text{ sq in.}$$

A $b'$ of 10 in. and a $d$ of 15 in. are selected.

Since two rows of bars will undoubtedly be necessary, the total height of beam must be $15 + 3\frac{1}{2} = 18\frac{1}{2}$ in., and the cross section below the slab is $10 \times 13\frac{1}{2}$ in., which gives a weight of stem of

$$\frac{10 \times 13.5}{144} \times 150 = 141 \text{ lb per ft}$$

approximately as assumed.

$$M = \tfrac{1}{8} \times 2470 \times 21^2 \times 12 = 1{,}635{,}000 \text{ in.-lb}$$
$$A_s = \frac{1{,}635{,}000}{20{,}000\ (15 - 2.5)} = 6.54 \text{ sq in.}$$

Three No. 9 and three No. 10 bars furnish 6.79 sq in.

In order to avoid the use of an excessive amount of tension reinforcement, it is often desirable to increase the depth of the beam where minimum headroom requirements do not control. In the present case by increasing the depth of the beam by $1\frac{1}{2}$ in. to $16\frac{1}{2}$ in. the revised weight of stem is 156 lb per ft; the revised weight per foot, 2490 lb; the revised bonding moment, 1,646,000 in.-lb; and the revised area of steel required, 5.88 sq in.

For all practical purposes of design, this approximate area of steel would be satisfactory and six No. 9 bars furnishing 6.00 sq in. could be selected as the reinforcement. However, if the true area required were to be determined, the procedure would be as follows: Assume that six No. 9 bars will be used. The flange width $b$ is limited in this case to $\frac{1}{4}$ span, or $\frac{1}{4}(21 \times 12) = 63$ in. Then

$$p = \frac{6.00}{63 \times 16.5} = 0.0058$$
$$pn = 12 \times 0.0063 = 0.070$$
$$\frac{t}{d} = \frac{5}{16.5} = 0.30$$

From Diagram 3 it appears that the neutral axis is just above the edge of the flange, and from Diagram 3, $j$ is obtained as 0.896 and the area of steel required is

$$A_s = \frac{1{,}646{,}000}{20{,}000 \times 0.896 \times 16.5} = 5.56 \text{ sq in.}$$

*Spacing of Vertical Stirrups.* At the support, the unit shear is

$$v = \frac{26{,}000}{10 \times 0.875 \times 16.5} = 181 \text{ psi}$$

of which 75 psi is resisted by the concrete, leaving 106 psi to be resisted by the stirrups.

$$V_c = 75 \times 10 \times 0.875 \times 16.5 = 10{,}800 \text{ lb} \qquad \text{(Art. 3.26)}$$
$$V' = 26{,}000 - 10{,}800 = 15{,}200 \text{ lb}$$

and with Nc. 3 U stirrups

$$s = \frac{2 \times 0.11 \times 20{,}000 \times 0.875 \times 16.5}{15{,}200} = 4.2 \text{ in.}$$

Since the shear decreases at the rate of 2490 lb per ft from the support toward the center of the beam, the required spacing of the stirrups will increase at the rate of $2490/15{,}300 \times 4.2 = 0.68$ in. per ft and no stirrups will be required between a point $15{,}200/2490 = 6.2$ ft from the center of the support and the center of the span.

Assuming the width of the support as 12 in., the spacing of stirrups required at the edge of the support is $4.2 + 0.3 = 4.5$ in. Beginning 2 in from the edge of the support, seven stirrups will be placed in the beam 4½ in. center to center[1]; then the last stirrup in this group will be 6 in. + 2 in. + $6 \times 4\frac{1}{2}$ in. = 35 in., or 2.92 ft from the center of the support where the required spacing is $4.2 + 2.92 \times 0.68 = 6.2$ in. Seven additional stirrups 6 in center to center will cover a distance of $2.92 + 3.5 = 6.42$ ft and provide all the necessary web reinforcement.

*Spacing of Bent-up Bars as Web Reinforcement.* While the allowable unit shear in the beam can according to the ACI Code be as great as $0.12f'_c$ where bent-up bars or inclined stirrups are used as web reinforcement, in the present design a decrease in the dimensions of the web of the beam would make it impractical to place the required amount of reinforcing steel in the beam.

Assume that some of the horizontal reinforcement will be bent up, making an angle of 30° with the horizontal to assist in resisting the diagonal tension forces. Since the ACI Code requires that at least one-third of the required positive reinforcement shall extend along the same face of the beam into the support a distance of 6 in , there are four bars which can be bent upward to resist diagonal tension forces, provided the remaining bars satisfy the bond requirements for the tension reinforcement. At the edge of the support, the total shear $V$ is $26{,}000 - 1240 = 24{,}760$. From Eq. (3.16),

$$\Sigma_0 = \frac{24{,}760}{250 \times 0.875 \times 16.5} = 6.90 \text{ in}$$

while two No. 9 bars furnish 7.08 in.

Two bars having an area 2.0 sq in. are $2.00 \div 6.00 = 33$ per cent of the reinforcement and from Diagram 1 are no longer required for tensile reinforcement at a distance of $0.21 \times 21 = 4.4$ ft from the support. Similarly, four bars with an area of 67 per cent of the reinforcement are no longer required at $0.09 \times 21 = 1.9$ ft from the support. These bars must be carried 12 diameters, or 14 in., beyond the point where they are no longer required for tensile reinforcement so that two bars may be

[1] A strict interpretation of the ACI Code would require that near the support where the shear is greater than $0.06f'_c$, the spacing of stirrups should not exceed $d/4$, or 4.12 in. However, the shear becomes equal to $0.06f'_c$ at 1.3 ft from the edge of the support, and the 4½ spacing rather than a 4-in. spacing is considered sufficient.

bent upward not farther away from the center of the support than 3.2 ft and two more not farther away from the center of the support than 0.7 ft.

From Eq. (3.13) the spacing of the bent-up bars required at the edge of the support is far greater than that allowed by the ACI Code for unit shears in excess of $0.06f'_c$. From Fig. 3.23 the maximum spacing of bars inclined at 30° with the horizontal is

$$s = \frac{3}{8}\left(\frac{16.5}{2} + \frac{16.5}{2} \times 1.732\right) = 8.5 \text{ in.}$$

Therefore, two bars can be bent upward 8.5 in. from the center of the support and two others 17 in. from the center of the support.

Referring to Fig. 3.31, the possible arrangement of the bent-up bars to assist in resisting the diagonal tension forces is shown. A distance of 4 in. above the mid-depth of the beam is shown in Fig. 3.31, which is

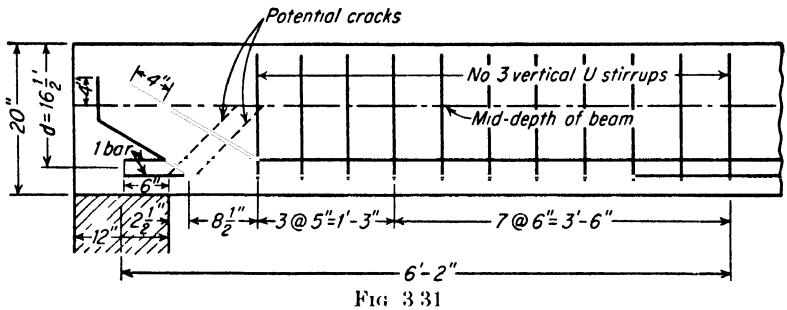



FIG. 3.31

ample distance to develop the bond for the slight stress carried by the bars. In order to provide at least two web reinforcing members crossing a potential crack (see Art. 3.34), a vertical stirrup must be placed about 11 in. from the edge of the support. At this point, 17 in., or 1.4 ft, from the center of the support, the required spacing is

$$4.2 + 1.4 \times 0.68 = 5.1 \text{ in.}$$

A spacing of 5 in. is used for three spaces and a 6-in. spacing to the point 6.2 ft from the center of the support beyond which web reinforcement is no longer required. This will require 11 stirrups, as shown in Fig. 3.31.

Since only three less vertical stirrups are required in this arrangement than are required when vertical stirrups alone are used, this saving is more than offset by the additional length of and the cost of bending the horizontal bars. In this case the use of vertical stirrups alone and the arrangement shown in Fig. 3.30 is to be preferred. In beams of greater dimension, especially continuous beams where both positive and negative moments must be provided for, bent bars can very often be used advantageously as web reinforcement.

IV. Design a simply supported, isolated T beam with a span of 30 ft-0 in. which must support a live load of 3000 lb per lin ft. Use working stresses as follows: $f_c = 650$ psi, $f_s = 16{,}000$ psi, $v = 120$ psi, $n = 15$.

Assume weight of beam = 950 lb per lin ft
Total load to be carried = 3950 lb per lin ft
$M = \frac{1}{8} \times 3950 \times 30^2 \times 12 = 5{,}330{,}000$ in.-lb
$V = 3950 \times {}^{30}\!/_{2} = 59{,}300$ lb

$$b'd \text{ (required)} = \frac{59{,}300}{120 \times {}^{7}\!/_{8}} = 565 \text{ sq in.}$$

Since $b'$ should preferably be from $\frac{1}{2}$ to $\frac{1}{3}d$, the values selected will be $b' = 16$ in. and $d = 36$ in. These are selected in preference to any other possible combination that falls within the limits stated above, in order to keep $b'$ in even inches and to secure as wide a beam as possible to allow for convenient placing of the reinforcement.

The thickness of the flange is usually made $\frac{1}{3}d$. Hence, $t$ will be taken as 12 in.

$$\frac{t}{d} = \frac{12}{36} = 0.333$$

$$k = \frac{15}{15 + 16{,}000/650} = 0.379 \qquad \text{and} \qquad kd = 0.379 \times 36 = 13.6 \text{ in.}$$

Therefore, the neutral axis is in the stem and the T-beam formulas apply.

From Eqs. (3.18) and (3.19), $z = 4.12$ in. and $j = 0.877$.

$$A_s = \frac{5{,}330{,}000}{16{,}000 \times 0.877 \times 36} = 10.5 \text{ sq in.}$$

The width of flange is determined from Eq. (3.24), all other quantities of which are known; the required width is 38.6 in. A width of 40 in. will be used. Seven No. 11 bars will be selected and placed in two rows. The total height of beam is $39\frac{1}{2}$ in., and the weight per foot 960 lb. The error in the assumed weight is only $\frac{1}{4}$ of 1 per cent of the total load, and the design is, therefore, satisfactory.

## Problems

1. If the beams of Example I, page 99 (see also Fig. 3.29), each support a concentrated load of 5000 lb at the mid-span in addition to the load from the slab as specified in that problem, determine the revised cross section required and the area of steel. Assume $f'_c = 2000$ psi and $f_s = 18{,}000$ psi.

2. For the T beam shown in Fig. 3.26, assume $b = 48$ in., $b' = 10$ in., $t = 4$ in., the depth below the flange 18 in., and the reinforcement four No. 7 bars in one row, the center of which is $2\frac{1}{2}$ in. above the bottom of the beam. Determine the resisting moment of the beam, with allowable unit stresses of 650 psi and 16,000 psi for the concrete and steel, respectively, and with $n = 15$.

3. If the span of the beams in Prob. 2 is 16 ft-0 in., the spacing of beams 7 ft-0 in., if they are of one span only, and if they rest freely on brick walls at the ends, what uniform live load may be placed on the slab without overstressing the beam in bending?

4. If the allowable unit shearing stress for the beam of Prob. 3 is 120 psi (assuming that adequate web reinforcement will be used), what uniform live load may be placed on the slab without overstressing the beam in shear?

5. If the thickness of the flange of the beam in Prob. 2 were increased to 5 in., the over-all height of the beam remaining the same, what would be the resisting moment of the beam?

## RECTANGULAR BEAMS REINFORCED FOR COMPRESSION

**3.46. Use of Beams Reinforced for Compression.** In general, it is economically advantageous to carry all the compression in a beam by the concrete and all the tension by the steel and to stress both materials to their maximum permissible values. Situations do arise, however, when a beam must be called upon to resist a moment larger than that which can be developed by the concrete in compression. This occurs primarily when the over-all size of the beam is limited by structural, architectural, or clearance requirements so that it is incapable of furnishing the required compression resisting moment. In such cases that part of the moment in excess of that resisted by the concrete is provided for by placing additional steel in the compression portion of the beam. A very important and frequent case of this type is discussed in Arts. 3.50 and 3.51.

If both steel and concrete are completely elastic, the stress in the compression steel is $n$ times the concrete stress at the same level, since the unit strains in the steel and the adjacent concrete are equal. This allows the compression steel stress to be computed. However, the stresses and strains in concrete are proportional only at relatively low stresses. This is shown in Fig. 4.2. At higher stresses the strains increase faster than the stresses. In view of the fact that the strains in the compression steel and the adjacent concrete remain equal, this means that at higher stress levels the unit stress in the steel, being proportional to the strain, will be larger than it would be if the concrete were a truly elastic material. This increase in the steel stress over that computed by assuming elastic behavior is accentuated by the fact that concrete to a certain extent keeps compressing under constant load or stress (flow or creep). In contrast, steel, if stressed below the yield point, maintains its length unchanged under constant stress. In a beam reinforced for compression this results in the fact that in the course of time the concrete, by minute flow, transfers part of its compression stress to the steel, and the actual stress in the steel becomes higher than that computed on the basis of elastic behavior.

Tests have shown that the compression steel can safely sustain these higher stresses if adequate support is provided by stirrups or other ties to

prevent buckling. The higher stress means a higher total compression force in the beam, and, therefore, contributes to the compression resisting moment. To take advantage of the added strength, the ACI Code states that:

The effectiveness of compression reinforcement in resisting bending may be taken at twice the value indicated from the calculation assuming a straight-line relation between stress and strain and the appropriate modular ratio, *but not of greater value than the allowable stress in tension.*

Compression steel in beams or girders shall be anchored by ties or stirrups not less than $\frac{1}{4}$ inch in diameter spaced not farther apart than 16 bar diameters, or 48 tie diameters. Such stirrups or ties shall be used throughout the distance where the compression steel is required.

**3.47. Formulas for Design.** The following analysis assumes elasticity in both steel and concrete for determining the position of the neutral axis. The corrections specified by the ACI Code (see Art. 3.46) are then applied to the stresses.

The notation used in the following derivations is as follows:

$M$ = total moment to be developed by beam
$M_1$ = moment that can be developed by limited cross section of concrete without compression reinforcement
$M_2$ = moment in excess of $M_1$ which must be developed by compression reinforcement, that is, $M_2 = M - M_1$
$A_{s_1}$ = area of tensile steel required to develop moment $M_1$
$A_{s_2}$ = area of additional tensile steel necessary to develop moment $M_2$
$A_s$ = total tensile steel area = $A_{s_1} + A_{s_2}$
$A'_s$ = total area of compressive steel
$f_s$ = unit stress in tensile steel
$f'_s$ = unit stress in compressive steel

Assuming that a rectangular beam, which will be called upon to resist a bending moment $M$, is limited in size to an effective cross section $bd$, the resisting moment $M_1$ of the concrete being less than the moment $M$, compressive steel of an amount $A'_s$ will be required in order to keep the unit stress in the concrete within the allowable limit.

The moment $M_1$ depends upon the concrete and equals (see Fig. 3.32 and Art. 3.4)

$$M_1 = \tfrac{1}{2} f_c kjbd^2 = Kbd^2 \tag{3.26}$$

Since the resisting moment of the tensile steel = $A_{s_1} f_s jd$, the area of steel required to provide sufficient tensile resistance fully to develop the strength of the concrete is

$$A_{s_1} = \frac{M_1}{f_s jd} \tag{3.27}$$

The values of $k$, $j$, and $K$ are computed from the same equations as for rectangular beams with only tensile reinforcement (Art. 3.4).

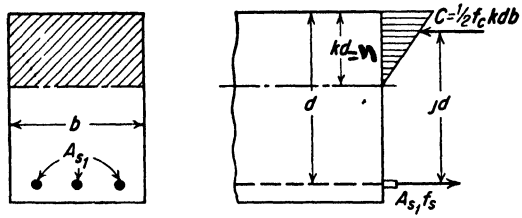



FIG. 3 32

A moment of an amount $= M_2 = M - M_1$ still remains to be provided for by the necessary amount of compressive steel and additional tensile steel $A_{s_2}$. The forces in these two quantities of steel form a couple, the lever arm of which is $d - d'$ (Fig. 3.33). The resisting

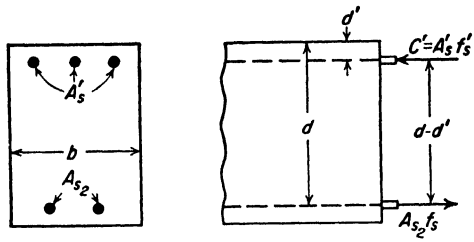



FIG. 3 33

moment of the couple is therefore $A_{s_2}f_s(d - d')$, and this must be sufficient to develop the moment $M_2$; hence

$$M_2 = A_{s_2}f_s(d - d')$$

from which

$$A_{s_2} = \frac{M_2}{f_s(d - d')} \tag{3.28}$$

The total tensile steel then equals

$$A_s = A_{s_1} + A_{s_2} \tag{3.29}$$

Since the beam must be in equilibrium,

$$A'_s f'_s = A_{s_2}f_s \tag{a}$$

From the assumption that unit stresses vary as the distance from the neutral axis,

$$\frac{f_s}{f'_s} = \frac{d - kd}{kd - d'}$$

from which

$$f'_s = f_s \frac{k - (d'/d)\dagger}{1 - k} \tag{b}$$

† This is the stress in the compression steel if the concrete were completely elastic.

However, since in view of the plasticity of the concrete the compressive steel may be stressed to twice this value, it follows that

$$f'_s = 2f_s \frac{k - (d'/d)}{1 - k} \tag{c}$$

Substituting in Eq. (*a*) the value of $f'_s$ from Eq. (*c*) in order to eliminate $f'_s$,

$$2A'_s f_s \frac{k - (d'/d)}{1 - k} = A_{s_2} f_s$$

from which

$$A'_s = \frac{A_{s_2}}{2} \frac{1 - k}{k - (d'/d)} \tag{3.30}$$

Since the forces in $A_{s_2}$ and $A'_s$ form a couple of magnitude $M_2$, they must be equal in absolute amount, that is, $A_{s_2} f_s = A'_s f'_s$, as shown in (*a*) for elastic stresses. Hence, if $A'_s$, as determined from Eq. (3.30), is smaller than $A_{s_2}$, it follows that $f'_s$ would be greater than $f_s$. The ACI Code specifies, however, that $f'_s$ shall not exceed the allowable stress for the tensile steel. To achieve this, it is therefore necessary in this case to make $A'_s$ equal to $A_{s_2}$.

Consequently, the required amount of compression steel is equal to the larger of the two values $A'_s$ or $A_{s_2}$, determined from Eqs. (3.28) and (3.30), respectively.[1]

Where the compressive steel area is determined by Eq. (3.30), since the additional tensile steel $A_{s_2}$ and $A'_s$ balance one another, the position of neutral axis is not changed and the values of $k$ and $j$ as previously determined for rectangular beams remain the same as if $A_{s_2}$ and $A'_s$ were not added. Where the value of $A'_s$, as determined from Eq. (3.30), has to be increased in order that the stress in the compressive steel does not exceed that in the tensile steel, a slight change in the position of the neutral axis occurs, but the error introduced is so small that it may be neglected.

**3.48. Formulas for Review.** The equations (for $k$, $j$, and the resisting moment) to be used in reviewing a rectangular beam with compressive reinforcement are derived in a manner similar to that used in deriving the corresponding equations for rectangular beams with tensile reinforcement only. The total tension in the steel $= T = A_s f_s = pbdf_s$. The total compression in steel and concrete $= C' + C$; the former (see Fig.

[1] Another method of determining the area of compression steel required is as follows: Eq. (*c*) may be written $f'_s = 2f_s[(kd - d')/(d - kd)]$ and a value determined for $f'_s$. $A'_s$ may then be obtained by using this value in the equation $A'_s = M_2/f'_s(d - d')$.

3.34) is equal to $A'_s f'_s$, and the latter is equal to $\frac{1}{2} f_c kbd$. Hence,

$$C' + C = A'_s f'_s + \tfrac{1}{2} f_c kbd = p'bdf'_s + \tfrac{1}{2} f_c kbd = bd(\tfrac{1}{2} f_c k + p'f'_s)$$

where $p$ is the ratio of tensile steel and $p'$ the ratio of compressive steel in terms of the effective cross section $bd$.

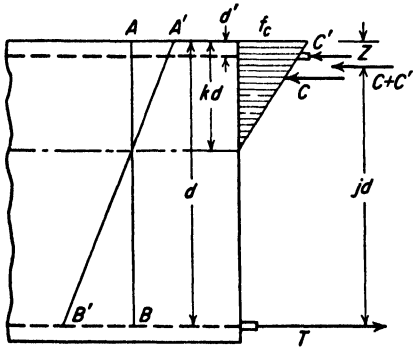



FIG. 3.34

For equilibrium, the total tension must equal the total compression; hence,

$$pf_s bd = bd(\tfrac{1}{2} f_c k + p'f'_s)$$

or

$$pf_s = \tfrac{1}{2} f_c k + p'f'_s \qquad (c)$$

From Fig. 3.34 it is seen that, since deformations vary as the distance from the neutral axis, if $AA'$ represents the deformation of the extreme compression fiber and $BB'$ the elongation of the tension steel,

$$\frac{AA'}{B\bar{B}'} = \frac{kd}{d - kd} \qquad (d)$$

If $AA'$ and $BB'$ are further assumed to be the deformations for a unit length of beam,

$$AA' = \frac{f_c}{E_c} \qquad \text{and} \qquad BB' = \frac{f_s}{E_s}$$

Hence,

$$\frac{AA'}{B\bar{B}'} = \frac{nf_c}{f_s} \qquad (e)$$

Equating $(d)$ and $(e)$ and solving for $f_c$,

$$f_c = \frac{f_s k}{n(1 - k)} \qquad (3.1a)$$

From Eq. $(c)$, Art. 3.47,

$$f'_s = \frac{2f_s[k - (d'/d)]}{1 - k} \qquad (3.31)$$

Substituting in Eq. $(c)$ the value of $f_c$ from Eq. (3.1$a$) and that of $f'_s$ from Eq. (3.31),

$$pf_s = \frac{f_s k^2}{2n(1 - k)} + \frac{2p'f_s[k - (d'/d)]}{1 - k}$$

from which

$$k = \sqrt{2n\left(p + \frac{2p'd'}{d}\right) + n^2(p + 2p')^2} - n(p + 2p') \qquad (3.32)$$

By taking the summation of moments of the compressive forces about the top of the beam, the position of the center of gravity of these forces may be determined. The resulting equation is

$$z = \frac{\frac{1}{3}kdC + d'C'}{C + C'} = \frac{\frac{1}{3}kd + d'(C'/C)}{1 + (C'/C)} \qquad (f)$$

Since $C'$, the total compressive stress in the steel, $= p'bdf'_s$ and $C$, the total compressive stress in the concrete, $= bdf_c k/2$,

$$\frac{C'}{C} = \frac{2p'f'_s}{f_c k}$$

Substituting from Eqs. (3.1*a*) and (3.31) the values of $f_c$ and $f'_s$,

$$\frac{C'}{C} = \frac{4p'f_s[k - (d'/d)]}{(1 - k)[f_s k^2/n(1 - k)]} = \frac{4p'n[k - (d'/d)]}{k^2}$$

Substituting this value of $C'/C$ in Eq. (*f*),

$$z = \frac{\frac{1}{3}k^3 d + 4p'nd'[k - (d'/d)]}{k^2 + 4p'n[k - (d'/d)]}$$

From Fig. 3 34, $jd = d - z$, or $j = 1 - (z/d)$, and therefore

$$j = \frac{k^2 - \frac{1}{3}k^3 + 4p'n[k - (d'/d)][1 - (d'/d)]}{k^2 + 4p'n[k - (d'/d)]} \qquad (3.33)$$

Where the critical stress in the compressive steel $f'_s$ is equal to the allowable stress in the tensile steel, Eq (3.31) reduces to

$$f'_s = f_s$$

and Eqs. (3.32) and (3.33) to

$$k = \sqrt{2n(p - p') + n^2(p - p')^2} - n(p - p') \qquad (3.32a)$$

and

$$j = \frac{k^2 - (k^3/3) + 2p'n(1 - k)[1 - (d'/d)]}{k^2 + 2p'n(1 - k)} \qquad (3.33a)$$

Equation (3.32*a*) is similar to Eq. (3.2). Therefore, $k$ may be obtained from Table 6, Appendix D, using $p - p'$ for $p$. Values of $j$ may be obtained from Diagram 8, Appendix D.

The resisting moment of the tension steel is found by taking moments about the center of gravity of the compressive forces from which

$$M_s = A_s f_s jd \qquad (3.34)$$

The resisting moment of the compression forces could be found by taking moments about the center of the tensile steel, but the resulting equation would be extremely cumbersome. It is simpler to use Eq.

(3.34) in conjunction with Eq. (3.1*a*), as explained in Art. 3.44, to determine the relative strength of the compression forces, as compared with the tension forces, from which relation the true resisting moment of the beam can be computed.

**3.49. Diagrams for Review of Beams Reinforced for Compression.** In order to simplify the computations for the review of beams of this type, Diagrams 4 to 8 (Appendix D) have been constructed. Diagrams 4 to 7 are based on Eqs. (3.32) and (3.33), while Diagram 8 is based on Eq. (3.33*a*). Since the values of $k$ and $j$ depend upon the relation between $pn$, $p'n$, and $d'/d$, the curves have been drawn with these quantities as variables. For intermediate values of $d'/d$, interpolation is necessary to determine the true values of $k$ and $j$.

From Eq. (3.31) when $f'_s = f_s$, $2[k - (d'/d)] = 1 - k$ and

$$k = \frac{1}{3} + \frac{2}{3}\left(\frac{d'}{d}\right)$$

For larger values of $k$, the stress in the compressive steel will control the design, for which case Diagram 8 has been drawn. Therefore, Diagrams 4 to 7 have not been carried beyond this limiting value of $k$. Where the intersection of the curve for $pn$ with the vertical for $p'n$ occurs outside the diagram, the value of $k$ should be obtained from Table 6 and the value of $j$ from Diagram 8.

**3.50. Types of Problems.** In designing a beam of limited cross section which supports a greater load than the compressive strength of the concrete permits, the rational process is to solve Eqs. (3.26) to (3.30) in order, thus determining the amounts of tensile and compressive steel required. A proper selection of the bars to be used may then be made.

Where the safe resisting moment is required, for maximum allowable values of $f_s$ and $f_c$, the values of $k$ and $j$ should first be obtained from Eqs. (3.32) and (3.33)[1] or from Diagrams 4 to 7, Appendix D. The value of $f_c$ should then be obtained from Eq. (3.1*a*) and the value of $f'_s$ from Eq. (3.31), using the allowable value of $f_s$ in these determinations.

In reviewing a beam with compressive reinforcement to determine the existing unit stresses under a given load, the value of $k$ and $j$ may be obtained from Eqs. (3.32) and (3.33) or from Diagrams 4 to 7, the value of $f_s$ from Eq. (3.34), $f_c$ from Eq. (3.1*a*), and $f'_s$ from Eq. (3.31). In detail:

1. Where both the values of $f_c$ and $f'_s$ are less than the allowable values, the resisting moment of the beam depends upon the tensile steel and can be obtained from Eq. (3.34).

2. Where the value of $f_c$ is greater than the allowable value, while the value of $f'_s$ is within the allowable limit, the resisting moment of the beam

[1] See exception under (3) below.

may be obtained by substituting the allowable value of $f_c$ in Eq. (3.1*a*) to obtain a corresponding value of $f_s$. With the value of $f_s$ thus determined, Eq. (3.34) gives the safe resisting moment of the beam.

3. Where the value of $f'_s$ is greater than the allowable value, while the value of $f_c$ is within the allowable limit, the corresponding value of $f_s$ may be determined from Eq. (3.31), using the value of $k$ from Eq. (3.32*a*) or from Table 6, Appendix D, using $(p - p')$ for $p$. Using the value of $f_s$ thus determined, Eq. (3.34), with the value of $j$ obtained from Eq. (3.33*a*) or Diagram 8, gives the safe resisting moment of the beam.

4. Where both the values of $f_c$ and $f'_s$ are greater than the allowable values, substitution of the allowable value of $f_c$ should be made in Eq. (3.1*a*) as in (2) above to obtain a corresponding value of $f_s$. Substitution of the allowable value of $f'_s$ in Eq. (3.31) as in (3) above should also be made to obtain another corresponding value of $f_s$. The smaller of these two values of $f_s$, with the appropriate value of $j$, substituted in Eq. (3.34) gives the safe resisting moment of the beam.

**3.51. Illustrative Examples.** I. A simply supported, reinforced concrete beam having a span of 20 ft-0 in. is limited in cross section to 8 by 18 in. The beam sustains a live load of 580 lb per lin ft; $f_s = 18{,}000$, $f_c = 1000$, and $n = 12$. Using $2\frac{1}{2}$ in. of insulation measured from the center of the bars, determine the area of steel required for tension ($A_s$) and for compression ($A'_s$).

$$\text{Weight of beam} = \frac{8 \times 18}{144} \times 150 = 150 \text{ lb per lin ft}$$

$$\text{Total load carried by beam} = 730 \text{ lb per lin ft}$$

$$M = \tfrac{1}{8} \times 730 \times 20^2 \times 12 = 438{,}000 \text{ in.-lb}$$

From Table 5, Appendix D, $K = 173$, $k = 0.400$, $j = 0.867$. Assuming that only one row of tensile steel will be required,

$$M_1 = 173 \times 8 \times (15.5)^2 = 333{,}000 \text{ in.-lb}$$

$$M_2 = 438{,}000 - 333{,}000 = 105{,}000 \text{ in.-lb}$$

$$A_{s_1} = \frac{333{,}000}{18{,}000 \times 0.867 \times 15.5} = 1.38 \text{ sq in.}$$

$$A_{s_2} = \frac{105{,}000}{18{,}000(15.5 - 2.5)} = 0.45 \text{ sq in.}$$

$$A_s = 1.38 + 0.15 = 1.83 \text{ sq in.}$$

$$A'_s = \frac{0.45}{2} \times \frac{1 - 0.400}{0.400 - (2.5/15.5)} = 0.57 \text{ sq in.}$$

As this area of steel is greater than $A_{s_2}$, the stress in the compressive steel will not exceed that in the tensile steel. Three No. 7 bars for tension and two No. 5 bars for compression are selected, the three bars being placed in one row as assumed. The areas furnished are 1.80 sq in. and

0.61 sq in., respectively. Two No. 5 bars rather than one No. 7 bar are selected for the compressive reinforcement in order to provide the most satisfactory arrangement for the anchorage of the stirrups or ties which are required.

II. A simply supported reinforced concrete beam having a span of 25 ft-0 in. is limited in cross section to 12 by 36 in. The beam sustains a live load of 2900 lb per lin ft; $f_s = 18{,}000$, $f_c = 1{,}125$, and $n = 12$. Using $2\frac{1}{2}$ in. of insulation measured from the center of the lower row bars, determine the area of steel required for tension and compression.

$$\text{Weight of beam} = \frac{12 \times 36}{144} \times 150 = 450 \text{ lb per lin ft}$$

$$\text{Total load carried by beam} = 3350 \text{ lb per lin ft}$$

$$M = \tfrac{1}{8} \times 3350 \times 25^2 \times 12 = 3{,}141{,}000 \text{ in.-lb}$$

From Table 5, $K = 207$, $k = 0.428$, $j = 0.857$. Assuming that two rows of bars will be required for the tensile steel,

$$M_1 = 207 \times 12 \times (32.5)^2 = 2{,}624{,}000 \text{ in.-lb}$$

$$M_2 = 3{,}141{,}000 - 2{,}624{,}000 = 517{,}000 \text{ in.-lb}$$

$$A_{s_1} = 2{,}624{,}000 \div (18{,}000 \times 0.857 \times 32.5) = 5.23 \text{ sq in.}$$

$$A_{s_2} = \frac{517{,}000}{18{,}000(32.5 - 2.5)} = 0.96 \text{ sq in.}$$

$$A_s = 5.23 + 0.96 = 6.19 \text{ sq in.}$$

$$A'_s = \frac{0.96}{2} \times \frac{1 - 0.428}{0.428 - (2.5/32.5)} = 0.78 \text{ sq in.}$$

However, if only 0.78 sq in. of steel were furnished for compression, the stress in this steel from Eq. (*c*), Art. 3.47, would be

$$f'_s = 2 \times 18{,}000 \left[\frac{0.428 - (2.5/32.5)}{1 - 0.428}\right] = 22{,}200 \text{ psi}$$

which is greater than the allowable. Therefore 0.96 sq in. of compression steel must be supplied.

Eight No. 8 bars are selected for tension and two No. 7 bars for compression, furnishing 6.28 sq in. and 1.20 sq in., respectively. The eight bars will be placed in two rows 2 in. center to center, with the center of the lower row $2\frac{1}{2}$ in. from the bottom surface of the beam. The two bars for compressive reinforcement will be placed in one row with their centers $2\frac{1}{2}$ in. from the top surface of the beam. This arrangement will satisfy the requirements of Art. 3.7 and allow for the anchoring of stirrups or ties above the compressive reinforcement and below the tensile reinforcement.

III. A reinforced concrete beam has a cross section 12 by 30 in. and is reinforced as follows: for tension, eight No. 7 bars in two rows, 2 in.

center to center, the center of the lower row being 2½ in. above the lower surface of the beam; and for compression, two No. 7 bars in one row, the center of which is 2½ in. below the upper surface of the beam; $f_s = 18{,}000$, $f_c = 800$, and $n = 15$. What is the safe resisting moment of the beam?

$$\frac{d'}{d} = \frac{2.5}{26.5} = 0.094$$

$$pn = 4.81 \times 15 \div 12 \times 26.5 = 0.227$$

$$p'n = \tfrac{1}{4}pn = 0.057$$

From Diagrams 4 and 5, $j = 0.872$ and $k = 0.429$. From Eq. (3.1*a*) for the allowable value of $f_c$,

$$f_s = \frac{800 \times 15(1 - 0.429)}{0.429} = 16{,}000 \text{ psi}$$

From Eq. (3.31) for the allowable value of $f'_s$,

$$f_s = \frac{18{,}000}{2}\left(\frac{1 - 0.429}{0.429 - 0.094}\right) = 15{,}300 \text{ psi}$$

This shows that the resisting moment is dependent upon the stress in the compression steel. Hence, the value of $k$ [Eq. (3.32*a*)] obtained from Table 6 for $p = 0.0151 - 0.0038 = 0.0113$ is 0.435, and $j$, from Diagram 8, is 0.867.

Substituting the allowable value of $f'_s$ in Eq. (3.31) as before,

$$f_s = \frac{18{,}000}{2}\left(\frac{1 - 0.435}{0.435 - 0.094}\right) = 14{,}900 \text{ psi}$$

and the safe resisting moment of the beam is

$$M = 4.31 \times 14{,}900 \times 0.867 \times 26.5 = 1{,}647{,}000 \text{ in.-lb}$$

## Problems

1. Assume that the simply supported beam in Fig. 3.14 is limited in size to an over-all cross section of 10 by 21 in. What reinforcement must be used, if $f_s = 18{,}000$ psi, $f_c = 1000$ psi, and $n = 12$? Assume that the tension steel will be placed in one row, the compression steel will be placed in one row, and the distance from the center of each row to the nearest surface of the beam is 2½ in.

2. A simply supported rectangular beam with a span of 20 ft-0 in. has an over-all cross section of 8 by 23 in. It is reinforced for compression with two No. 6 bars, in one row, the center of which is 2½ in. from the upper surface of the beam, and for tension with four No. 7 bars in two rows, 2 in. center to center, the center of the lower row being 2½ in. above the lower surface of the beam. What is the resisting moment of the beam, if $f_c = 1000$ psi, $f_s = 20{,}000$ psi, and $n = 12$?

3. If the beam in Prob. 2 supports a single concentrated load at the mid-span, what is the maximum safe value of this load?

4. A simply supported beam 10 by 22½ in. in cross section has a span of 19 ft-0 in. and is reinforced for tension with six No. 6 bars in two rows 2 in. center to center, the center of the lower row being 2½ in. above the lower surface of the beam. The beam

also has compression reinforcement consisting of two No. 5 bars with their centers 2 in. from the upper surface of the beam. If $f_s = 18{,}000$ psi, $f_c = 800$ psi, and $n = 15$. what uniform live load per foot can the beam sustain?

**3.52. Continuous T Beams.** In a continuous T beam the flange is in compression only in the region of positive moment, *i.e.*, in the center portions. The negative moments at the supports cause the bottom fibers to be in compression. At these points, therefore, the action of the member is that of a rectangular beam of width $b'$, that is, the width of the compression portion of the concrete. For this reason a much smaller concrete compression area is available at the supports for negative moments than is present in the center for positive moments. In addition, the negative support moments are generally considerably larger than the positive span moments (see Art. 3.6). Finally, the external shear force is also greatest at the support. It is seen, therefore, that the critical sections of such beams are those at the support, and design must begin there. First, the web area necessary to resist the shear is selected. Next, the amount of compression and tension reinforcement which is required to resist the maximum negative moment is determined. At the point of maximum positive moment, only tension reinforcement is necessary and its area is determined as explained in Art. 3.44. Since, in many cases, the same bars are used to provide reinforcement at the supports and in the center of the span, they must be satisfactorily arranged to satisfy both requirements with economy in mind.

The review of an existing design of a continuous T beam can be made by the same methods previously described for the review of T beams and beams reinforced for compression.

**3.53. Illustrative Example.** A continuous T beam of three spans is supported by and constructed monolithically with the columns and girders as shown in Fig. 3.35. The 4-in. slab which forms the flange of the T beam is also constructed integrally with the beams. The slab supports a live load of 100 psf. The beams are spaced 12 ft-0 in. center to center. Design the beam using a 2500-psi concrete, intermediate-grade steel, and with a maximum allowable unit shearing stress of $0.06f'_c$.

Total load on slab = 100 + 50 = 150 psf
Load from slab to beam = 150 × 12 = 1800 lb per lin ft
Assumed weight of stem of beam = 180 lb per lin ft
Total load on beam = 1980 lb per lin ft

At first interior support, the shear is

$$1.15\frac{wl'}{2} = \frac{1.15 \times 1980 \times 20}{2} = 22{,}800 \text{ lb}$$

$$b'd = \frac{22{,}800}{150 \times \frac{7}{8}} = 174 \text{ sq in.}$$

with $b' = 10$ in., $d = 17.5$ in. Assuming that two rows of bars will be required both at the bottom and top of the beam, the total depth of the beam is 21 in.,[1] and the section below the slab is 17 in. A 10- by 17-in. section weighs 177 lb per lin ft, which agrees with the assumed weight. Although the shear at other supports is less than that of the first interior support, the section of the beam will be kept constant for all spans in order to simplify the construction.

At the left end of *B1* it will be assumed that the ratio of column stiffness to beam stiffness does not exceed 8 and that, therefore, the negative bending moment at that point is $\frac{1}{16}wl'^2$ (see Art. 3.6).

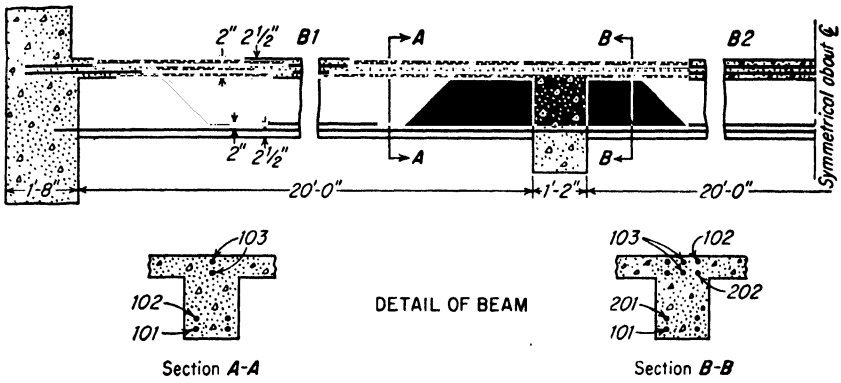



Fig. 3.35

The several bending moments and areas of tensile reinforcement required at centers of spans and edges of support (Fig. 3.35) are computed as follows:

*B1*:

Edge of exterior support: $M = \frac{1}{16} \times 1980 \times 20^2 \times 12$
$= 594{,}000$ in.-lb
$A_s = 1.96$ sq in.

Center of span: $M = \frac{1}{14} \times 1980 \times 20^2 \times 12 = 679{,}000$ in.-lb
$A_s = 2.24$ sq in.

Edge of first interior support: $M = \frac{1}{10} \times 1980 \times 20^2 \times 12$
$= 950{,}000$ in.-lb
$A_s = 3.14$ sq in.

*B2*:

Center of span: $M = \frac{1}{16} \times 1980 \times 20^2 \times 12 = 594{,}000$ in.-lb
$A_s = 1.96$ sq in.

[1] In this example no consideration has been given to the relative arrangement of the reinforcement in beam and girder. For the desired arrangement to provide ease of placement, see Art. 8.29.

At the edges of all supports compressive reinforcement will usually be required in the bottom of the beam. This may necessitate some modification of the tensile reinforcement at the supports, but any such modification is usually not made in actual design, since the available commercial sizes of bars generally furnish excess steel areas over and above the theoretical requirements.

The resisting moment of the rectangular beam without compressive steel is

$$M_1 = 196 \times 10 \times (17.5)^2 = 600{,}000 \text{ in.-lb}$$

and the area of tensile steel ($A_{s_1}$) required for this moment is 1.98 sq in. The additional tensile steel ($A_{s_2}$) at the edges of the supports is

*B*1:

Exterior support, none since 600,000 in.-lb is greater than 594,000 in.-lb

Interior support: $A_{s_2} = \dfrac{350{,}000}{20{,}000(17.5 - 3.5)} = 1.25$ sq in. Total 3.23 sq in.[1]

The corresponding area of compressive steel required is

$$\frac{1.25}{2} \times \frac{1 - 0.103}{0.403 - (3.5/17.5)} = 1.84 \text{ sq in.}$$

The reinforcement scheduled in Fig. 3.38 and further shown in Figs. 3.35 and 3.36 supplies tensile steel in the following amounts

*B*1:

Exterior support: 2.40 sq in.
Center of span: 2.40 sq in.
Interior support: 3.28 sq in.

*B*2:

Center of span: 2.08 sq in.

and compressive reinforcement at the interior support of 2.40 sq in. While no compressive reinforcement is required at the exterior support, the two straight No. 7 bars of the positive reinforcement are carried through into the support a distance of 6 in., in order to satisfy the requirements of the ACI Code (see Art. 3.35).

The two No. 7 bent bars in *B*1 supply 50 per cent of the positive-moment reinforcement. From Diagram 2, Appendix D, these bars may be bent upward a distance of $0.23 \times 20 = 4.6$ ft from the face of the

[1] This area is in close agreement with that previously determined from the total bending moment.

support. Allowing for the 12 bar diameters required by the ACI Code, the point of bending is 3.7 ft from the support. At the exterior support, the two No. 7 bent bars provide 50 per cent of the negative-moment reinforcement and may be bent downward at a distance of

$$0.07 \times 20 + 0.9 = 2.3 \text{ ft}$$

from the support. At the interior support, the two No. 7 bent bars supply 37 per cent of the negative-moment reinforcement and may be bent downward $0.08 \times 20 + 0.9 = 2.5$ ft from the support. The two No. 6 bent bars from *B*2 provide 27 per cent of the negative-moment reinforcement, or with the two No. 7 bent bars furnish 64 per cent of the total reinforcement, and may be discontinued $0.15 \times 20 + 0.8 = 3.8$ ft from the

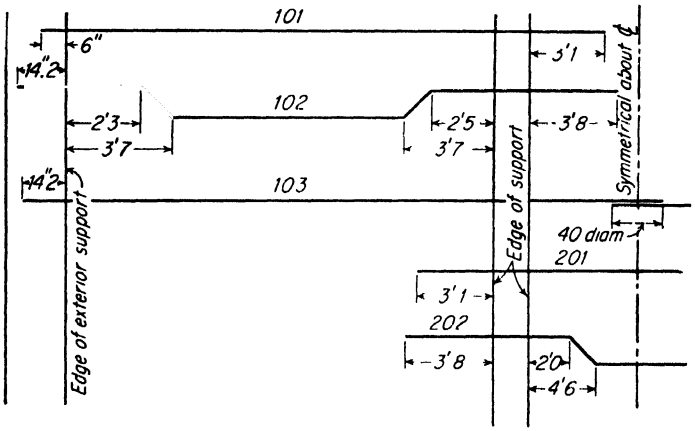



Fig. 3.36

support. These No. 6 bars may for the same reason be bent downward in *B*2 at $1.2 + 0.8 = 2.0$ ft from the support, and the two No. 7 bent bars from *B*1 may be discontinued in *B*2, 3.8 ft from the support. The two No. 6 bent bars in *B*2 may be bent upward $0.27 \times 20 - 0.8 = 4.6$ ft from the support. The compressive reinforcement at the interior support consists of four No. 7 bars. From Diagram 2 each pair must extend across the support a distance of $0.11 \times 20 + 0.9 = 3.1$ ft. The theoretical points of bend or cutoff are shown in Fig. 3.36, the marks on the several bars referring to the schedule of Fig. 3.38, where the actual points of bend or cutoff are shown.

*Bond.* The actual stress in the tensile reinforcement in the top of *B*1 at the edge of the exterior support is $1.96/2.40 \times 20{,}000 = 16{,}300$ psi, and the force to be developed in a No. 7 bar is $0.60 \times 16{,}300 = 9780$ lb, which requires a length of bar of $9780/(250 \times 2.75) = 14.2$ in., while 15 in. is furnished by the reinforcement as scheduled. The necessary

development by bond of the bars over the interior support is accomplished by their extended lengths, as scheduled.

*Web Reinforcement.* For beams of this type with longitudinal reinforcement bent up at one point only, the saving in the number of stirrups by considering the bent portion of these bars as part of the web reinforcement is negligible. In this example stirrups will be designed to provide all the diagonal tensile reinforcement required.

At the interior support the shear is 22,800 lb. The amount of shear to be taken by the concrete is $10 \times 17.5 \times \frac{7}{8} \times 50 = 7900$ lb, leaving

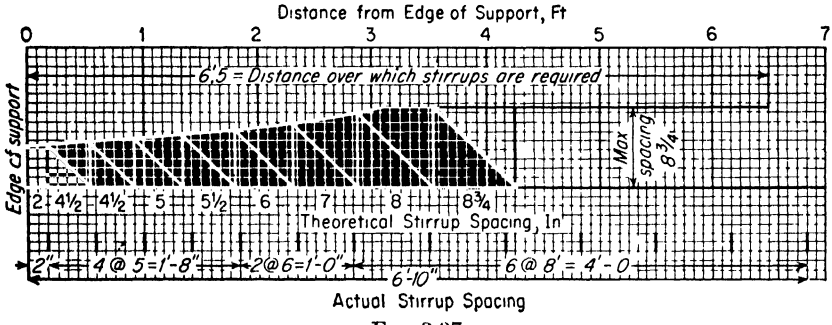



FIG. 3.37

14,900 lb to be taken by the web reinforcement. Assuming the shear to decrease uniformly at the rate of 2300 lb per ft, the required spacing of No. 3 U stirrups is:

| | |
|---|---|
| At the support, $\frac{0.22 \times 20{,}000 \times \frac{7}{8} \times 17.5}{14{,}900}$. | 4.52 in. |
| 1 ft from support | 5.34 in. |
| 2 ft from support | 6.51 in. |
| 3 ft from support | 8.12 in. |
| 4 ft from support | 11.80 in. |

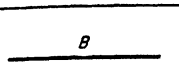
Web reinforcement is required for a distance of $14{,}900 \div 2300 = 6.5$ ft from the support, while the maximum spacing of stirrups allowed by the ACI Code (see Fig. 3.23) is $\frac{1}{2}d = 8\frac{3}{4}$ in.

The curve of Fig. 3.37 is plotted from the stirrup spacings computed above and the required theoretical spacings are determined by the construction there shown. In order to simplify the construction, fewer different spacings are actually used, as shown in Fig. 3.37. A satisfactory spacing for stirrups can usually be obtained after a little experience from the spacings computed at 1- or 2-ft intervals without the construction of the curve of Fig. 3.37.

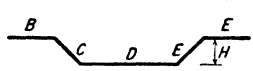
Where the live load is large in comparison with the dead load, stirrup spacings should be computed with the live load covering only that por-

tion of the span to the right of the point in question. In the present example the shear computed in this manner at 6.5 ft from the support is 8300 lb, which is only slightly greater than the 7900 lb taken by the con-

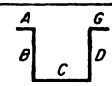
Reinforcing Schedule




Type 1




Type 3




Type S1

| No | Mark | Size W | Size D | Placing Notes | Reinforcing No | Reinforcing Designation | Reinforcing Length | Reinforcing Mark | All dimensions are out to out Sketch | Type | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | B1 | 10 | 21 | Bottom | 2 | #7 | 33 10 | 101 | — | 1 | | | | | | | | |
| | | | | Upper layer | 2 | #7 | 27 0 | 102 | ⌒ | 3 | | 36 | 18 | 18 | 18 | 76 | | 12 |
| | | | | Top | 2 | #7 | 33 10 | 103 | — | 1 | | 33 10 | | | | | | |
| | | | | Stirrups 1 at 2, 4 at 5, 2 at 6, 6 at 8 | 26 | | | | ⊔ | S1 | 5 | 16 | 7 | 16 | | | 5 | |
| 1 | B2 | 10 | 21 | Bottom | 2 | #7 | 28 6 | 201 | — | 1 | | | | | | | | |
| | | | | Upper layer | 2 | #6 | 31 8 | 202 | ⌒ | | | 70 | 15 | 120 | 15 | 70 | | 10 |
| | | | | Top | 2 | #7 | | See | 103 | B1 | | | | | | | | |
| | | | | Stirrups 1 at 2, 4 at 5, 2 at 6, 6 at 8 | 26 | | | | ⊔ | S1 | 5 | 16 | 7 | 16 | | | 5 | |

Note Stirrup spacing bars, top of beam not scheduled

FIG. 3.38

crete. The additional stirrup provided beyond this point is sufficient to provide for this unusual condition.

Figures 3.35 and 3.38 give all the details necessary for the construction of the beams of this design.

# SECTION 4

# COLUMNS

**4.1. Types of Columns.** Concrete compression members whose unsupported length is more than four times the least dimension of the cross section are classified as columns. Such members should not be built without reinforcement of some type. In modern construction four types of reinforced concrete columns are used, namely:

1. Columns reinforced with longitudinal steel and closely spaced spirals.
2. Columns reinforced with longitudinal steel and lateral ties.
3. Composite columns, in which a structural-steel or cast-iron column is thoroughly encased in a concrete core of Type 2.
4. Combination columns, in which a structural-steel column is wrapped with wire and encased in at least $2\frac{1}{2}$ in. of concrete over all metal except rivet heads.

Types 1 and 2 are more generally used, Types 3 and 4 being economical with heavy construction loads or extremely heavy permanent loads.

Pipe columns, in which a steel pipe is filled with concrete, are sometimes used.

**4.2. Columns with Spiral and Longitudinal Reinforcement.** Whenever a material is subjected to compression in one direction, there will be an expansion in the direction perpendicular to the compression axis. Where this expansion is resisted, lateral compressive stresses are developed, which tend to neutralize the effect of the longitudinal compressive stress and thus increase resistance against failure. This is the principle involved in the use of spiral or hooped reinforcement (see Fig. 4.1). Within the limit of elasticity the hooped reinforcement is much less effective than longitudinal reinforcement. Such reinforcement, however, raises the ultimate strength of the column, because the hooping delays ultimate failure of the concrete. The concrete continues to compress and to expand laterally, thus increasing the tension in the bands, while final failure occurs upon the excessive stretching or breaking of the hooping. Thus a somewhat higher working stress may be employed on the concrete contained within such hooping that on a concrete not so con-

fined. Tests show that about 1 per cent of closely spaced spiral hooping increases the resistance to ultimate failure sufficiently to allow a reasonable increase in the working stress in the concrete.

As long as the bond between the steel and the concrete is effective, the two materials will deform equally, and the intensities of the stresses will be proportional to their moduli of elasticity. That is, since the deformation $\lambda_c$ of the concrete must be equal to the deformation $\lambda_s$ of the steel,

$$\lambda_c = \frac{f_c}{E_c} = \lambda_s = \frac{f_s}{E_s}$$

Therefore

$$f_s = nf_c$$

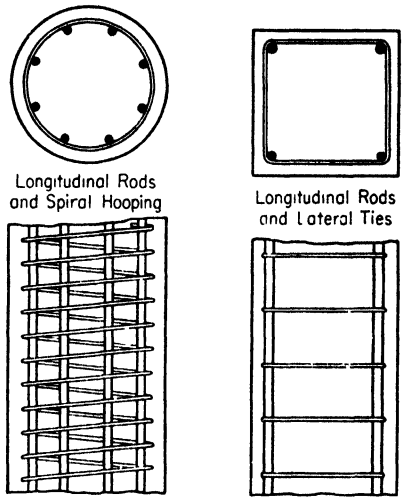



FIG. 4.1

Let $A_c$ = net area of concrete in this section, measured to the outside diameter of the spiral
$A_s$ = area of longitudinal steel in this section
$A_g$ = over-all, or gross, area of concrete section
$p_g$ = steel ratio $A_s/A_g$
$f_s$ = unit compressive stress in steel
$f_c$ = unit compressive stress in concrete
$P$ = total strength of reinforced column

Then

$$\begin{aligned} P = f_c A_g + f_s A_s &= f_c(A_g - p_g A_g) + f_c n p_g A_g \\ &= f_c A_g[1 + (n - 1)p_g] \\ &= f_c[A_g + (n - 1)A_s] \end{aligned}$$

The analysis just given, which showed that $f_s = nf_c$, was based on the assumption that both concrete and steel are elastic materials. To understand the reason why test results do not agree with this analysis, it is necessary to examine the actual behavior of these materials under load. In Fig. 4.2 are given typical stress-strain diagrams for steel and concrete. As a concrete cylinder or a short piece of reinforcing rod is compressed in a testing machine, it shortens as the load is increased. The diagrams show the unit shortening, or strain, $\lambda$ in inch per inch plotted against the unit compression stress.

While the steel is seen to be elastic, *i.e.*, to have a straight stress-strain graph, almost to its yield point (40,000 psi minimum for intermediate-

grade steel), this is seen to be far from true for concrete. For the standard rate of loading employed in cylinder tests, the graph is seen to be reasonably straight up to about 50 per cent of $f'_c$; from then on it becomes increasingly curved. The cylinder does not actually fracture when $f'_c$ is reached, but it continues to resist decreasing stress until it finally fractures at unit strains of the order of 0.004 to 0.008. If the load is applied

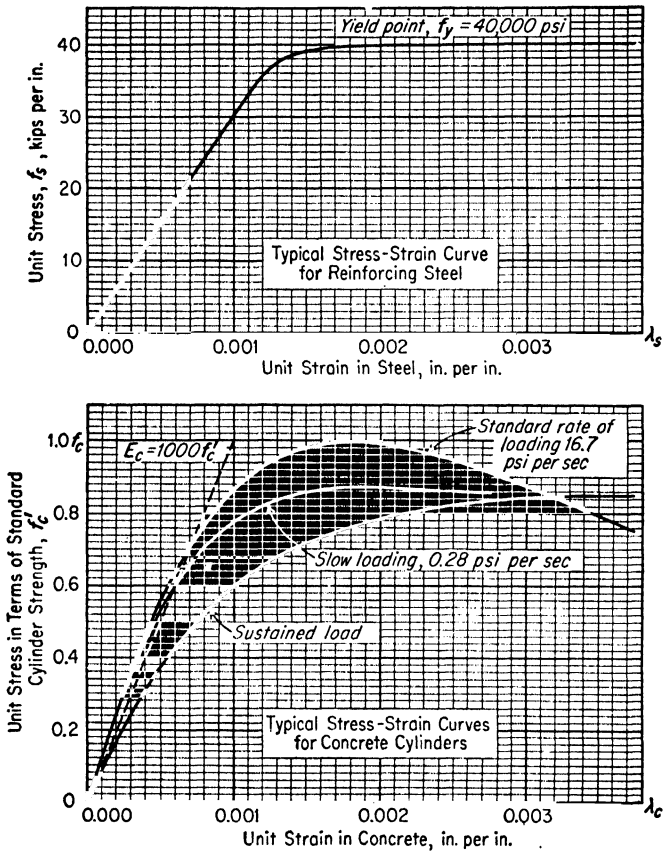



FIG. 4.2. The concrete curves of this figure are taken from C. S. Whitney, "Application of Plastic Theory of the Design of Concrete Structures," *J. Boston Soc. Civil Engrs.*, vol. 25, no. 1, 1948.

about one-sixtieth as fast, the concrete behaves differently. The strains are seen to be considerably larger, and the concrete, after considerable deformation under constant load (flow), fractures at about 87 per cent of $f'_c$. The rate of loading for which this curve is plotted is about that employed in column testing. Finally, if the load in the test is kept constant over considerable lengths of time, the curve designated "sustained

load" is obtained. This curve illustrates the property of concrete known as flow, creep, or plasticity. If, for example, a stress equal to 50 per cent of $f'_c$ is applied at the standard rate, the strain is equal to 0.00045. If, now, the load is kept constant, the concrete keeps compressing at a decreasing rate. When it finally stops compressing, the strain is seen to be 0.00075, almost twice the initial value. At higher stresses this effect is even more pronounced. Under such sustained loading the concrete is seen to resist only about 85 per cent of $f'_c$. If a unit stress of this magnitude is applied, the graph shows that the concrete will not fracture immediately, but, after flowing a very considerable amount, will eventually give way.

In contrast to this actual performance in test, the behavior assumed in the previously given elastic analysis is represented by the broken line.

The implications of this situation on the actual strength of columns are easily seen from the data assembled in the following table, which can be read directly off Fig. 1.2.

For a column made of concrete with $f'_c = 3000$ psi and reinforced with intermediate-grade steel, the table shows for three increasing values of the applied compression stress (*a*) the value of the steel stress obtained from elastic analysis, that is, $f_s = nf_c$; (*b*) the strain $\lambda$ in the concrete in a slow column test, and the corresponding stress in the steel at that strain; and (*c*) the strain $\lambda$ in the concrete and the corresponding stress in the steel which would be obtained in a building, *i.e.*, under sustained loading. It will be remembered that the unit strains $\lambda$ in concrete and steel are equal in such columns, so that corresponding stress values are located on the same vertical in both graphs of Fig. 4.2.

| Concrete stress | $f_s = nf_c$ | Slow loading | | Sustained load | |
|---|---|---|---|---|---|
| | | $\lambda$ | $f_s$ | $\lambda$ | $f_s$ |
| 1,500 | 15,000 | 0.00045 | 13,500 | 0.00075 | 22,500 |
| 2,400 | 24,000 | 0.0011 | 33,000 | 0.0022 | 40,000 |
| 2,550 ($=0.85f'_c$) | 25,500 | 0.0013 | 38,500 | 0.0035 | 40,000 |

It is evident that at a concrete stress equal to half the ultimate, *i.e.*, 1500 psi, the theoretical steel stress $f_s = nf_c$ lies between the actual values for slow loading and for sustained loads. At 80 per cent of $f'_c$, or 2400 psi, however, the actual steel stresses for both types of loading are seen to exceed very considerably the value indicated by elastic analysis. Finally, when the carrying capacity of the concrete, $0.85f'_c = 2550$ psi, is reached, the actual stresses in the steel are, in this case, 50 to 60 per cent higher than indicated by elastic theory.

In view of the fact that concrete is not an elastic material and is subject to creep or flow, it follows that the stresses in compression steel just prior to failure are considerably higher than if the concrete were elastic. It is this fact, verified by many tests, which is the reason for allowing increased stresses in the compression reinforcement of beams (see Art. 3.46).

For columns, the actual strength is seen to be simply the sum of the ultimate strength of the concrete and of the yield strength of the reinforcement. In the above numerical example, when the concrete stress reaches 2550 psi and the concrete finally fails, the total load on the column is $P_{ult} = 2550A_g + 40{,}000A_s$. In general, then, the ultimate strength of a reinforced concrete column under sustained load is

$$P_{ult} = 0.85f'_cA_g + f_yA_s$$

In view of this well-established behavior, present-day column design is no longer based on the proposition of elasticity, but rather on the above equation for $P_{ult}$. The permissible design load in the ACI Code is derived from that equation by dividing the two terms on the right by factors of safety. A larger factor is applied to the concrete strength than to the steel strength, since the quality of concrete in narrow forms filled with reinforcing steel is not as easily controlled as that of the steel itself. For this reason the ACI Code specifies for spiral reinforced columns,

$$P = 0.225f'_cA_g + f_sA_s \tag{4.1}$$

where $f'_c$ is the ultimate strength of the concrete in pounds per square inch and $f_s$ the working stress in the longitudinal reinforcement (16,000 psi for intermediate-grade and 20,000 psi for hard-grade steel).

The longitudinal reinforcement should consist of at least six bars no smaller than No. 5, and its effective cross-sectional area should be not less than 1 per cent or more than 8 per cent of the gross area of the column section. The ratio of spiral reinforcement $p'$ must be not less than

$$p' = 0.45(R - 1)\frac{f'_c}{f'_s}$$

where $p'$ = ratio of volume of spiral reinforcement to volume of spiral core (out to out of spirals)

$R$ = ratio of gross area of column to core area

$f'_s$ = useful limit stress of spiral reinforcement, 40,000 psi for hot-rolled bars of intermediate grade, 50,000 psi for hard grade, and 60,000 psi for cold-drawn wire

The center-to-center spacing of spirals should not exceed one-sixth the core diameter. The clear spacing between spirals should not exceed

3 in. or be less than $1\frac{3}{8}$ in., or $1\frac{1}{2}$ times the maximum size of the coarse aggregate. The spirals should be continuous from the top of the slab or footing to the level of the lowest reinforcement in the slab or beam above. They should be held firmly in place and true to line by at least two vertical spacers for spirals 20 in. or less in diameter, three vertical spacers for spirals from 20 to 30 in. in diameter, and four vertical spacers for larger columns. The spiral bars or wire should not be less than $\frac{1}{4}$ in. in diameter, and the thickness of concrete outside the spiral reinforcement should not be less than $1\frac{1}{2}$ in.

Where the reinforcing bars are spliced by lapping, the length of the lap should be 20 bar diameters for intermediate- or hard-grade steel, provided that the concrete has a strength of 3000 psi or more. With concrete of lesser strength, the lengths given above should be increased one-third. The above values are for deformed bars; lap lengths of plain bars should be increased by 100 per cent. Where changes in the cross section of a column occur, the offset of the bars should be made where there is lateral support, such as a column capital, floor slab, or metal ties or spirals. The slope of the inclined portion of the bars should not exceed 1 in 6, and the bars above and below should be parallel with the axis of the column.

**4.3. Columns with Longitudinal Reinforcement and Lateral Ties.** Tests show that columns without spirals develop lower strengths in both the concrete and the steel. In such columns the widely spaced, relatively weak ties do not afford the concrete the lateral support which a close-spaced spiral would provide, which is the reason for the lower strength of concrete in tied columns. In addition, the steel in tied columns is laterally supported only at relatively large intervals at which ties are placed. For this reason it is possible that at high stress the part of the reinforcing bars between two consecutive ties will buckle outward by bursting the protective concrete layer. This behavior has been observed in many tests and is the reason why the reinforcing steel in tied columns shows lower strength than in spiral reinforced columns. Accordingly, the ACI Code specifies the safe axial load for this type of column as 0.8 of that for a spirally reinforced column, or

$$P = 0.18f'_c A_g + 0.8A_s f_s$$

The longitudinal reinforcement should consist of at least four bars no smaller than No. 5, and its effective cross-sectional area should be not less than 1 per cent or more than 4 per cent of the gross area of the column section. There should be a clear distance between the longitudinal bars and the face of the column of $1\frac{1}{2}$ in. plus the diameter of the tie.

The longitudinal bars are held in alignment during construction by lateral ties, as illustrated in Figs. 4.1 and 4.3. These ties should be made

*of wire at least ¼ in. in diameter,*[1] *and the vertical distance between ties or sets of ties should not exceed 16 bar diameters, 48 tie diameters, or the*

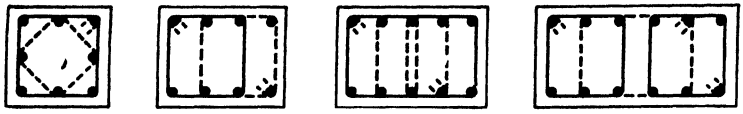


FIG. 4.3

least dimension of the column. When the number of bars in a column exceeds four, the ties should be so detailed as to prevent the outward bending of every bar at each tie interval. The methods of accomplishing this are illustrated in Fig. 1.2.

Splices in vertical reinforcement should be made in the same manner as that described in the previous article.

**4.4. Composite, Combination, and Pipe Columns.** Though columns of these types are not strictly reinforced concrete columns, the addition of concrete does, under certain conditions, add to the strength of the column.

A *composite column*, consisting of a structural-steel or cast-iron column, whose cross-sectional area does not exceed 20 per cent of the gross area of the column thoroughly encased in concrete, reinforced with both longitudinal and spiral reinforcement, may sustain a load

$$P = 0.225A_c f'_c + A_s f_s + f_r A_r$$

where $A_c$ = net area of concrete section = $A_g - A_s - A_r$
$A_s$ = cross-sectional area of longitudinal-bar reinforcement
$A_r$ = cross-sectional area of steel or cast-iron core
$f_r$ = allowable unit stress in metal core: 16,000 psi for a steel core; 10,000 psi for a cast-iron core

A *combination column* consisting of a structural-steel column encased in concrete at least 2½ in. in thickness over all metal (except rivet heads) and reinforced by welded wire mesh wrapped completely around the steel column may sustain a load

$$P = A_r f'_r \left(1 + \frac{A_g}{100A_r}\right)$$

where $f'_r$ is the allowable unit stress for an unencased steel column and the remaining notation is the same as for a composite column. A *steel pipe*

[1] There is no rational method of determining the size of wire that should be used for a lateral tie. A safe rule to follow is to use wire of such diameter that the area of its section is not less than 2 per cent of the section of the longitudinal reinforcement held in place by the tie.

*filled with concrete* may sustain a load

$$P = 0.225f'_c A_c + f'_r A_r$$

where $f'_r$ = average unit stress in metal core

$$= \left(18{,}000 - 70\frac{h}{K}\right)F$$

$h$ = unsupported length of column

$K$ = least radius of gyration of metal-core section

$F = \dfrac{\text{yield point of pipe}}{45{,}000}$

**4.5. Unsupported Length and Limiting Dimensions.** The strength of compression members of any material depends not only on the ultimate compressive strength of the material proper but also on the dimensions of the members. Members which are very slender that is, long in relation to their least cross-sectional dimension, fail by buckling rather than by direct crushing of the material. Their strength, therefore, decreases with increasing slenderness ratio $h/r$ ($r$ = radius of gyration) or $h/d$ ($d$ = least cross-sectional dimension). It is customary in specifications to establish a limiting ratio for $h/d$ above which this loss of strength becomes pronounced and must be considered in design. Columns whose $h/d$ ratio is smaller than that limit are known as short columns. On the basis of extensive tests, the ACI Code limits a short column to one whose unsupported length $h$ is not greater than 10 times its least lateral dimension. Columns with $h/d$ larger than 10 are known as long columns. Their strength is reduced by the possibility of buckling, the more so, the larger $h/d$ becomes. For long columns, the ACI Code gives the working load on columns whose length is greater than 10 times the least dimension as

$$P' = P\left(1.3 - 0.03\frac{h}{d}\right)$$

where $P$ is the total safe load on a column of the same section where the $h/t$ (or $d$) ratio is less than 10 and where $t$ or $d$ is the least dimension of the column and $h$ is the unsupported length of the column.

This same code limits the minimum diameter to 12 in. for principal members or, in the case of rectangular columns, a minimum thickness of 10 in. and a minimum gross area of 120 sq in. The difficulty of making uniform deposition of the concrete in a form of smaller dimensions is obvious, especially with several longitudinal bars with their spirals or ties in the form. In addition, a smaller column has very little reserve strength to withstand possible shocks not allowed for in the design, and if at any time damaged by fire, the loss in effective section is relatively large.

The unsupported length $h$ of a column is the distance between those points at either end where lateral support is present in at least two directions, making an angle of 90° or nearly 90° with one another. Therefore, it follows that the unsupported length is:

1. In flat-slab construction, the clear distance between the floor and the underside of the capital.

2. In beam-and-slab construction, the clear distance between the floor and the underside of the shallowest beam framing into the column.

3. In floor construction with beams in one direction only, the clear distance between floor slabs.

4. In cases where the columns are supported between floors by struts or beams, the unsupported length may be considered decreased, provided these struts or beams meet the column at approximately the same elevation and make horizontal angles of approximately 90° with one another.

5. When haunches are used on beams or struts, the unsupported length may be considered to be reduced by the depth of the haunch, provided the haunch is as wide as the beam and at least half the width of the column.

**4.6. Flexural Stresses in Columns.** The previous articles have dealt with columns subject to direct axial load only. While there are many cases where this is the only type of load sustained by the column, there are many more cases where the maximum stress developed in the column is a combination of axial stress and bending (see Sec. 5).

Bending moments are produced in columns (*a*) by reactions from eccentrically placed beams; (*b*) by the loads on brackets or cantilevers; (*c*) by the eccentricity of the columns themselves, a condition which often occurs in the wall columns of a building where the sections of the columns are changed at some floor levels, while the exterior faces of the columns are kept in line throughout the height of the structure; (*d*) by the application of a direct horizontal force or of a force having a horizontal component; (*e*) by the transfer from slabs or girders built monolithic with the columns of unbalanced moments due to the loads on the slabs or girders.

With conditions such as are described in (*a*), (*b*), and (*c*), the amount of moment produced in the column is easily determined, for the amount of load and the eccentricity of its center of application are known. A condition such as that described in (*d*) is caused by the wind pressure on the walls of a building, but, on account of the massiveness and rigidity of the structure, it is not usual to calculate the wind stresses in the frame of any but high and narrow buildings of reinforced concrete. A moment caused by the direct application of any other type of horizontal force is not common, but in such cases the moment is usually directly determinate. With the conditions described in (*e*), the column is a component part of a rigid frame made up of columns and slabs or columns and

girders. The distribution of moments in rigid frames will be considered in Secs. 5 and 6.

**4.7. Column Tables.** Tables 7 and 8, Appendix D, give the safe concentric loads on the concrete and on the longitudinal bars of reinforced concrete columns, based on the ACI Code. Values are given for the minimum amount of longitudinal steel. Loads for greater percentages of reinforcement are proportioned.

Table 9 gives the area of section, weight per foot, and moment of inertia of circular and octagonal sections and these same functions for rectangular sections 1 in. in width for use with rectangular columns. The values of the moment of inertia are for use where bending combined with axial stress occurs in a column, as described in Sec. 5.

Table 10 gives the size and pitch of spirals for various column diameters which satisfy the provisions of the ACI Code given in Art. 4.2. These values are based on a concrete thickness outside the spirals of $1\frac{1}{2}$ in. A greater thickness of concrete protections requires slightly more spiral reinforcement.

**4.8. Illustrative Example.** I. A circular column reinforced with longitudinal steel and spiral hooping has an unsupported length of 14 ft-0 in. and sustains a direct axial load of 250,000 lb. The ultimate strength of the concrete is 3000 psi and hard-grade steel reinforcement is to be used. Design the column.

From Table 7 the following selections are made:

| Diam. of col | Load carried by concrete | Load by steel | Weight of col (Table 9) | Net load, kips | $p_g$ | $A_s$, sq in | Bars selected |
|---|---|---|---|---|---|---|---|
| 20 | 212 | 63 | 5 | 270 | 0.010 | 3.14 | 8 No. 6 |
| 19 | 191 | 63 | 4 | 250 | 0.011 | 3.12 | 8 No. 6 |
| 18 | 172 | 82 | 4 | 250 | 0.016 | 4.08 | 10 No. 6 |
| 17 | 153 | 100 | 3 | 250 | 0.022 | 4.99 | 9 No. 7 |
| 16 | 136 | 117 | 3 | 250 | 0.029 | 5.83 | 8 No. 8 |
| 15 | 119 | 134 | 3 | 250 | 0.038 | 6.72 | 9 No. 8 |
| 14 | 104 | 148 | 2 | 250 | 0.048 | 7.39 | 10 No. 8 |

All these selections satisfy the specifications of the ACI Code. The 20-in. column has the minimum amount of reinforcement allowed by the specifications; but since it requires no more steel in obtainable sizes than the 19-in. column, the latter is theoretically more economical. However, metal column forms are usually available only in diameters of even integral inches, so that the 20-in. column would usually be chosen.

If it were desirable to have the column as small as possible, the 14-in.

column could be selected.[1] Care must be taken when large percentages of steel are used that sufficient space is left between the bars. Usually about 2 per cent of longitudinal steel furnishes the most desirable column. For the 20-in. column, a $\frac{3}{8}$-in. spiral with a pitch of 2 in. (hot-rolled) or $2\frac{3}{4}$ in. (cold-drawn) is selected (Table 10).

II. Select a column reinforced with longitudinal steel and lateral ties with an unsupported length of 14 ft-0 in. to carry an axial load of 240,000 lb. The concrete strength is 3000 psi and intermediate-grade steel is to be used. Furthermore, the column is to be as small as possible, and its least dimension cannot exceed 14 in.

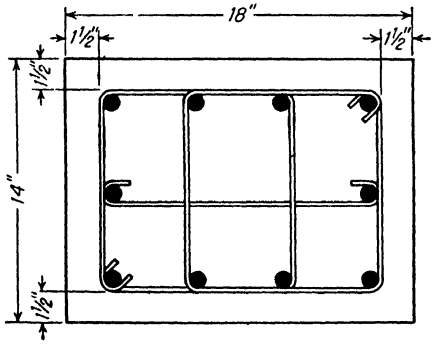



FIG. 4.4

This column is a long column since $h = 12$, and its safe load is

$$(1.3 - 0.03 \times 12) = 0.94 \text{ times that of a short column}$$

From Table 8 it appears that a 14- by 18-in. column will be required, which weighs (Table 9) $18.8 \times 14 \times 14 = 2700$ lb, or approximately 3 kips. The total load to be carried is 243 kips, and a short column must be selected which is capable of sustaining $243 \div 0.94 = 259$ kips. From Table 8, the concrete can sustain 136 kips, leaving 123 kips for the steel, which requires a percentage of $123 \div 32 = 3.8$. This is slightly under the maximum of 4 per cent and requires an $A_s$ of

$$0.038 \times 14 \times 18 = 9.58 \text{ sq in.}$$

This is furnished by ten No. 8 bars. The bars must be held in place by ties. The arrangement in Fig. 4.4 shows three sets of ties. Applying

[1] Such a column would, however, be a long column, since $h/d = 12$, and the safe load would be $(1.3 - 0.03 \times 12) = 0.94$ of that for a short column. $252 \div 0.94 = 268$. This leaves 164 kips to be carried by the reinforcing bars, requiring

$$164 \div 31 = 5.3 \text{ per cent}$$

or an $A_s$ of $0.053 \times 154 = 8.16$ sq in. This requires additional steel, or eleven No. 8 bars.

the rule given in the footnote on page 128, each tie should have a sectional area of $0.02 \times 4.000 = 0.08$ sq in. Wire $\frac{5}{16}$ in. in diameter furnishes approximately this cross section and is selected. Ties are placed 14 in. center to center according to the specifications given on page 128.

## Problems

1. Design the column of Example 11 as a square column using the minimum amount of reinforcement.

2. Design a column with an unsupported length of 20 ft-0 in. to sustain a direct axial load of 300,000 lb; $f'_c = 2500$; $f_s = 16{,}000$.

(*a*) As a spirally reinforced column with $p_g = 0.02$.

(*b*) As a tied column with $p_g = 0.02$.

3. What safe load can be sustained by a spiral column 22 in. in diameter reinforced with twelve No. 7 bars?

(*a*) If $f'_c = 3750$, $f_s = 16{,}000$, and $h = 15$ ft.

(*b*) If $f'_c = 2000$, $f_s = 20{,}000$, and $h = 20$ ft.

4. What safe load can be supported by a tied column 12 by 16 in. reinforced with four No. 8 bars?

(*a*) If $f'_c = 2500$, $f_s = 16{,}000$, and $h = 16$ ft

(*b*) If $f'_c = 3000$, $f_s = 20{,}000$, and $h = 10$ ft.

# SECTION 5

# BENDING AND AXIAL STRESS

**5.1. General Theory.** In the two preceding sections, members in bending and members sustaining direct axial stress have been treated separately. In some structural members both types of stresses occur simultaneously and the combined stress due to bending and direct axial load becomes the critical working stress. The more usual cases are:

1. A beam subject to inclined forces, or a beam acting as a strut between its supports.
2. A column sustaining in addition to its axial load a bending moment caused by one of the conditions mentioned in Art. 4.6.
3. An arch rib, where the arch thrust acts other than parallel to and along the arch axis.

In all these cases, the resultant stress is a combination of that produced by bending and that produced by the direct load acting along the axis of the member.

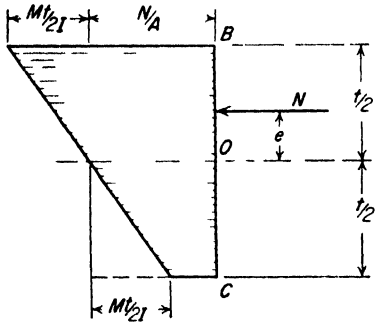


FIG. 5.1

Let Fig. 5.1 represent a plain concrete section $BC$. The resultant of all the forces $N$ is applied at a distance $e$ from the centroid $O$ of the section. If the force $N$ were applied at the point $O$, the intensity of the stress over the whole section $BC$, whose area is $A$, would be uniform and equal to $N/A$. Since, however, the force $N$ is not applied at the center of the section $O$, it produces a moment about the point $O$ equal to $Ne$, that is, the force $N$ applied at a distance $e$ from the gravity axis may be replaced by an equal force applied at $O$ and a couple whose moment is $Ne$. The intensity of the stress at the extreme fibers of the section produced by this moment is $M \times t/2 \div I$, where $I$ is the moment of inertia of the section about an axis $O$ perpendicular to the plane of the paper. The total intensity of the

compression at the edge $B$ is then

$$f_c = \frac{N}{A} + \frac{Mt}{2I}$$

and at the edge $C$

$$f_c = \frac{N}{A} - \frac{Mt}{2I}$$

If the stress $f_c$ is a negative quantity, it shows that the stress produced by flexure is greater than that produced by the direct action of $N$, and the resultant stress at the edge $C$ is tension.

In a reinforced concrete member it is presupposed that the bond between the steel and the concrete remains intact under stress. If elastic action is assumed, the steel in the compression side of a member subject to bending and axial stress can withstand a stress only sufficient to

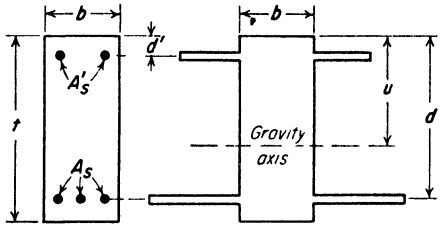



FIG. 5.2

make it deform equally with the concrete or $n$ times the stress in the concrete ($nf_c$). This steel might then be replaced by $n$ times the amount of concrete at the same distance from the axis of the section. Such a section is known as the transformed section (see Art. 3.12).

Referring to Fig. 5.2, let

$A_s$ = area of steel near tension surface

$A'_s$ = area of steel near compressive surface

$p = \frac{A_s}{bt}$, $p' = \frac{A'_s}{bt}$, $p_g = \frac{A_s + A'_s}{bt}$

$u$ = distance from compressive surface to gravity axis of transformed section

$A_t$ = area of transformed section

$I_c$ = moment of inertia of transformed section

It may be seen that

$$A_t = bt + (n-1)(A_s + A'_s)$$

$$I_t = I_c + (n-1)I_s$$

$$u = \frac{(t/2) + p(n-1)d + p'(n-1)d'}{1 + p(n-1) + p'(n-1)}$$

$$I_c = \tfrac{1}{3}b[u^3 + (t-u)^3]$$

Neglecting the moments of inertia of the bars about their own diameters,

$$I_s = A_s(d - u)^2 + A'_s(u - d')^2$$

If the reinforcement is symmetrical, then $u = t/2$ and $I_c = \frac{1}{12}bt^3$, and $I_s = 2A'_s[(t/2) - d']^2$.

If the eccentricity $e/t$ is within certain limits, then compression exists over the whole section. For greater eccentricities, there will be tension over a part of the section. If it is assumed that the concrete takes no tension, the analyses for these two cases are quite different. The limiting value of $e/t$, which results in zero stress on the tension surface, is dependent on the relative amounts of steel and concrete and the ratio of the moduli of elasticity of the two materials.

There are four general conditions of bending and axial stress:

1. Rectangular sections with compression over the whole section.
2. Rectangular sections with tension over part of the section.
3. Circular sections with compression over the whole section.
4. Circular sections with tension over part of the section.

These four conditions will be discussed in the following articles. *Only symmetrical reinforcement will be considered in this section, and the total steel area will be referred to as* $A_s$, the $A_s + A'_s$, or the $2A'$ referred to in the discussion of Fig. 5.2.

**5.2. Rectangular Sections. Compression over the Whole Section.** The maximum unit stress in the concrete may be computed as though the members were homogeneous, and is

$$f_c = \frac{N}{A} + \frac{Mt}{2I} \tag{5.1}$$

This equation may be written

$$f_c = \frac{N}{bt + (n - 1)A_s} + \frac{Mt}{2[I_c + (n - 1)I_s]} \tag{5.2}$$

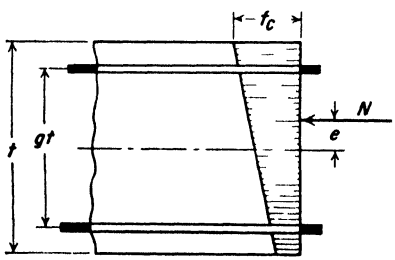



FIG. 5.3

The unit flexural stress in the concrete in the plane of the reinforcement is $Mgt/2I$ (see Fig. 5.3). On the compressive side of the member this stress is an addition to the direct axial stress, while on the tensile side of the member it is a subtraction from the axial stress, as shown in the previous article. The maximum unit stress in the steel is then

$$f'_s = \frac{N}{A_t} + \frac{nMgt}{2I}$$

which is less than $nf_c$, since $gt$ is less than $t$. It is therefore always within the limits of a reasonable value for $f'_s$, provided $f_c$ is within allowable limits

Where compression exists over the whole section, Eq. (5.2) furnishes an exact solution. For an approximate solution, Eq. (5.2) may be written

$$f_c = \frac{N}{bt + nA_s} + \frac{Mt}{2(I_c + nI_s)} \tag{5.3}$$

The error in the above equation is not large, involving only the difference in the last term of each of the denominators between $n$ and $n - 1$. From this equation diagrams may be prepared that use the same parameters as diagrams for the solutions of cases where tension exists over a portion of the section.

Equation (5.3) may be written

$$\begin{aligned} f_c &= \frac{N}{bt + p_g nbt} + \frac{Ne(t/2)}{\frac{1}{12}bt^3 + p_g nbt(gt/2)^2} \\ &= \frac{N}{bt}\left[\frac{1}{1 + p_g n} + \frac{e(t/2)}{(t^3/12) + p_g n(gt/2)^2}\right] \\ &= \frac{N}{bt}\left[\frac{1}{1 + p_g n} + \frac{e}{t}\,\frac{6}{1 + 3p_g nq^2}\right] \end{aligned} \tag{5.4}$$

By allowing the expression within the brackets in Eq. (5.4) to be known as $K$, the lower portions of Diagrams 18 to 20, Appendix D, have been plotted from this equation for definite values of $g$, with $e/t$ and $p_g n$ as variables. By entering the appropriate diagram with these variables as arguments the value of $K$ may be obtained for use in the equation

$$f_c = \frac{NK}{bt} \tag{5.5}$$

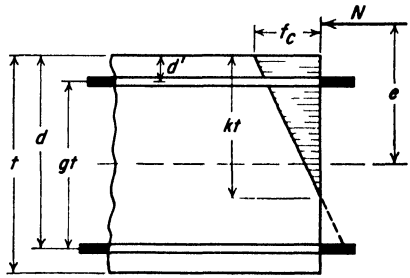


FIG. 5.4

**5.3. Rectangular Sections. Tension over Part of the Section.** (Fig. 5.4). Where the second term of Eq. (5.2) is greater than the first, it indicates tension over a part of the section. Unless this tension is so small that the concrete can take its proportionate part, the analysis of the previous article is not applicable With any appreciable tension on the tension surface of the member, the section becomes cracked and all the tension must be resisted by the steel.[1]

[1] The ACI Code allows the section to be assumed uncracked as long as the ratio of $e/t$ is less than unity.

By reference to Fig. 5.4,

$$f'_s = nf_c\left(1 - \frac{d'}{kt}\right) \tag{5.6}$$

and

$$f_s = nf_c\left(\frac{d}{kt} - 1\right) \tag{5.7}$$

Since the total resultant stress = $N$,

$$\frac{1}{2}f_c bkt + \frac{A_s}{2}f'_s - \frac{A_s}{2}f_s = N$$

Substituting the values of $f'_s$ and $f_s$, from Eqs. (5.6) and (5.7), replacing $A_s$ by $p_g bt$, and with the relation that $d + d' = t$,

$$N = \frac{f_c bt}{2}k^2 + \frac{2np_g k - np_g}{k} \tag{5.8}$$

or

$$f_c = \frac{N}{bt}\left(\frac{2k}{k^2 + 2np_g k - np_g}\right) \tag{5.9}$$

This equation is similar in form to Eq. (5.4), but the position of the neutral axis must be determined before it can be used.

Since the moment of the stresses about the gravity axis = $M$,

$$\frac{1}{2}f_c bt\left(\frac{t}{2} - \frac{kt}{3}\right) + \frac{f'_s A_s}{2}\left(\frac{t}{2} - d'\right) + \frac{f_s A_s}{2}\left(d - \frac{t}{2}\right) = M$$

and eliminating $f'_s$ and $f_s$ as before

$$\frac{M}{bt^2 f_c} = \frac{np_g(t - 2d')^2}{4kt^2} + \frac{k}{12}(3 - 2k) \tag{5.10}$$

Since $M = Ne$, the right side of Eq. (5.8) may be multiplied by $e$ and this value substituted for $M$ in Eq. (5.10). Substituting $gt$ for $t - 2d'$, the following equation results:

$$k^3 - 3\left(\frac{1}{2} - \frac{e}{t}\right)k^2 + 6np_g k\frac{e}{t} = 3np_g\left(\frac{e}{t} + \frac{1}{2}g^2\right) \tag{5.11}$$

In Eq. (5.9) the expression within the parentheses is designated as $K$, resulting in an equation similar to Eq. (5.5). For constant values of $g$, values of $k$ and $np_g$ may be substituted in Eq. (5.11) and values of $e/t$ obtained. Substituting these same values of $k$ and $np_g$ in Eq. (5.9) the corresponding values of $K$ are obtained. The upper portions of Diagrams 18 to 20 have been plotted from such computations.

**5.4. Circular Sections. Compression over the Whole Section.** The reinforcement in a circular section is practically always symmetrical and,

although it is in the form of bars, it may be considered to be in the form of a hollow cylinder whose average diameter is equal to the diameter of the circle on which the bars are placed, the cross-sectional area of the wall of the cylinder being equal to the sum of the cross-sectional area of the bars (see Fig. 5.5).

Following the same procedure as for rectangular sections, with $r$ equal to $t/2$,

$$f_c = \frac{N}{\pi r^2 + (n-1)A_s} + \frac{Mr}{I_c + (n-1)I_s}. \tag{5.12}$$

Considering the longitudinal bars replaced by a hollow cylinder whose wall is $A_s$, $I_s = A_s(gt)^2/8$, where $gt$ is the average diameter of the cylinder

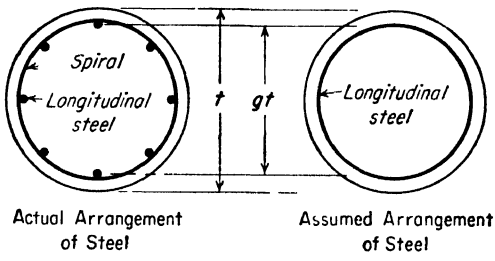



FIG. 5.5

or the diameter of the circle on which the bars are placed. Then Eq. (5.12) becomes

$$f_c = \frac{N}{\pi r^2 + (n-1)A_s} + \frac{Mr}{\pi r^4/4 + (n-1)A_s(gt)^2/8} \tag{5.13}$$

These equations give an exact solution where there is no tension on the section. Equation (5.12) may be used in conjunction with the values given in Tables 9 and 12, Appendix D, or Eq. (5.13) may be used by making the proper numerical substitutions.

Equation (5.13) can be reduced[1] to the form

$$f_c = \frac{N}{\pi r^2}\left[\frac{1}{1 + p_g n} + \frac{c}{t} \times \frac{2}{0.25 + 0.5pn(gt)^2}\right]$$

and designating the expression within the brackets as $K$,

$$f_c = \frac{NK}{\pi r^2} \tag{5.14}$$

Following the same procedure described in Art. 5.2 for rectangular sections, the lower portions of Diagrams 15 to 17, Appendix D, have been constructed from which the value of $K$ may be determined.

[1] The substitution of $n$ for $n - 1$ is accomplished here for the same reason described in Art. 5.2.

**5.5. Circular Sections. Tension over Part of the Section.** A direct solution by equations similar to those shown for rectangular sections cannot be made for tension over part of a circular section. The expression of relations becomes very complicated. However, by carrying the derivation partially to completion, values of some of the variables may be assumed and from these other values determined. In such a manner the upper portions of Diagrams 15 to 17, Appendix D, have been constructed.

**5.6. Allowable Stresses for Combined Compression and Bending.** The maximum permissible stress in the concrete of a column may be increased when the combined effect of bending and axial stress is considered.[1]

The allowable concrete stress for axially loaded spirally reinforced columns is $0.225f'_c$, while for tied columns 80 per cent of this value is permitted. On the other hand, for beams in bending without axial compression the allowable concrete stress is $0.45f'_c$. Members subject to both axial compression and bending occupy an intermediate position between these two extremes. For small eccentricities, their allowable stresses must be close to those for axially loaded columns, while for large eccentricities, when the state of simple bending is approached, they must be close to those for beams. Tests have confirmed that higher concrete stresses can be sustained when bending is present than for axial compression, since, with bending, the maximum stress occurs only at the extreme compression fiber and decreases rapidly toward the axis.

For this reason the ACI Code provides allowable stresses which increase with increasing eccentricity, *i.e.*, the ratio $e/t$. For zero eccentricity they are identical with those allowed for axially loaded columns. They become equal to $0.45f'_c$ for $e/t = 1$, at which value this code assumes the section to become cracked, *i.e.*, to act in the same manner as a beam.

The ACI Code allows a concrete stress on the compressive side of the column of

$$f_p = f_a \frac{t + De}{t + CDe} \tag{5.15}$$

where $f_a$ = average allowable unit stress in the concrete of an axially loaded reinforced concrete column $= \dfrac{0.225f'_c + f_s p_g}{1 + (n-1)p_g}$ for spiral columns and 0.8 of that value for tied columns

$D = l^2/2R^2$, where $R$ = least radius of gyration of the section

$C = f_a/0.45f'_c$

$t$ and $e$ represent the same dimensions indicated in Art. 5.1 and Eq. (5.3)

[1] The unincreased unit stress must not be exceeded when the maximum axial load produces no bending.

Since $D$ appears in both numerator and denominator of Eq. (5.15), it involves but a slight error if the transformed steel area is neglected. Diagrams 9 to 14, Appendix D, are plotted on this basis with $e/t$ and $p_g$ as arguments.

Where a more exact determination is required, $f_a$ may be computed by the equation given above or taken from Table 11, Appendix D.

$$D = \frac{t^2}{2R^2}$$

may be computed[1] or taken from Table 14, and with $C' = f_a \div 0.45f'_c$, substitution in Eq. (5.15) gives an exact value of the allowable stress $f_p$.

**5.7. Design of Columns Subject to Bending and Axial Stress.** The procedures outlined in Arts. 5.2 to 5.6 provide the means of checking the adequacy of columns of given dimensions to resist loads and moments of known value. In contrast, the problem facing the designer is that of selecting the dimensions and the reinforcement of a column acted upon by a given axial force and subject to some bending. The magnitude of the bending moment, in many instances, in turn depends on the size of the column. In such cases the correct column size can be obtained only by successive trials. The approach explained in this article serves to reduce the number of trials to a minimum.

Referring to the fundamental equation for the maximum unit stress in the concrete [Eq. (5.1)], since

$$\begin{aligned} A &= A_g[1 + (n - 1)p_g] \\ M &= Ne \\ I &= AR^2 \end{aligned}$$

[1] For a column with lateral ties,

$$\begin{aligned} I &= AR^2 = \tfrac{1}{12}bt^3 + (n - 1)btp_g^2t^2 \\ &= bt^3[\tfrac{1}{12} + (n - 1)p_g^2] \\ R^2 &= \frac{bt^3[\tfrac{1}{12} + (n - 1)p_g^2]}{bt + (n - 1)pbt} \\ &= \frac{t^2[\tfrac{1}{12} + (n - 1)p_g^2]}{1 + (n - 1)p} \\ D &= \frac{1 + (n - 1)p}{\tfrac{1}{6} + \tfrac{1}{2}(n - 1)pg^2} \end{aligned}$$

Similarly, the value of $D$ for a round column section with spirals is

$$D = \frac{1 + (n - 1)p}{\tfrac{1}{8} + \tfrac{1}{4}(n - 1)p_g^2}$$

and for a square section with spirals

$$D = \frac{1 + (n - 1)p}{\tfrac{1}{6} + \tfrac{1}{4}(n - 1)p_g^2}$$

and

$$D = \frac{t^2}{2R^2}$$

this equation may be resolved into

$$f_c = \frac{N}{A_g}\left[\frac{1 + (De/t)}{1 + (n-1)p_g}\right] \tag{5.16}$$

Equating this value to the allowable stress $f_p$ [Eq. (5.15)] and reducing,[1]

$$P = N\left(1 + \frac{CDe}{t}\right) \tag{5.17}$$

This equation indicates that a column subject to flexural stresses may be designed for an equivalent axial load $P$. In the above equation the only known quantity for all cases is the total load to be carried $N$. The two general cases to be considered are (1) where the only bending moment to be sustained is that caused by an eccentric load and (2) where the bending moment to be sustained is caused mainly by the unbalanced loading or unequal span lengths of the horizontal members framing into the column.[2]

*Bending Moment Caused by Eccentric Load Only.* Where the dimensions of the column have been fixed by other considerations and the eccentric load is known in position and magnitude, the problem is considerably simplified. Both $e$ and $t$ are now known. For preliminary design, the product $CD$ in Eq. (5.17) may be assumed as 3 for tied columns and 5 for spiral columns. The value of $P$ in Eq. (5.17) may be computed and the preliminary amount of reinforcement determined. The value of $f_c$ may be computed for this trial reinforcement by use of Eqs. (5.2) or (5.12) or from Diagrams[3] 15 to 20, and the allowable value of $f_p$ determined from Diagrams 9 to 14, or computed from Eq. (5.15) with or without the assistance of Tables 11 and 14. Revision of the required steel area and redetermination of $f_c$ and $f_p$ must be made if necessary.

Where the dimensions of the column have not been previously fixed, an assumption of column dimensions must be made. Reference to Eq.

$$\frac{N}{A_g}\left[\frac{1 + (De)/t}{1 + (n-1)p_g}\right] = \frac{0.225f'_c + f_s p_g}{1 + (n-1)p_g}\left(\frac{t + De}{t + cDe}\right)$$

and since $A_g\,(0.225f'_c + f_s p_g) = P$,

$$N\left(1 + \frac{De}{t}\right) = P\left(\frac{t + De}{t + CDe}\right) \quad \text{and} \quad P = N\left(1 + \frac{CDe}{t}\right)$$

[2] There may also be an additional bending moment from an eccentric load.

[3] Where no tension exists in the section, Eq. (5.2) or (5.12) should be used.

(5.17) indicates that, with the approximate values for the product $CD$ of 3 for tied columns and 5 for spiral columns suggested above, where $e/t$ is close to the limit of 1.0 allowed by the ACI Code, the size of the column would be approximately four times that required for direct load for tied columns and approximately six times that required for the direct load for spiral columns. Smaller values of $e/t$ indicate a smaller increase in the size of the column required. Approximations of $M$, $e$, and $t$ should be made in order to select a reasonable size of column for trial.

*Approximate Determination of Bending Moment Caused by the Unbalanced Moment of the Horizontal Members Framing into the Column.* An

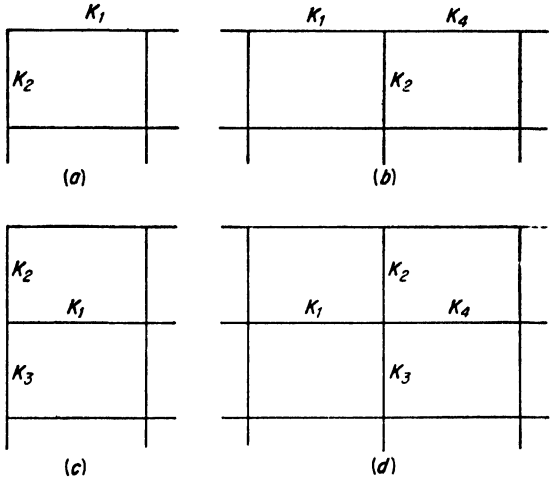



FIG. 5.6

accurate determination of bending moments induced in columns by abutting beams and girders requires rather elaborate computations. Some of the simpler methods are discussed in Sec. 6. In some cases, particularly in multistory, multibay building frames, the following approximate method is often sufficiently accurate. Its justification is given in Sec. 6. The amount of bending moment introduced into the column depends upon the relative stiffness of the several members meeting at the joint. The stiffness is $EI$ divided by $l$ or $h$, $l$ being the center-to-center span of a horizontal member and $h$ the unsupported height of a column. Since $E$ is usually a constant for all the members and only relative stiffness is to be considered, the stiffness $K$ may be taken as $I/l$ or $I/h$. The stiffness of a member actually depends upon the degree of fixit at its far end, but for a trial analysis and for simple cases the far end may be considered fixed. The ACI Code states that in computing relative stiffnesses the reinforcement in slabs, beams, girders, and columns may be neglected. In computing the $I$ of a T section, it is recommended

that the full width of the flange be considered and the $I$ computed about the lower surface of the flange. For flat slabs, the section to be considered is the column strip.

The Code also states that columns shall be designed to resist the axial forces from loads on all floors, plus the maximum bending due to loads on a single adjacent span of the floor under consideration.

The four conditions encountered in column design, where the column is a member of a rigid building frame, are illustrated in Fig. 5.6. In ($a$) the moment to be resisted by the column is $M[K^2/(K_1 + K_2)]$, where $M$ is the fixed-end[1] moment in the horizontal member. In ($b$) the moment to be resisted by the column 2 is $M_2[K^2/(K_1 + K_2 + K_4)]$, where $M_2$ is the difference of the fixed-end moments of the two horizontal members. In ($c$) the moment to be resisted by column 2 is $M_1[K_2/(K_1 + K_2 + K_3)]$, and in ($d$), $M_2[K_2/(K_1 + K_2 + K_3 + K_4)]$. From the above it may be stated that the moment to be resisted by a column is

$$\frac{MK}{\Sigma K}$$

where $M$ is the unbalanced fixed-end moment of the horizontal member or members, its multiplier $K$ the relative stiffness of the column under consideration, and $\Sigma K$ the sum of the relative stiffnesses of all the members meeting in the joint.

In designing this type of column, neither the value of $c$ nor of $t$ is known, and $c$ cannot be evaluated until the relation between the moment to be resisted by the column and the direct load sustained is known. For conditions such as ($a$) of Fig. 5.6, the column may have a size of from two to four times that of the column required to carry the direct load without bending moment, depending upon the relative stiffnesses of the beam and column. For long-span beams and lightly loaded columns, the larger figure is recommended in assuming a column for trial, while for heavily loaded columns and short-span beams the smaller figure will be more liable to result in a satisfactory assumption. After selecting a size of column for trial, the procedure is as follows:

[1] The fixed-end moments for the usual cases are as follows:

Uniform loading: $\frac{1}{12} wl^2$

Center-point loading: $\frac{1}{8} Pl$

Third-point loading: $\frac{2}{9} Pl$

Quarter-point loading: $\frac{5}{16} Pl$

where $l$ is the span center to center of supports, $w$ the uniform load per foot of span, and $P$ the concentrated load. For fixed-end moments for other loadings and a further analysis of the distribution of moments in a frame, see Sec. 6.

1. Compute the $K$ of this column.
2. Determine the moment to be resisted by the column.
3. Determine $e = M/N$.
4. Assuming $CD$ as before, compute $P$ from Eq. (5.17) and select the amount of reinforcement.
5. Determine $f_c$ either by Eq. (5.2) or (5.12) or from Diagrams 15 to 20.
6. Determine $f_p$ and revise the steel area if necessary.

The above method will usually result in a satisfactory design. However, if the amount of reinforcement required is unsatisfactory, the size of the column may have to be slightly modified and $f_c$ and $f_p$ redetermined.

For conditions such as (*c*) of Fig. 5.6., select a column capable of carrying approximately twice the full direct load, and proceed as above. For conditions such as are illustrated in (*b*) and (*d*), select a column from Table 7 or 8 with any desired percentage of reinforcement capable of carrying the full direct load, compute $M$ and $e$, and proceed as above. Usually the column required to sustain the direct load will prove satisfactory, but if not, a slight modification of the steel area can be made.

**5.8. Illustrative Examples.** I. A column reinforced with longitudinal steel and lateral ties is to support a direct axial load of 200,000 lb and in addition an eccentric load of 30,000 lb near the top of the column. The center of bearing of the 30,000-lb load is 8 in. from the outside face of the column. The unsupported height of the column is 19 ft-0 in. The ultimate strength of the concrete is specified as 2500 psi, and intermediate-grade steel is to be used for reinforcement.

From Table 8, Appendix D, a column 20 by 20 in. with 1 per cent reinforcement will sustain a total load of 180 + 51 = 231 kips, and with 2 per cent, 282 kips. Selecting a 20- by 20-in. column for trial, from Table 9, its weight is 20 × 20.8 = 416 lb per ft, and the total load on the column is 200 + 30 + 19 × 0.416 = 238 kips. The value of $h/t = (19 \times 12)/20 = 11.4$, so that as a long column it can sustain $1.3 - 0.03 \times 11.4 = 0.96$ of the safe load of a short column, and the equivalent short-column load is 238/0.96 = 248 kips.

$$M = 30{,}000(10 + 8) = 540{,}000 \text{ in.-lb}$$

$$e = \frac{540{,}000}{230{,}000} = 2.35 \text{ in.} \qquad \text{and} \qquad \frac{e}{t} = 0.12$$

Substituting in Eq. (5.17) with a trial value of $CD = 3$,

$$P = 230(1 + 3 \times 0.12) = 313 \text{ kips}$$

The column selected would require (Table 8)

$$\frac{313 - 180}{100} \div 51 = 2.61 \text{ per cent of reinforcement, or } 10.44 \text{ sq in.}$$

Eight No. 10 bars furnishing 10.12 sq in., arranged as shown in Fig. 5.7, are selected for trial. From Tables 9 and 13 the moment of inertia of the column section is $20 \times 667 + 11 \times 6 \times 71 = 18{,}000$ in.$^4$ and

$$f_c = \frac{230{,}000}{400 + 11 \times 10.12} + \frac{540{,}000 \times 10}{18{,}000} = 750 \text{ psi}$$

With $e/t = 0.12$ and $p_g = 10.12/400 = 0.0253$, from Diagram 13, $f_p = 750$ psi and the design is satisfactory.

The more exact determination of $f_p$ is as follows:

$$f_a = 605 \text{ psi (Table 11)}$$
$$C = 605 \div 1125 = 0.538$$
$$R^2 = 18{,}000 \div 511 = 35.3$$
$$D = 400 \div (2 \times 35.3) = 5.68$$
$$f_p = 605 \times \frac{20 + 5.68 \times 2.35}{20 + 0.538 \times 5.68 \times 2.35} = 748 \text{ psi}$$

II. A column reinforced with longitudinal steel and lateral ties is to support a direct axial load of 250,000 lb and in addition an eccentric load

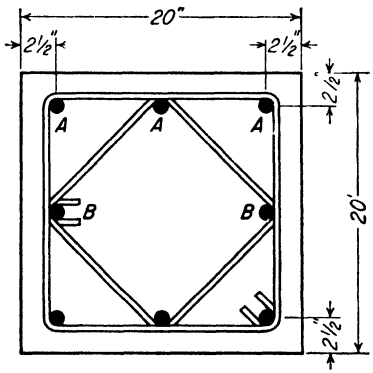



Fig 5.7

of 75,000 lb near the top of the column. The center of bearing of the 75,000-lb load is 9 in. from the outside face of the column, whose unsupported length is 17 ft-0 in. The ultimate strength of the concrete is specified as 3000 psi, and hard-grade steel is to be used for reinforcement.

Neglecting the weight of the column, from Table 8 it is seen that an 18- by 18-in. column with approximately 3 per cent of steel

$$(175 + 3 \times 52 = 331)$$

would be required to support the total load as an axial load. With this size column, however, $M = 75{,}000 \times 18 = 1{,}350{,}000$ in.-lb,

$$e = 1{,}350{,}000 \div 325{,}000 = 4.15 \text{ in.}$$

and $e/t = 0.231$. Substitution in Eq. (5.17) indicates a column capable of supporting approximately 1.69 times the total load of 325,000 lb would be required. It therefore appears desirable to select a somewhat larger column for trial. With a 22- by 22-in. column,

$$M = 75{,}000 \times 20 = 1{,}500{,}000 \text{ in.-lb}$$

$e = 1{,}500{,}000 \div 325{,}000 = 4.62$ in.; and $e/t = 0.21$. $P$, from Eq. (5.17), $= 325(1.63) = 530{,}000$ lb. A 22- by 22-in. column would require $\dfrac{530{,}000 - 261{,}000}{100} \div 77 = 3.50$ per cent, or 16.9 sq in. of reinforcement.

Selecting twelve No. 11 bars, $A_s = 18.75$ sq in. With these bars spaced symmetrically on the four sides of the column, the moment of inertia of the column section (Tables 9 and 13) is

$$22 \times 887 + 9 \times 8 \times 113 + 9 \times 4 \times 13 = 28{,}100 \text{ in.}^4$$

and

$$f_c = \frac{325{,}000}{484 + 9 \times 18.75} + \frac{1{,}500{,}000 \times 11}{28{,}120} = 1082 \text{ psi}$$

The allowable stress $f_p$ is determined as follows:

$$f_a = 860 \text{ psi (Table 11)} \qquad C = 860 \div 1350 = 0.637$$

$$R^2 = \frac{28{,}100}{653} = 43.0 \qquad D = 481 \div (2 \times 43.0) = 5.63$$

$$f_p = 860 \times \frac{22 + 5.63 \times 4.62}{22 + 0.637 \times 5.63 \times 4.62} = 1070 \text{ psi}$$

The design is considered satisfactory.

III. In the frame of Fig. 5.8, the beams, 5, 6, 7 and 8, are T beams with a flange section of 60 by 5 in. and a web section below the slab of

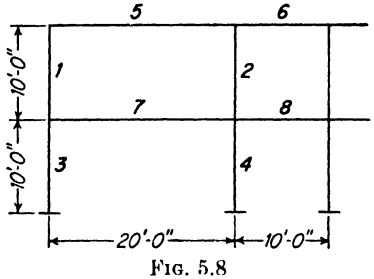



FIG. 5.8

10 by 17 in. The direct load sustained by column 1 is 66,000 lb, and the total fixed-end moment at the left end of beam 5 is 1,320,000 in.-lb. Design column 1 using a 2500-psi concrete and intermediate-grade steel.

The direct load of 66,000 lb would require only the minimum size column allowed by the ACI Code (10 by 12 in.). Assuming a column capable of sustaining a load 2½ times as great, or 165,000 lb, a column

14 by 16 with between 2 and 3 per cent of reinforcement would be required (Table 8).

The gravity axis of the concrete section of the beams is 1.5 in. below the bottom of the flange. The moment of inertia about this axis is

$$\frac{1}{3} \times 60 \times 5 + 60 \times 5 \times 1.5^2 + \frac{10}{3} \times (1.5)^3 + \frac{10 \times (15.5)^3}{3} = 15{,}600 \text{ in.}^4$$

and the relative stiffness is

$$15{,}600 \div 240 = 65$$

The moment of inertia of the concrete section of the column is

$$16 \times 229 = 3664 \text{ in. (Table 9)}$$

and its relative stiffness is

$$3664 \div 120 = 31$$

The moment to be resisted by the column is

$$\frac{31}{31 + 65} \times 1{,}320{,}000 = 426{,}000 \text{ in.-lb}$$

$$e = \frac{M}{N} = 426{,}000 \div 66{,}000 = 6.46 \text{ in.} \qquad \frac{e}{t} = 0.46$$

From Eq. (5.17), assuming $CD$ as 3, $P = 66(1 + 3 \times 0.46) = 157{,}000$ lb. A 14- by 16-in. column would require $\frac{157{,}000 - 101{,}000}{100} \div 29 = 1.93$ per cent reinforcement to sustain this load. Eight No. 7 bars with an area of 4.81 sq in. furnish 2.15 per cent of reinforcement and are selected for trial.

The moment of inertia of the column section is

$$16 \times 229 + 6 \times 11 \times 12 = 4456 \text{ in.}^4$$

and

$$f_c = \frac{66{,}000}{224 + 11 \times 4.81} + \frac{426{,}000 \times 7}{4456} = 908 \text{ psi}$$

$$f_a = 586 \text{ psi} \qquad C = 555 \div 1125 = 0.520$$
$$R^2 = 4456 \div 277 = 16.1 \qquad D = 196 \div 2 \times 16.1 = 6.09$$
$$f_p = 586\left(\frac{14 + 6.09 \times 6.46}{14 + 0.520 \times 6.09 \times 6.46}\right) = 908 \text{ psi}$$

IV. Refer to Fig. 5.8, design column 4, considering that this column will be reinforced with longitudinal reinforcement and spirals, with the same strength concrete and the same grade of steel specified for Example III. With equal loads on all floors the total load supported by column 4 is approximately three times that of column 1, or 178,000 lb, which would

require an 18-in. column with approximately 1 per cent of steel. With the live load removed from beam 8, the load to be supported at the top of column 4 is 154,000 lb and the unbalanced moment at the top of the column from the beams 7 and 8 is 1,230,000 in.-lb.

By previous design, column 2 is a 14-in. column, the moment of inertia of the concrete section is 1886 in.$^4$, and the relative stiffness is 16. Similarly the relative stiffness of the 18-in. column is 43, while the relative stiffness of beam 7 is 65, as in Example III, and that of beam 8 is 130.

The moment to be resisted by the column is

$$\frac{13}{65 + 130 + 16 + 43} \times 1{,}230{,}000 = 208{,}000 \text{ in.-lb}$$

$$e = 208{,}000 \div 154{,}000 = 1.35 \text{ in.} \qquad \frac{e}{t} = 0.075$$

From Eq. (5.17),

$$P = 154{,}000(1 + 5 \times 0.075) = 212{,}000 \text{ lb}$$

The percentage of steel required is

$$\frac{212{,}000 - 143{,}000}{100 \times 41} = 1.68, \text{ or } 4.28 \text{ sq in.}$$

Seven No. 7 bars furnish 3.21 sq in. for a percentage of 1.65 and are selected for trial. The moment of inertia of the column is

$$5153 + 1.65 \times 593 = 6133 \text{ in.}^4$$

(Tables 9 and 12) and

$$f_c = \frac{154{,}000}{255 + 11 \times 4.81} + \frac{208{,}000 \times 9}{6133} = 805 \text{ psi}$$

$$f_a = 699 \qquad C = 699 \div 1125 = 0.621 \qquad R^2 = 6133 \div 308 = 19.9$$

$$D = 324 \div (2 \times 19.9) = 8.14$$

$$f_t = 699\left(\frac{18 + 1.35 \times 8.14}{18 + 0.621 \times 1.35 \times 8.14}\right) = 818 \text{ psi}$$

## Problems

1. Design column 3 of Fig. 5.8, assuming the same fixed-end moment in beam 7 as was used in Example III for beam 5.

2. A column reinforced with longitudinal steel and lateral ties is to support a direct axial load of 250,000 lb and in addition an eccentric load of 50,000 lb near the top of the column with its center of bearing 10 in. from the outside face of the column. The unsupported height of the column is 16 ft-0 in. A 2500-psi concrete and hard-grade steel are to be used.

3. Design the column of Prob. 2 as a square column reinforced with longitudinal bars and spirals.

# SECTION 6

# CONTINUOUS BEAMS AND FRAMES

**6.1. Continuity of Reinforced Concrete Structures.** The individual members which compose a steel or timber structure are fabricated or cut separately and joined together by rivets, bolts, welds, or nails. Unless the joints are especially designed for rigidity, they are too flexible to transfer moments of significant magnitude from one member to another. In contrast, in reinforced concrete structures as much of the concrete as is practical is poured in one single operation and, in addition, reinforcing steel is not terminated at the ends of a member but is carried over at the joints into adjacent members. At construction joints, special care is taken to bond the new concrete to the old by carefully cleaning the latter, carrying reinforcement across the joint, and other means. As a result, such structures represent monolithic or continuous units. Their behavior under load is shown in Fig. 6.1.

With simple joints, as provided in steel construction by many types of riveted and bolted connections, only the beam 3-4 would bend in the frame of Fig. 6.1*a*, while all other members would remain essentially straight. With rigid joints, such as in a reinforced concrete or welded steel frame, the distortion caused by a load on one single member is seen to spread to all other members of the frame, although the magnitude of deformation decreases with increasing distance from the loaded member. Consequently a beam, such as 1-2, even though free from load, bends in view of the fact that its ends are rotated by the amounts of $\alpha_1$, and $\alpha_2$, as shown in Fig 6.1*c*. Since curvature is proportional to bending moment ($1/r = M/EI$), this and all other members are subject to bending stresses, even though they are not loaded.

If horizontal forces, such as wind, act on the same frame, it deforms as shown in Fig. 6.1*b*. Here, too, all members of the frame distort, even though the forces act only on the left side; the amount of distortion is seen to be about the same for all members, regardless of their distance from points of load, in contrast to the case of vertical loading. A member, such as 2-4, even though not directly loaded, will experience bending moments caused by the distortions imposed on it, which are shown in Fig. 6.1*d*. It is seen that in addition to the rotations $\alpha_2$ and $\alpha_4$ one

end is displaced horizontally with respect to the other by the amount $d$, resulting in an additional rotation $d/l$. If, in addition, a member carries transverse load, such as beam 3-4, the end rotations $\gamma$ caused by the load must be superposed on those due to the rotations $\alpha$ caused by the continuity at the joints (Fig. 6.1*e*). The final end slopes $\theta$ of each individual member are seen to be the sums of these individual contributions $\alpha$, $d/l$, and $\gamma$.

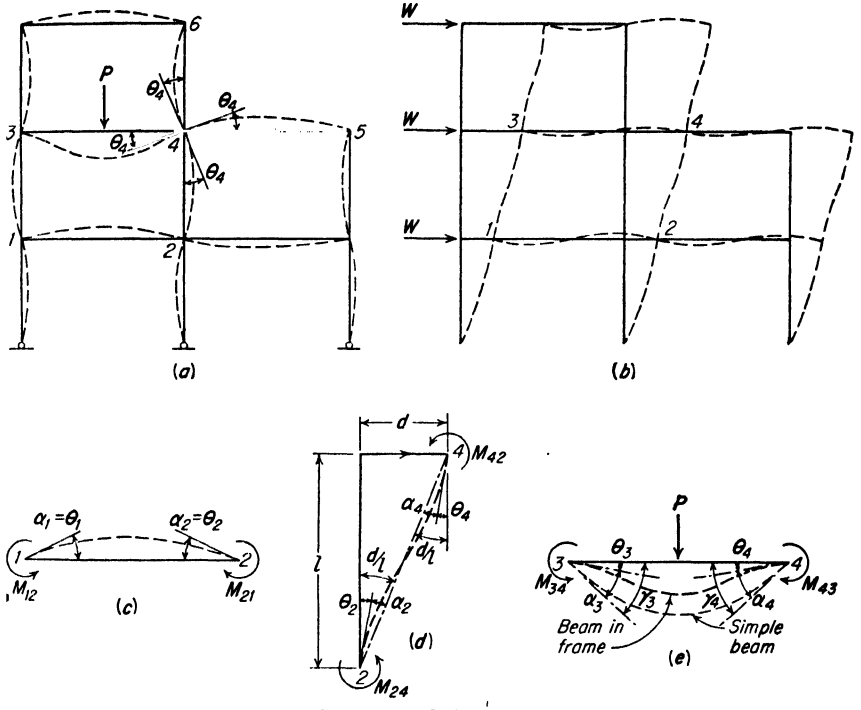



FIG. 6.1. Deflections and slopes of rigid frames.

In statically determinate structures, such as simple beams, the deflected shape and the moments and shears depend only on the type and magnitude of the loads and the dimensions of the member. In contrast, inspection of Fig. 6.1 shows that in statically indeterminate frames the deflection curve of any member, in addition to the loads, depends on the end slopes $\theta$ whose magnitudes, in turn, depend on the distortion of adjacent, rigidly connected members. In particular, for a rigid joint such as joint 4 (Fig. 6.1*a*), the slopes $\theta$ of all abutting members must be the same and equal to the rotation of the joint. For a correct design of such frames, it is evidently necessary to determine moments and shears with due consideration to the effect of the continuity at the joints.

A number of methods have been developed for such determinations.

Those that were available in the early days of concrete design were rather lengthy and tedious. To save time, it became customary to determine moments in continuous structures by using approximate coefficients, such as those given in Art. 3.6. Such coefficients are satisfactory if used within their specified limitations, *i.e.*, for continuous beams with spans not differing by more than 20 per cent and with ratios of live to dead load not larger than 3:1. Their use under any other condition is unsafe, and even within their specified limitations they often result in uneconomical design.

With the advent of more practical and efficient methods of analysis, it has become customary to ascertain moments and shears in continuous structures with a high degree of accuracy and to use approximate moment

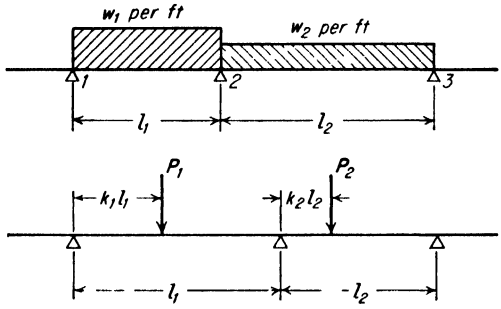



FIG. 6.2

coefficients only within their specified limits and when a closer design would not result in significant economy.

In the following articles the practically useful methods of analysis are discussed as they apply to relatively simple and more or less standard structures. For the more intricate details of this important phase of design, the reader is referred to special texts on analysis of statically indeterminate structures

**6.2. Moments in Continuous Beams.** The calculation of moments, shears, and reactions for continuous beams is based on the theorem of three moments. Considering all supports on the same level, the two fundamental equations are (see Fig. 6.2): For uniform loads,

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\tfrac{1}{4} w_1 l_1^3 - \tfrac{1}{4} w_2 l_2^3 \qquad (6.1)$$

For concentrated loads,

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\Sigma P_1 l_1^2 (k_1 - k_1^3) - \Sigma P_2 l_2^2 (2k_2 - 3k_2^2 + k_2^3) \qquad (6.2)$$

In this simplest form the two equations apply to the most frequent case, that of uniform cross section.

By using the equation applicable to the particular case, the bending

moments at all the supports may be determined, the reactions computed, and finally, the bending moment at any section of the beam may be obtained. From Eq. (6.2), influence lines may be plotted for the moment at any section of the beam, and the loading determined which will produce the maximum moment in that section. In the case of uniform load and equal spans, the exact coefficients of $wl^2$ for the theoretical maximum moments are tabulated below.

**TABLE 6.1. Moments in Continuous, Equal-span Beams with Uniformly Distributed Load**

| Number of spans | Intermediate spans and supports | | | | End spans and second support | | | |
|---|---|---|---|---|---|---|---|---|
| | At center, positive moment | | At support, negative moment | | At center, positive moment | | At support, negative moment | |
| | Dead load | Live load | Dead load | Live load | Dead load | Live load | Dead load | Live load |
| 2 | | | | | 0 070 | 0 095 | 0 125 | 0 125 |
| 3 | 0 025 | 0 075 | | | 0 080 | 0 100 | 0 100 | 0 117 |
| 4 | 0 036 | 0 081 | 0 071 | 0 107 | 0 071 | 0 098 | 0 107 | 0 120 |
| 5 | 0 046 | 0 086 | 0 079 | 0 111 | 0 072 | 0 099 | 0 105 | 0 120 |
| 6 | 0 043 | 0 084 | 0 086 | 0 116 | 0 072 | 0 099 | 0 106 | 0 120 |
| 7 | 0 044 | 0 084 | 0 085 | 0 111 | 0 072 | 0 099 | 0 106 | 0 120 |

These coefficients are derived for simple (knife-edge) supports and for the most unfavorable distribution of live loads. The coefficients of the ACI Code (see Art. 3.6) were derived from exact coefficients such as those in the table, with due consideration to the maximum ratio of live to dead load (3:1), maximum allowable span differences (20 per cent), the fact that reinforced concrete beams are never simply supported but either rest on supports of considerable width, such as walls, or are built monolithically with columns, and other factors. Since all these influences are considered, the ACI coefficients are necessarily quite conservative, so that actual, accurate moments in any particular design are likely to be considerably smaller than indicated by these coefficients. Hence, in many reinforced concrete structures significant economy can be effected by making a precise analysis of the moments in beams and girders. This becomes mandatory for beams and slabs with spans differing by more than 20 per cent and/or sustaining unsymmetrical loads. Such analyses are more expeditiously carried out by the method of moment distribution (Art. 6.7) than by the use of Eqs. (6.1) and (6.2).

**6.3. Moment-area Principles.** It was indicated in Art. 6.1 that in continuous structures moments and shears caused by imposed deformations, such as the rotation $\alpha$ or the displacements $d$, superpose on those

caused by the direct effect of the load acting on the particular member. To analyze this type of action, it is therefore necessary to establish the relations that exist between bending moments and the corresponding deformations. These are most easily expressed in the form of the two moment-area principles.

In Fig. 6.3*a* let the straight line *ab* represent the original, unloaded shape of an elastic member (beam, girder, column, or the like) and the curve *aABb* its deformed shape or elastic curve. In view of the relation $1/r = M/EI$, this distortion is connected with associated bending moments. Let Fig. 6.3*b* represent the corresponding $M/EI$ diagram, that is, the graph of the bending moments divided by the respective values of $EI$. If tangents are drawn to the elastic curve at two points $A$ and $B$, the moment-area principles state:

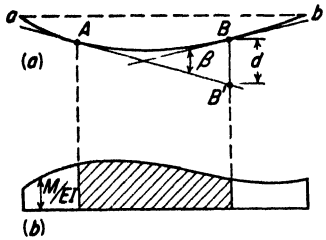


FIG. 6.3

1. *The change of slope $\beta$ between the tangents to the elastic curve at two points A and B is equal to the area of the portion of the $M/EI$ diagram between the two points.*

2. *The distance $d$ of point B from the tangent at point A is equal to the moment about the perpendicular through B of the area of the portion of the $M/EI$ diagram between the two points.*

The corresponding portion of the $M/EI$ diagram is shaded in Fig. 6.3*b*. Note that the distance $d$ is measured perpendicular to the original, undistorted position *ab* of the member (*i.e.*, the line connecting its supports)

For a derivation of these two important principles the reader is referred to any text on the strength of materials

**6.4. Slope-deflection Method.** In computing moments and shears on isolated beams, the ends are usually considered as freely supported or as fixed. However, there is, even in isolated beams, often an intermediate condition. In frames in which the joints are rigid such an intermediate condition is commonly encountered. No matter what the degree of restraint, once the end moments are known, moments and shears at any point of the member can be computed from the equations of statics. As is evident from Art. 6.1, the end slopes $\theta$ depend on (1) the rotation $\alpha$ of the end tangents to the elastic curve from the position they would have if the member were freely supported (see Fig. 6.1*c*); (2) the relative displacement $d$ of the supports and the corresponding rotation $d/l$ (see Fig. 6.1*d*); (3) the rotations $\gamma$ which the direct, transverse loads (if any) would cause if the members were freely supported (see Fig. 6.1*e*). The final end slopes $\theta$ are the sums of these individual rotations (see Fig. 6.1*d* and *e*).

In a freely supported beam, the end moments are zero and the end slopes $\gamma$ depend on the load and can be computed in the usual manner, *e.g.*, by the moment-area principles (see Art. 6.3). In a fixed-ended beam the end slopes are zero and the end moments are known as the "fixed-end moments" $M^f$. If the beam is an intermediate condition (Fig. 6.1*e*), the end slopes $\theta$ obviously have intermediate values (between 0 and $\gamma$) and the end moments $M$ have corresponding intermediate values between $M^f$ and 0.

Once the end slopes $\theta$ are known, the end moments can be determined. Hence, the term "slope deflection" has been applied to this method of analysis, which was developed independently by A. Bendixen in Germany (1914) and Prof. G. A. Maney in the United States (1915).

In the discussion that follows, the relations between the end slopes and the end moments are first investigated separately for the three described types of rotations. Two subscripts are used for moments acting on one end of a member, the first indicating the "near" end on which it acts and the other the "far" end. Thus (see Fig. 6.4) $M_{12}$ indicates the moment acting on end 1 of beam 1-2.

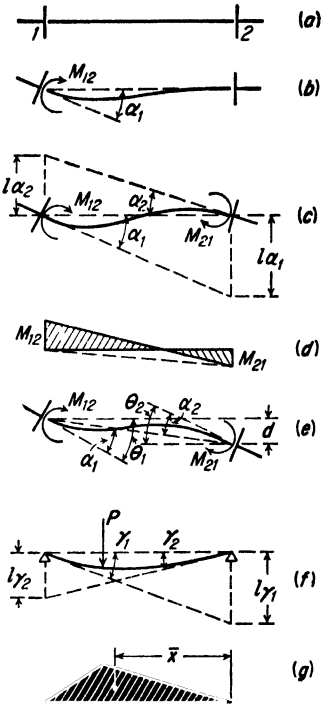


FIG. 6.4

In Fig. 6.4*a*, a member 1-2 has been isolated from the rest of a rigid frame by passing imaginary cuts around the joints at its two ends. The member then represents a freely supported simple beam. The effect of the adjacent frame members on this beam is to impose certain rotations and deflections on its ends, as can be seen from Fig. 6.1. These effects will be studied one by one.

In (*b*) the left support, 1, of the beam has been rotated by the action of moments outside the span 1-2. This rotation causes bending in the beam and produces compression in the upper fiber at 1. In (*c*) the right support, 2, has also been rotated in the same direction by the action of moments outside the beam. This produces tension in the upper fiber at 2. Since the stresses in the upper fiber at 1 and 2 are of opposite sign, there is evidently a point of inflection in the beam, and the moment diagram for the beam will be similar to that shown in (*d*).

Since the angles of rotation are small, the distance at 2 from a tangent

to the axis of the beam at 1 is $l\alpha_1$, where $l$ is the span 1-2. Then by the principle of moment areas, $l\alpha_1$ is equal to the moment of the $M/EI$ diagram between 1 and 2 about 2. This moment is evidently equal to the algebraic sum of the moments of the shaded areas about 2, *i.e.*,

$$l\alpha_1 = \frac{1}{EI}\left(\frac{M_{12}l}{2} \times \frac{2l}{3} - \frac{M_{21}l}{2} \times \frac{l}{3}\right) \quad \text{or} \quad \alpha_1 = \frac{l}{6EI}(2M_{12} - M_{21})$$

Similarly,

$$\alpha_2 = \frac{l}{6EI}(2M_{21} - M_{12})$$

In (*e*) the joint 2 has been moved downward a vertical distance $d$ from its original position, the horizontal distance between joints 1 and 2 remaining constant. This has increased the angles between the tangents to the member 1-2 at 1 and 2 and the horizontal by an amount $d/l = R$. If then $\theta_1 = \alpha_1 + R$ and $\theta_2 = \alpha_2 + R$,

$$\theta_1 = \frac{l}{6EI}(2M_{12} - M_{21}) + R \tag{6.3}$$

and

$$\theta_2 = \frac{l}{6EI}(2M_{21} - M_{12}) + R \tag{6.4}$$

from which

$$M_{12} = \frac{2EI}{l}(2\theta_1 + \theta_2 - 3R) \tag{6.5}$$

and

$$M_{21} = \frac{2EI}{l}(2\theta_2 + \theta_1 - 3R) \tag{6.6}$$

Equations (6.5) and (6.6) are the fundamental slope-deflection equations for the moments at the ends of a member sustaining no intermediate loads, in terms of the relative change in slope and displacement of its ends.

In the foregoing derivation the following sign convention has been used:

1. When the rotation of the end tangent is clockwise, the angle measuring this rotation is a positive angle.
2. The deflection of one end with respect to the other is measured normal to the original position of the member and is positive when measured in the same direction as positive angles.
3. External end moments are positive when they tend to rotate the end of the member on which they act in a clockwise direction.

For illustration, in Fig. 6.4*a* to *e* the sense of all rotations, deflections, and moments is positive.

If a transverse loads acts on a member, such as shown in Fig. 6.4*f*, the corresponding rotations $\gamma$ must be added to $\alpha$ and $R$ to result in the final

rotations $\theta$. From Fig. 6.4*f* it is seen that for vertical load and by the sign convention just given, $\gamma_1$ is positive, while $\gamma_2$ is negative. If $F$ is the area of the moment curve for the simple beam (Fig. 6.4*g*), and $\bar{x}$ the distance of its centroid from support 2, from the second moment-area principle,

$$l\gamma_1 = \frac{F\bar{x}}{EI} \qquad \text{and} \qquad l\gamma_2 = -\frac{F(l-\bar{x})}{EI}$$

If these equations are divided by $l$ and the resulting angles $\gamma$ added to those of Eqs. (6.3) and (6.4), one obtains for the case of members carrying transverse loads

$$\theta_1 = \frac{l}{6EI}(2M_{12} - M_{21}) + R + \frac{F}{EI}\frac{\bar{x}}{l} \tag{6.7}$$

and

$$\theta_2 = \frac{l}{6EI}(2M_{21} - M_{12}) + R - \frac{F}{EI}\left(\frac{l-\bar{x}}{l}\right) \tag{6.8}$$

from which

$$M_{12} = \frac{2EI}{l}(2\theta_1 + \theta_2 - 3R) - \frac{2F}{l^2}(3\bar{x} - l) \tag{6.9}$$

and

$$M_{21} = \frac{2EI}{l}(2\theta_2 + \theta_1 - 3R) + \frac{2F}{l^2}(2l - 3\bar{x}) \tag{6.10}$$

These are the complete equations of the slope-deflection method. By comparison with Eqs. (6.5) and (6.6) it is seen that the last term of Eqs. (6.9) and (6.10) accounts for the influence of tranverse loads on the member under consideration. Its physical significance is easily established in the following manner. If the member of Fig. 6.4 were fixed at its ends, $\theta_1 = \theta_2 = R = 0$. In this case $M_{12}$ and $M_{21}$ would represent the fixed-end moments of the beam. Equating to zero the indicated terms, it is seen that these fixed-end moments are, respectively, equal to the last terms of Eqs. (6.9) and (6.10). These fixed-end moments will be designated by $M^f$.

If, in addition, the rigidity $I/l$ of the member is designated by $K$ and subscripts $n$ and $f$ are used, respectively, for the "near" and "far" end of the member, Eqs. (6.9) and (6.10) can be written in the following general form:

$$M_{nf} = 2EK(2\theta_n + \theta_f - 3R) + M^f_{nf} \tag{6.11}$$

The sign of $M^f_{nf}$ is established from Rule (3), given before. For instance, the load of Fig. 6.4*f* produces fixed-end moments which tend to rotate the left end of the member counterclockwise and the right end clockwise. Hence, the former is negative, the latter positive, as is correctly reflected in Eqs. (6.9) and (6.10). In the general equation (6.11)

they must be substituted with these same signs. A different way of stating the same rule, which some may find easier to remember, is:

4. The fixed-end moment term in Eq. (6.11) is positive if the load on a member, independent of the member, tends to rotate counterclockwise around the end under consideration. Conversely, if the load tends to rotate clockwise around the end, the fixed-end moment is negative.

(It should be noted that Rule 4 which applies, in particular, to fixed-end moments is only a different way of stating the general rule, Rule 3.)

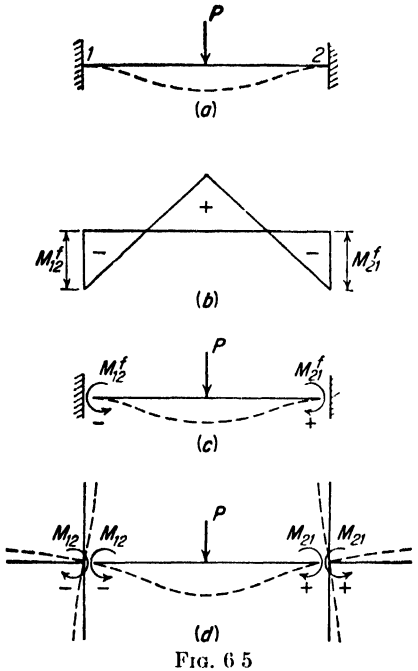



FIG. 6.5

It will be realized that this sign convention of the slope-deflection method differs from the usual convention of signs for beam moments. It is customary to designate a beam moment as positive if it causes the beam to bend in such a manner that the concave side faces upward. By this convention the signs of the bending moments of a fixed beam are those of Fig. 6.5*b*. If the beam is cut at the ends, the action of the fixed supports must be replaced by the fixed-end moments $M_{12}^f$ and $M_{21}^f$, acting as shown in Fig. 6.5*c*. Both these moments are negative by the beam sign convention. It is seen that $M_{12}^f$ acts counterclockwise *on the member* and is therefore also negative by the slope-deflection sign convention. $M_{21}^f$, however, is seen to act clockwise *on the member* and, consequently, is a positive moment by the slope-deflection convention. If this beam were a member of a rigid frame, by the principle of action and reaction it would exert equal and opposite moments on the joints (Fig. 6.5*d*). In consequence, the slope-deflection sign convention indicates that a moment *acting on a joint* is positive if it tends to rotate *the joint* counterclockwise, negative if clockwise. (The reader will find that some texts on structural analysis use a sign convention for rotations, deflections, and moments other than that given in Rules 1 to 4. The choice of such a convention is entirely a matter of the designer's preference. One is as good as the other, provided it is consistent in itself and is used consistently.) The determination of the correct sign is greatly facilitated by the use of rough sketches of the deflected shape of the structure, such as Figs. 6.1 and 6.5. By means of such sketches the signs of the moments

throughout the frame are easily determined even before an analysis is begun. In fact, the approximate shape of the final moment diagrams can be obtained directly from such sketches, observing that by the beam sign convention a moment changes sign when the curvature changes direction, as is seen directly from Fig. 6.5*b*. Sketches of this type, therefore, are an inestimable aid to the designer.

The slope-deflection equations enable one to compute the moment at both ends of all members of a rigid-joint structure by making use of the following two obvious requirements:

1. Condition of joint equilibrium: The sum of all moments $M_{nf}$ acting on a joint is zero.

2. Condition of joint continuity: The rotations $\theta_n$ of the tangents of all members meeting at a joint are equal.

In these statements $n$ refers to the "near end," *i.e.*, the end of the member at the joint in question. For an illustration of the second condition, see joint 4 in Fig. 6.1*a*.

**6.5. Restraint Conditions and Fixed-end Moments.** The general slope-deflection equations, (6.7) to (6.11), were derived to hold for all conceivable situations, that is, for any relative magnitudes of $\theta$ and $M$ at both ends, and for any value of $R$. The value of $\theta$ and $M$ in any particular case depends on the conditions of restraint at both ends,

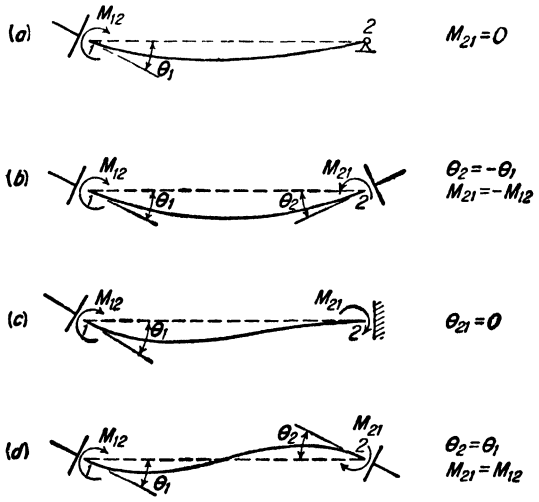



FIG. 6.6

hinged, fixed, or partially restrained. In addition to the general case of partial, unequal restraint at both ends, for which the general equations were derived, there are four special conditions of restraint which are often met in structural analysis. These are shown in Fig. 6.6 and

are (*a*) far end hinged, (*b*) symmetrical bending, (*c*) far end fixed, and (*d*) antisymmetrical bending. $R$ equal to zero is assumed for these four simple cases, and the corresponding conditions are shown on the right of Fig. 6.6. By making use of these conditions, simplified equations are easily derived for these four frequent cases.

For instance, for Case *a*, noting that $M_{21} = 0$ at the hinged end, Eqs. (6.9) and (6.10) become

$$\begin{aligned} M_{12} &= 2EK(2\theta_1 + \theta_2) - M^f_{12} \\ 0 &= 2EK(2\theta_2 + \theta_1) + M^f_{21} \end{aligned}$$

From the second equation

$$\theta_2 = \left(\frac{\theta_1}{2} + \frac{M^f_{21}}{4EK}\right)$$

If this value is substituted in the first equation, one obtains

$$M_{12} = 3EK\theta_1 - \left(M^f_{12} + \frac{M^f_{21}}{2}\right)$$

In particular, if no loads act on the member proper so that bending is caused only by the end moments applied through the joints from adjacent members, $M^f_{12} = M^f_{21} = 0$. In this case the last equation indicates

$$M_{12} = 3EK\theta_1$$

while the next to last results in

$$\theta_2 = \frac{\theta_1}{2}$$

In a similar manner, such special, simple relations are easily derived for the other three cases. The results are tabulated in Table 6.2.

If the far end of a member, say end 2, is hinged and subject to joint deflection, one has

$$\begin{aligned} M_{12} &= 2EK(2\theta_1 + \theta_2 - 3R) \\ 0 &= 2EK(2\theta_2 + \theta_1 \quad 3R) \end{aligned}$$

from which

$$M_{12} = 3EK\theta_1 - 3EKR$$

Consequently, *Table 6.2 can also be used if joint deflections are present*, by adding $-6EKR$ to the moments $M_{12}$ and $M_{21}$ for Cases *b* to *d*, and $-3EKR$ to $M_{12}$ of Case *a*.

The use of these relations is easily seen from the following problem. Suppose it were required to find the moment at the fixed end of a beam whose other end is hinged and which carries transverse load. The condition that the left end is fixed while the right end is hinged is obtained by making $\theta_1 = 0$ in Case *a* of Table 6.2. Then $M_{12}$ is the moment at

## TABLE 6.2. Moments for Different Conditions of Restraint

| Case (see Fig. 6.6) | Relation between $\theta$ angles, no load on span | Moment at near end in terms of $\theta_1$, no load on span | Moment at far end in terms of $\theta_1$ and $\theta_2$, no load on span | Moment at near end in terms of $\theta_1$, intermediate loads on span | Moment at far end in terms of $\theta_1$ and $\theta_2$, intermediate loads on span |
|---|---|---|---|---|---|
| *a* Far end hinged | $\theta_2 = -\frac{1}{2}\theta_1$ | $M_{12} = 3EK\theta_1$ | $M_{21} = 0$ | $M_{12} = 3EK\theta_1 - \left(M'_{12} + \frac{M'_{21}}{2}\right)$ | $M_{21} = 0$ |
| *b* Symmetrical bending | $\theta_2 = -\theta_1$ | $M_{12} = 2EK\theta_1$ | $M_{21} = -2EK\theta_1$<br>$M_{21} = 2EK\theta_2$ | $M_{12} = 2EK\theta_1 - M'_{12}$ | $M_{21} = -2EK\theta_1 + M'_{21}$<br>$M_{21} = 2EK\theta_2 + M'_{21}$ |
| *c* Far end fixed | $\theta_2 = 0$ | $M_{12} = 4EK\theta_1$ | $M_{21} = 2EK\theta_1$ | $M_{12} = 4EK\theta_1 - M'_{12}$ | $M_{21} = 2EK\theta_1 + M'_{21}$ |
| *d* Antisymmetrical bending | $\theta_2 = \theta_1$ | $M_{12} = 6EK\theta_1$ | $M_{21} = 6EK\theta_1$<br>$M_{21} = 6EK\theta_2$ | $M_{12} = 6EK\theta_1 - M'_{12}$ | $M_{21} = 6EK\theta_1 + M'_{21}$<br>$M_{21} = 6EK\theta_2 + M'_{21}$ |

the fixed end of that beam and is seen to be equal to

$$M_{12} = -\left(M'_{12} + \frac{M'_{21}}{2}\right)$$

This equation indicates that the moment at the fixed end of a beam whose other end is hinged can be expressed in terms of the fixed-end moments of that beam, *i.e.*, the end moments for both ends fixed.

The practical application of the slope-deflection equation obviously requires the knowledge of the fixed-end moments $M'_{nf}$ for the particular design loading. It was shown that these fixed-end moments are equal to the last terms of Eqs. (6.9) and (6.10) from which they can be computed for any type of loading. For the most frequent arrangements of load, values of $M'_{12}$ and $M'_{21}$ for the left and right end, respectively, are given in Table 6.3. The table also contains for the same loadings the moments at the fixed end if the other end is hinged, computed from the last equation.

**6.6. Application of the Slope-deflection Method.** The practical workings of the slope-deflection method are most easily understood by its application to concrete examples of beams and frames. Each of the following five problems illustrates and develops one of the main features of this method. These features are summarized at the end of this article.

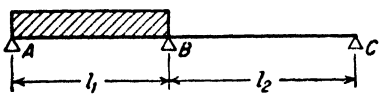



FIG. 6.7

*Illustrative Examples.* I. Determine the moments in the two-span continuous beam of Fig. 6.7. In this case only the ends of both spans at $B$ are partially restrained by an unknown amount. The outer ends are hinged, so that, instead of the general equation (6.11), the special equations of Case *a* of Table 6.2 can be applied. Consequently, with the pertinent value of $M'_{12} + M'_{21}/2$ obtained from Table 6.3,

$$M_{BA} = 3EK_1\theta_B + \frac{w_1 l_1^2}{8}$$

and

$$M_{BC} = 3EK_2\theta_B$$

Since there is equilibrium at the joint $B$,

$$M_{BA} + M_{BC} = 0$$

or

$$(3EK_1 + 3EK_2)\theta_B + \frac{w_1 l_1^2}{8} = 0$$

and

$$\theta_B = -\frac{1}{3EK_1 + 3EK_2} \cdot \frac{w_1 l_1^2}{8}$$

**TABLE 6.3. Fixed-end Moments $M^f$ and Moments at Fixed Ends for Beams with Other End Hinged, for Various Types of Loading**

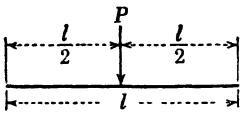
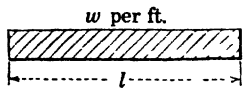
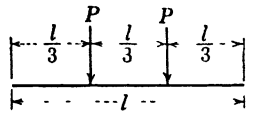
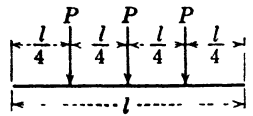
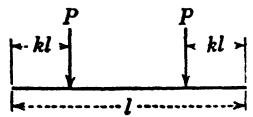
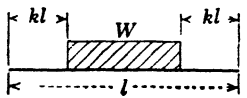
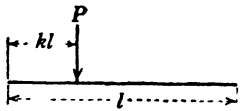
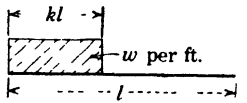
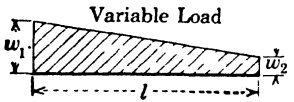
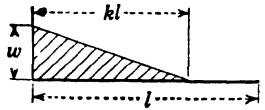
| Type of loading | Left fixed-end moment $M^f_{12}$ | Right fixed-end moment $M^f_{21}$ | Moment at left fixed end, right end hinged $M^f_{12} + M^f_{21}/2$ | Moment at right fixed end; left end hinged $M^f_{21} + M^f_{12}/2$ |
|---|---|---|---|---|
|   | $\frac{1}{8}Pl$ | $\frac{1}{8}Pl$ | $\frac{3}{16}Pl$ | $\frac{3}{16}Pl$ |
|   | $\frac{1}{12}wl^2$ | $\frac{1}{12}wl^2$ | $\frac{1}{8}wl^2$ | $\frac{1}{8}wl^2$ |
|   | $\frac{2}{9}Pl$ | $\frac{2}{9}Pl$ | $\frac{1}{3}Pl$ | $\frac{1}{3}Pl$ |
|   | $\frac{5}{16}Pl$ | $\frac{5}{16}Pl$ | $\frac{15}{32}Pl$ | $\frac{15}{32}Pl$ |
|   | $Pkl(1-k)$ | $Pkl(1-k)$ | $\frac{3}{2}Pkl(1-k)$ | $\frac{3}{2}Pkl(1-k)$ |

**TABLE 6.3. Fixed-end Moments $M^f$ and Moments at Fixed Ends for Beams with Other End Hinged, for Various Types of Loading** (*Continued*)

| Type of loading | Left fixed-end moment $M^f_{12}$ | Right fixed-end moment $M^f_{21}$ | Moment at left fixed end, right end hinged $M^f_{12} + M^f_{21}/2$ | Moment at right fixed end; left end hinged $M^f_{21} + M^f_{12}/2$ |
|---|---|---|---|---|
|  | $\frac{Wl}{12}(1 + 2k - 2k^2)$ | $\frac{Wl}{12}(1 + 2k - 2k^2)$ | $\frac{Wl}{8}(1 + 2k - 2k^2)$ | $\frac{Wl}{8}(1 + 2k - 2k^2)$ |
|  | $Pkl(1 - k)^2$ | $Pk^2(1 - k)$ | $\frac{1}{2}P(1 - k)^2(2 - k)kl$ | $\frac{1}{2}P(1 - k^2)k^2l$ |
|  | $wl^2k^2(3k^2 - 8k + 6)$ | $\frac{wl^2k^3}{12}(4 - 3k)$ | $\frac{1}{8}wl^2k^2(2 - k)^2$ | $\frac{1}{8}wl^2k^2(2 - k^2)$ |
|  | $\frac{l^2}{60}(3w_1 + 2w_2)$ | $\frac{l^2}{60}(2w_1 + 3w_2)$ | $\frac{l^2}{120}(8w_1 + 7w_2)$ | $\frac{l^2}{120}(7w_1 + 8w_2)$ |
|  | $\frac{wk^2l^2}{60}(10 - 10k + 3k^2)$ | $\frac{wk^3l^2}{60}(5 - 3k)$ | $\frac{wk^2l^2}{120}(20 - 15k + 3k^2)$ | $\frac{wk^2l^2}{120}(10 - 3k^2)$ |

Substituting the value of $\theta_B$ in the equations for $M_{BA}$ and $M_{BC}$,

$$M_{BC} = -M_{BA} = -\frac{K_2}{K_1 + K_2}\left(\frac{w_1 l_1^2}{8}\right)$$

If the load covers the span $l_2$ instead of the span $l_1$, a similar analysis gives

$$M_{BA} = -M_{BC} = \frac{K_1}{K_1 + K_2}\left(\frac{w_2 l_2^2}{8}\right)$$

For both spans sustaining loads $w_1$ and $w_2$, respectively,

$$M_{BC} = -M_{BA} = -\left[\frac{K_2(w_1 l_1^2) + K_1(w_2 l_2^2)}{8(K_1 + K_2)}\right]$$

which for equal loads, equal spans, and equal moments of inertia becomes $-\frac{1}{8}wl^2$, the negative moment over the center support of a beam of two equal spans resting freely on its supports.

The maximum positive span moment does not occur at the middle of the loaded span since end moments act only on ends $B$ of either span. Its value is found from the condition that the maximum moment occurs at the point where the shear force is zero. Taking as an illustration the case of equal loads, spans, and moments of inertia,

$$R_A = \frac{wl}{2} - \frac{(wl^2)}{8}\left(\frac{1}{l}\right) = \frac{3wl}{8}$$

Designating by $x$ the distance from $A$ at which $V = 0$,

$$V = \frac{3wl}{8} \qquad wx = 0$$

from which $x = 3l/8$. At this point the maximum positive span moment is

$$M = \frac{3wl}{8}\frac{3l}{8} - w\frac{3l}{8}\frac{3l}{2 \cdot 8} = \frac{9wl^2}{128}$$

Using the beam sign convention, the support moment is $M_B = -0.125wl^2$ and the maximum positive moment $+M_{max} = 0.070wl^2$, which are the values recorded in Table 6.1.

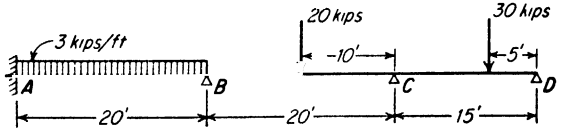



FIG. 6.8

II. Determine the support moments in the three-span, continuous beam of Fig. 6.8. In this beam span, $AB$ is fixed at $A$, corresponding to

Case *c* of Table 6.2, while span *CD* is hinged at *D*, corresponding to Case *a*. Span *BC* is partially restrained at both ends, so that for this span Eq. (6.11) must be used.

All moments of inertia are assumed equal. To simplify computations, it is convenient to express all values of $K$ in terms of the $K$ value for one of the members. Thus

$$K_{AB} = \frac{I}{20} \qquad K_{BC} = \frac{I}{20} = K_{AB} \qquad K_{CD} = \frac{I}{15} = 1.35K_{AB}$$

Using the information of Tables 6.2 and 6.3, the fixed-end moments

$$M^f_{AB} = -M^f_{BA} = -100 \text{ ft-kips}; \quad M^f_{BC} = -M^f_{CB} = -50 \text{ ft-kips};$$
$$M^f_{CD} = -66.7 \text{ ft-kips}$$

and

$$M_{AB} = 2K_{AB}\theta_B - 100$$
$$M_{BA} = 4K_{AB}\theta_B + 100$$
$$M_{BC} = 2K_{BC}(2\theta_B + \theta_C) - 50 = 4K_{AB}\theta_B + 2K_{AB}\theta_C - 50$$
$$M_{CB} = 2K_{BC}(2\theta_C + \theta_B) + 50 = 4K_{AB}\theta_C + 2K_{AB}\theta_B + 50$$
$$M_{CD} = 3K_{CD}\theta_C - 66.7 = 4K_{AB}\theta_C - 66.7$$

The condition that joints *B* and *C* be in equilibrium requires

$$M_{BA} + M_{BC} = 0, \text{ i.e., } 8K_{AB}\theta_B + 2K_{AB}\theta_C + 50 = 0$$
$$M_{CB} + M_{CD} = 0, \text{ i.e., } 2K_{AB}\theta_B + 8K_{AB}\theta_C - 16.7 = 0$$

Solving these two simultaneous equations, one obtains

$$K_{AB}\theta_B = -7.23 \qquad K_{AB}\theta_C = +3.89$$

The end moments of the three spans are then obtained by substituting these values in the equations for $M_{AB}$, $M_{BA}$, etc., for example,

$$M_{AB} = 2(-7.23) - 100 = -114.5 \text{ ft-kips}$$

Similarly $M_{BA} = +71.1$, $M_{BC} = -71.1$, $M_{CB} = +51.1$, $M_{CD} = -51.1$ ft-kips. The signs of these resulting moments, of course, are those of the slope-deflection convention. In terms of the beam sign convention all support moments are negative.

III. The portal frame of Fig. 6.9*a* is subject to a wind load of 20 kips; determine the moments. As in Example II it is convenient to express the rigidities of all members as multiples of one value of $K$. Thus, for the given frame, $K_1 = I_1/20$, $K_2 = 4I_1/40 = 2K_1$, $K_3 = K_1$.

Since changes of length of members due to tension or compression forces cause only negligible effects on the magnitude of moments, it can be assumed that joints *B* and *C* displace the same amount *d* as shown on Fig. 6.9*a*. Consequently, $R = d/20$ for both *AB* and *CD*. Since the frame is geometrically symmetrical and the top displacements of both

legs are equal, both legs distort in precisely the same way and amount, causing the beam $BC$ to deflect in an antisymmetrical shape. (This would not be true if, for example, $I_3 \neq I_1$, or if the lengths of both legs were not equal.) Hence, Case $d$ of Table 6.2 applies for member $BC$, while Case $c$ applies for $AB$ and $CD$, since the bottom ends are fixed. The fixed-end moments are zero for all members, since no loads are applied between ends, but sidesway $R$ is present in members $AB$ and $CD$. With this information the slope-deflection equations are

$$M_{AB} = M_{DC} = 2EK_1\theta_B - 6EK_1R$$
$$M_{BA} = M_{CD} = 4EK_1\theta_B - 6EK_1R$$
$$M_{BC} = M_{CB} = 6EK_2\theta_B = 12EK_1\theta_B$$

The condition of joint equilibrium, $M_{BA} + M_{BC} = 0$, gives

$$16EK_1\theta_B - 6EK_1R = 0$$

To obtain a second equation for the unknown quantities $\theta_B$ and $R$, use is made of the fact that the sum of the horizontal reactions of both legs must be equal and opposite to the applied horizontal load. In Fig. 6.9*b* are shown the moments and horizontal forces on each leg. At this stage the moments and forces are not known either as to magnitude or to direction. For this reason they are all shown in the positive sense, *i.e.*, moments clockwise, and forces so as to result in positive deflections. In this manner one ensures that the moments and forces resulting from the analysis are obtained with correct sign. Considering the equilibrium of each leg,

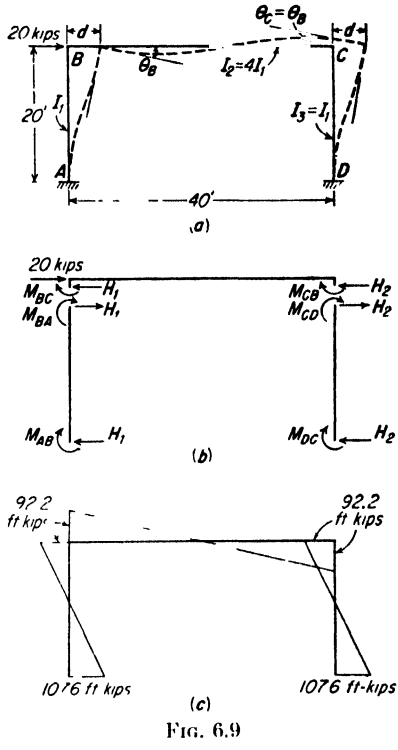



FIG. 6.9

$$H_1 = -\frac{M_{AB} + M_{BA}}{20}$$

$$H_2 = -\frac{M_{DC} + M_{CD}}{20}$$

The shears or reactions at the bottom of the legs must be in equilibrium with the applied horizontal load or

$$H_1 + H_2 - 20 = 0$$

This expression is often referred to as the shear or bent equation. Substitution of the values $H_1$ and $H_2$ results in

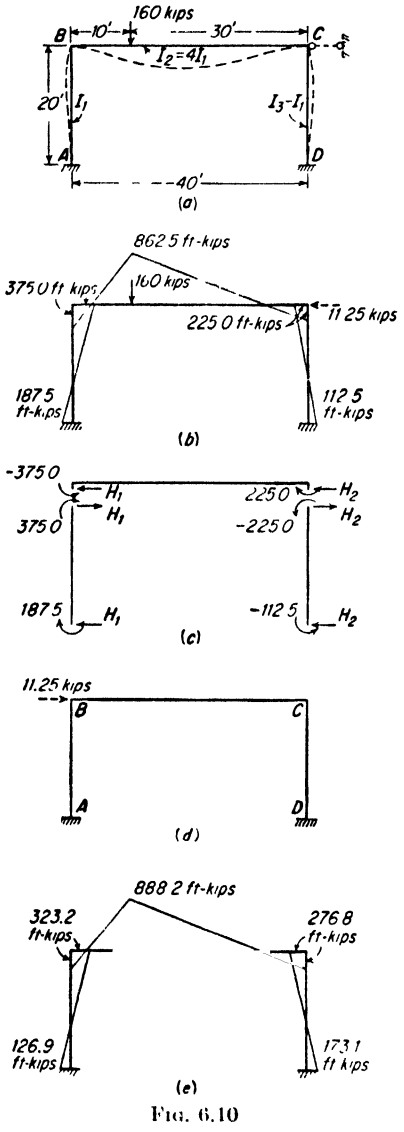



Fig. 6.10

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 400 = 0$$

If the values of the moments are substituted in this expression, one obtains a second equation in term of $\theta_B$ and $R$, namely,

$$3EK_1\theta_B - 6EK_1R + 100 = 0$$

Simultaneous solution with

$$16EK_1\theta_B - 6EK_1R = 0$$

gives

$$EK_1\theta_B = 7.68 \qquad EK_1R = 20.5$$

If these values are substituted in the original moment equations, the following moments result:

$$M_{AB} = M_{DC} = -107.6 \text{ ft-kips}$$
$$M_{BA} = M_{CD} = -92.2 \text{ ft-kips}$$
$$M_{BC} = M_{CB} = 92.2 \text{ ft-kips}$$

The moment diagram is as shown in Fig. 6.9c.

IV. The frame of Example III is loaded as shown in Fig. 6.10a. From Table 6.3, with $k = \frac{1}{4}$, the absolute values of the fixed-end moments are $M^f_{BC} = 900$ ft-kips and $M^f_{CB} = 300$ ft-kips. With proper attention to signs, and noting that the bottoms of both legs are fixed, the slope-deflection equations are

$$M_{AB} = 2EK_1\theta_B$$
$$M_{BA} = 4EK_1\theta_B$$
$$M_{BC} = 8EK_1\theta_B + 4EK_1\theta_C - 900$$
$$M_{CB} = 8EK_1\theta_C + 4EK_1\theta_B + 300$$
$$M_{CD} = 4EK_1\theta_C$$
$$M_{DC} = 2EK_1\theta_C$$

Joint equilibrium requires

$$M_{BA} + M_{BC} = 0 \qquad \text{and} \qquad M_{CB} + M_{CD} = 0$$

which, upon substitution of the moment values, results in the two following simultaneous equations:

$$12EK_1\theta_B + 4EK_1\theta_C - 900 = 0$$
$$4EK_1\theta_B + 12EK_1\theta_C + 300 = 0$$

which give

$$EK_1\theta_B = 93.75 \qquad \text{and} \qquad EK_1\theta_C = -56.25$$

If these are substituted in the original moment equations, the following values result:

$$M_{AB} = 187.5 \qquad M_{BA} = 375.0 \qquad M_{BC} = -375.0$$
$$M_{CB} = 225.0 \qquad M_{CD} = -225.0 \qquad M_{DC} = -112.5$$

The corresponding moment diagram is shown in Fig. 6.10*b*.

Since no horizontal loads act on the frame, equilibrium requires that the sum of the horizontal reactions or shears at the bottom of the legs be zero. In Fig. 6.10*c* are shown the moments acting at the ends of the legs. Since these moments are known from previous computations, they are shown in the correct sense, while the as yet unknown shears $H$ are indicated as acting in the positive direction. Considering the equilibrium of both legs, it is seen that

$$H_1 + H_2 = -\frac{(375.0 + 187.5)}{20} + \frac{(225.0 + 112.5)}{20}$$
$$= -11.25 \text{ kips} \neq 0$$

Consequently, the frame is not in equilibrium under the computed distribution of moments. It would be in equilibrium if an additional support were present at the level of the beam, as shown in broken lines in Fig. 6.10*a*, which would prevent horizontal displacement at that level. The reaction in that fictitious support must equilibrate the sum of the shears in the legs, and is therefore as shown in broken lines on Fig. 6.10*b*. This condition is known as "sidesway prevented" or "no sidesway." If such a support were physically present, as it sometimes is in bridge frames, the moment diagram of Fig. 6.10*b* would be correct.

In a free-standing frame, however, no such horizontal support exists. To realize this condition of zero horizontal load, it is necessary to apply to the frame a horizontal force equal and opposite to the fictitious, horizontal reaction. This means that the correct moments in the free-standing frame are the sum of those computed for no sidesway (Fig. 6.10*b*) plus those resulting from a horizontal force equal to the sum of the horizontal leg reactions (Fig. 6.10*d*).

The latter loading is identical in character with that of Example III, except that the load is now 11.25 kips instead of 20 kips. No new analysis is needed since all moments are simply (11.25/20) times the moments

of Example III. Superposition of these sidesway moments on those previously obtained for no sidesway results in the final moments as follows:

$$M_{AB} = 187.5 - 107.6 \times \frac{11.25}{20} = 126.9$$

$$M_{BA} = 375.0 - 92.2 \times \frac{11.25}{20} = 323.2$$

$$M_{BC} = -375.0 + 92.2 \times \frac{11.25}{20} = -323.2$$

$$M_{CB} = 225.0 + 92.2 \times \frac{11.25}{20} = 276.8$$

$$M_{CD} = -225.0 - 92.2 \times \frac{11.25}{20} = -276.8$$

$$M_{DC} = -112.5 - 107.6 \times \frac{11.25}{20} = -173.1$$

It can easily be checked that the frame is now in equilibrium, *i.e.*, the sum of the moments at each joint is zero, and the sum of the horizontal reactions at the bottom is zero. These checks should always be made to guard against errors.

The final moment diagram is shown in Fig. 6.10*c*. Comparison with Fig. 6.10*b* shows the sizable difference which obtains between moments computed for no sidesway and those actually occurring in a free-standing frame, *i.e.*, with free sidesway. A correct evaluation of the actual sidesway conditions is therefore a prerequisite for correct analysis. The influence of sidesway is the more pronounced the more the frame or the distribution of loads, or both, depart from symmetry. Indeed, if the given portal frame were symmetrically loaded, sidesway would not occur and the moments determined for no sidesway would be correct. It is the off-center location of the load of Fig. 6.10*a* which causes the frame to sidesway to the right. The direction of this sidesway is that corresponding to the horizontal force of Fig. 6.10*d*, which can also be realized from the fact that to prevent this sidesway it was found that a reaction acting to the left (Fig. 6.10*b*) would be required.

It is of some interest to realize the actual magnitude of such horizontal sidesway displacements. In Example III it was found that for a horizontal load of 20 kips, $EK_1R = 20.5$. Since $R = d/20$, one obtains for the horizontal load of 11.25 kips of Fig. 6.10*c*, the sidesway displacement $d = 20(20.5/EK_1)(11.25/20)$ if all quantities are expressed in kips and feet. If a column section of 24 by 24 in. is required to resist the computed moments and loads, and if $E = 3 \times 10^6$ psi, one finds, with proper attention to units, that the sidesway $d = 0.127$ in., *i.e.*, just over $\frac{1}{8}$ in. It is seen that even such small sidesway displacements cause sizable changes

in moments. Therefore the assumption of "no sidesway" is justified only if horizontal displacement of the actual structure is prevented by effective connection to a practically unyielding mass, such as solid rock, or to another structure which is very much more rigid than the frame proper.

V. Direct analysis of the frame of Example IV. To obtain the "sidesway correction moments" in Example IV, use was made of the previously determined moments caused by a horizontal joint load (Example III). This procedure is advisable if a frame is subject to a variety of load conditions, such as dead load, several types of live load, wind, etc. In this case only one determination for sidesway must be made, which is best done for some convenient horizontal load, such as unity, or 10 kips. Sidesway moments for any of the various load conditions are then found by simple proportion, as in Example IV.

If only a single loading is to be investigated, it is frequently simpler to use the complete slope-deflection equations directly, including the sidesway terms. For the frame of Example IV, remembering that sidesway displacements do not occur in member $BC$, and that $R$ is the same for $AB$ and $CD$, these equations are

$$\begin{aligned}
M_{AB} &= 2EK_1\theta_B - 6EK_1R \\
M_{BA} &= 4EK_1\theta_B - 6EK_1R \\
M_{BC} &= 8EK_1\theta_B + 4EK_1\theta_C - 900 \\
M_{CB} &= 8EK_1\theta_C + 4EK_1\theta_B + 300 \\
M_{CD} &= 4EK_1\theta_C - 6EK_1R \\
M_{DC} &= 2EK_1\theta_C - 6EK_1R
\end{aligned}$$

The requirement of equilibrium results in the joint equations

$$M_{BA} + M_{BC} = 0 \qquad \text{and} \qquad M_{CB} + M_{CD} = 0$$

If only vertical loads act on the frame, as in Example IV, the shear or bent equation is

$$\frac{M_{AB} + M_{BA}}{20} + \frac{M_{CD} + M_{DC}}{20} = 0$$

If, in addition, a horizontal force, such as the 20 kips of Example III, acts simultaneously with the vertical load, the shear or bent equation is

$$\frac{M_{AB} + M_{BA}}{20} + \frac{M_{CD} + M_{DC}}{20} + 20 = 0$$

Substitution of the expressions for $M_{AB}$, $M_{BA}$, etc., into the joint equations and the bent equation results in three simultaneous equations for $\theta_B$, $\theta_C$, and $R$. Once these are determined, the moments are computed as usual.

*Example I* shows that in structures without sidesway the requirement of joint equilibrium (joint equation) permits one to determine the rotation of the particular joint. Once this is known, the moments are determined from the basic, slope-deflection equations.

Example II illustrates the fact that, without sidesway, as many simultaneous equations are obtained as there are unknown joint rotations, except that rotations of hinged ends need not be determined if use is made of the information of Table 6.2. Example III shows that if horizontal forces cause sidesway in a frame, use must be made of the requirement that the sum of horizontal forces and reactions is zero (shear or bent equation) to obtain the necessary number of equations. The example also shows the simplification which obtains if Table 6.2 is used in case of antisymmetrical deformation. Example IV demonstrates that sidesway is caused not only by horizontal forces but also by asymmetry of loading and/or shape. It gives a method by which the effect of sidesway is determined by a separate sidesway correction, superposed on the solution of the frame without sidesway. Finally, Example V shows that instead of such a separate correction the effect of sidesway can be included directly in the first analysis. It also illustrates the fact that in frames with sidesway one obtains as many simultaneous equations as there are unknown joint rotations $\theta$ and unknown sidesway displacements $d$.

## Problems

For the following structures (*a*) compute the end moments, (*b*) draw complete moment and shear diagrams, and (*c*) sketch deflected shape

1 The beam of Fig. 6.8, loaded as shown, but with the support $C$ settled $\frac{1}{4}$ in. relative to the line $ABD$. Assume a cross section of 12 by 24 in

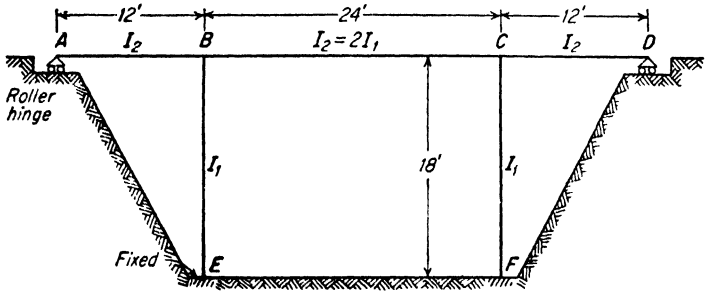



FIG. 6.11. Rigid-frame bridge.

2. The bridge frame of Fig. 6.11, subject to a uniform dead load of 1 kip per ft over the entire length. (*Hint:* the symmetry of the structure can be used to simplify computations.)

3. The bridge frame of Fig. 6.11, subject to a horizontal force of 1 kip, applied along the axis of the beam. (Such forces are caused by the inertia of vehicles when brakes are applied.)

4. The bridge frame of Fig. 6.11, subject to a uniform live load of 1 kip per ft on span $AB$.

**6.7. Moment-distribution Method.** The slope-deflection method is a powerful and relatively simple tool for determining moments in continuous structures, as was illustrated in Art. 6.6. It was shown that this method results in as many simultaneous equations as there are unknown joint rotations and translations (not counting the rotations of hinged joints). If there are not more than three or four such unknowns, the method does not present any practical difficulties. However, in

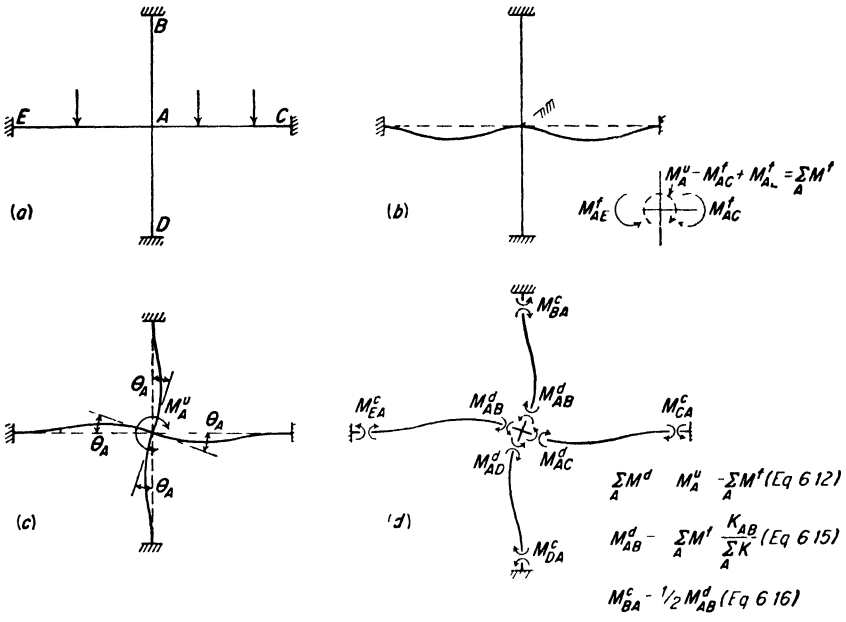



FIG. 6.12. Moment distribution at one joint (procedure and formulas).

many structures, such as multistory, multibay frames, there may be anywhere from five to twenty or more joints, each with an unknown rotation and many with additional, unknown translations. If the slope-deflection method were used to analyze such structures, the work and time involved in solving large numbers of simultaneous equations would be prohibitive. To overcome this difficulty, Prof. Hardy Cross developed and published the moment-distribution method in 1932. In view of the great saving of time achieved by it, this method is now very widely used. It is different from most other types of analysis in that no equations are set up and solved unless sidesway is involved. Instead, the final moments are obtained by successive, numerical approximation.

Suppose that the four members shown on Fig. 6.12*a* were part of a building frame. For the time being the ends $B$, $C$, $D$, and $E$ will be considered fixed. A temporary fixation against rotation is now introduced at joint $A$, indicated symbolically in Fig. 6.12*b*. For this state all moments in the structure are known, since the moments at both ends of both girders are simply the fixed-end moments for the respective spans and can be obtained from Table 6.3, while the columns are not subject to any moments.

In general, $M^f_{AC}$ and $M^f_{AE}$ will not be equal and opposite so that an *unbalanced moment* $M^u_A = M^f_{AC} + M^f_{AE}$ or in general

$$M^u_A = \Sigma_A M^f$$

acts on the joint. The right side of this equation stands for the sum of all fixed-end moments acting on the joint. This unbalanced moment is resisted by the temporary, fictitious fixation.

If this fixation is now removed, *i.e.*, the joint is released, it will rotate in the direction of $M^u_A$, as indicated in Fig. 6.12*c*, until equilibrium is established. Further rotation is then resisted by the bending rigidity of all members meeting at the joint, each of which exerts on the joint a moment that counteracts the rotation, as shown on Fig. 6.12*d*. For equilibrium to establish itself, the sum of these moments must be equal and opposite to the unbalanced moment, so that the sum of all moments at the joint is zero. Hence

$$\Sigma_A M^d = M^d_{AB} + M^d_{AC} + M^d_{AD} + M^d_{AE} = -\Sigma_A M^f \qquad (6.12)$$

These moments are known as the *distributed moments*, since they represent the fractions of the balancing moment, $-\Sigma_A M^f$, distributed to the respective abutting members.

In Fig. 6.12 all moments are shown in the direction in which they actually act in this particular example, regardless of sign. In connection with the moment-distribution method, *the same sign convention will be used as was established for the slope-deflection method in Art.* 6.4. In particular, the distributed moments in the free-body sketches of the members and joints of Fig. 6.12*d* are all seen to be positive in this particular case, since they act clockwise on their respective members, or counterclockwise on joint $A$.

Since joint $A$ is rigid, the angles $\theta_A$ of all members (Fig. 6.12*c*) must be equal. From Case *c* of Table 6.2, for a member whose far end is fixed, the angle at the near end

$$\theta_1 = \frac{M_1}{4EK} \qquad (6.13)$$

Hence, canceling $4E$, the condition that all end angles are equal can be written as

$$\frac{M^d_{AB}}{K_{AB}} = \frac{M^d_{AC}}{K_{AC}} = \frac{M^d_{AD}}{K_{AD}} = \frac{M^d_{AE}}{K_{AE}} \tag{6.14}$$

Using this relation to express all distributed moments in Eq. (6.12) in terms of $M^d_{AB}$, one obtains, if the first term is multiplied by $(K_{AB}/K_{AB})$,

$$M^d_{AB}\left(\frac{K_{AB}}{K_{AB}} + \frac{K_{AC}}{K_{AB}} + \frac{K_{AD}}{K_{AB}} + \frac{K_{AE}}{K_{AB}}\right) = -\Sigma_A M^f$$

from which

$$M^d_{AB} = -\Sigma_A M^f \frac{K_{AB}}{\Sigma_A K} \tag{6.15}$$

Similarly,

$$M^d_{AC} = -\Sigma_A M^f \frac{K_{AC}}{\Sigma_A K}, \text{ etc.}$$

The quantities

$$\frac{K_{AB}}{\Sigma_A K}, \text{ etc.}$$

are known as the *distribution factors* since they indicate the fraction of the balancing moment $-\Sigma M^f$ which is distributed to each individual member.

The distributed moments, acting at the near end of each member (at the joint), cause it to bend as shown on Fig. 6.12*d* and consequently result in a moment at the fixed, far end. By comparing $M_{12}$ and $M_{21}$ of Case *c*, Table 6.2, it is seen that the moment $M_{21}$ at the fixed, far end is one-half of the moment $M_{12}$ applied at the near end. Hence

$$\begin{aligned} M^c_{BA} &= \tfrac{1}{2} M^d_{AB} \\ M^c_{CA} &= \tfrac{1}{2} M^d_{AC} \end{aligned} \tag{6.16}$$

The moments $M^c_{AB}$, etc., are known as the *carry-over moments*, since they are carried over from the near to the far end, and the factor $\frac{1}{2}$ is known as the *carry-over factor*. From the free-body sketches of Fig. 6.12*d* it is seen that the carry-over moments have the same sign as the distributed moments at the other end of the member. In the figure they are positive, since they act clockwise on their respective members, or counterclockwise on their joints $B$ to $E$.

The *final moments* at either end of any member, then, are the sums of the original fixed-end moments (where such exist, as in the girders) plus the distributed moments or the carry-over moments, as the case may be. Thus,

$$M_{AC} = M^f_{AC} + M^d_{AC} = M^f_{AC} - \Sigma_A M^f \frac{K_{AC}}{\Sigma_A K}$$

$$M_{AB} = M^d_{AB} = -\Sigma_A M^f \frac{K_{AB}}{\Sigma_A K}$$

$$M_{EA} = M^f_{EA} + M^c_{EA} = M^f_{EA} - \frac{1}{2}\Sigma_A M^f \frac{K_{AE}}{\Sigma_A K} \qquad (6.17)$$

$$M_{DA} = M^c_{EA} = -\frac{1}{2}\Sigma_A M^f \frac{K_{AD}}{\Sigma_A K}$$

and similarly for the other four end moments.

It is seen that the second of Eqs. (6.17) is identical with the expression used in Art. 5.7 for determining moments at the ends of columns in building frames. In fact, approximate values of such moments are often obtained in design practice by considering the far ends of all members abutting to a joint as fixed (as in Fig. 6.12). In this case the indicated expression is obtained for the moments at the near ends of the two columns.

The above procedure may seem complicated if written in general terms. The following numerical example will indicate its actual simplicity. Since, in the moment-distribution method, all computations are carried out numerically rather than in the form of equations, it is important to organize the computing sheets in a systematic manner in order not to get lost in a maze of numbers. Various such schemes are in use, at the computer's preference. The authors believe that the scheme employed in the following examples is simple and not conducive to errors.

The left, upper sketch in Fig. 6.13 shows the dimensions and loads of the frame. The fixed-end moments (from Table 6.3) and the unbalanced joint moments are computed as shown. Next, the stiffnesses $K$ and the distribution factors $K/\Sigma K$ are determined as indicated. The moment distribution proper is carried out on a separate sketch, not necessarily drawn to scale. It is advantageous to indicate each member on this sketch by two parallel lines. The designations of the joints can then be placed as indicated. The stiffness $K$ of each member is inscribed between the lines at mid-length and the distribution factor is written near the joint to which it pertains (here joint $A$). Next, the fixed-end moments are shown at the sections where they act. It is advantageous to write all moments in consecutive spans alternatingly on one and the other side of the member. This provides the necessary space to enter all subsequent moments. In the given frame no fixed-end moments act on $AD$ and $AB$. The balancing moment (equal and opposite to the unbalanced moment $M^u_A$) is now distributed to the abutting members according to Eq. (6.15); that is, $M^d_{AB} = 8.0 \times 0.1 = 0.8$, $M^d_{AC} = 8.0 \times 0.4 = 3.2$, etc.

The carry-over moments at the far ends are one half of the distributed moments [Eq. (6.16)], the carry-over being indicated by arrows. Finally, the final moments at each end of all members are obtained by summing the individual entries. Once the distribution is completed, joint $A$ is in equilibrium, *i.e.*, the sum of the final moments is zero:

$$14.1 + 0.8 - 16.8 + 1.6 = 0$$

This check should always be made to guard against errors. At the ends of members $AB$ and $AD$ the respective $M^f$ and $M^c$ are the final moments since no fixed-end moments are present.

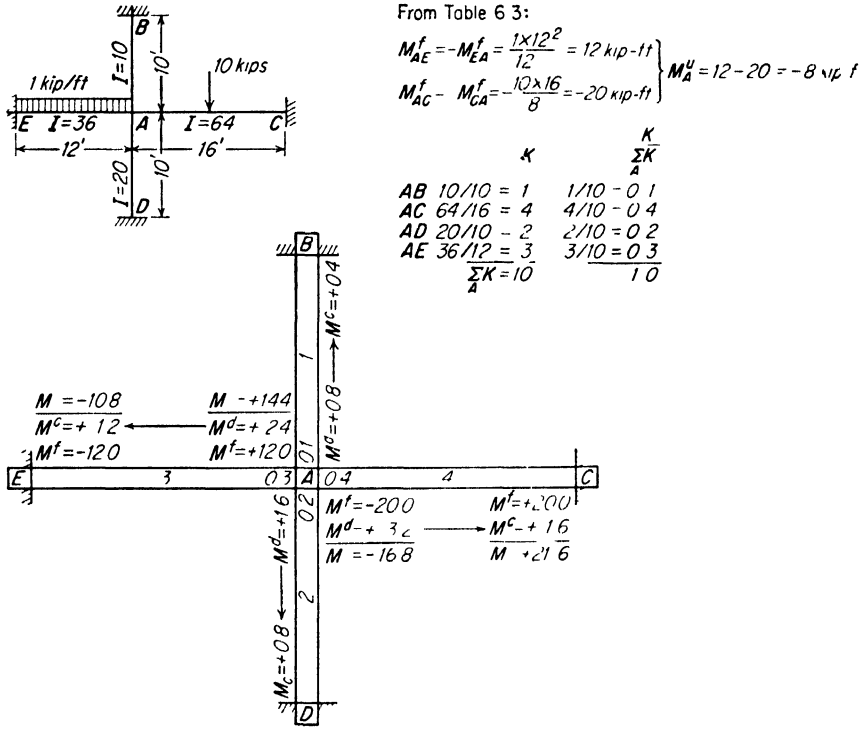



FIG. 6.13. Moment distribution at one joint (calculations)

It is seen that all eight end moments were determined without use of equations, by a very simple, numerical process. With some practice the moment designations ($M^f$, $M^c$, etc.) as well as the carry-over arrows can be omitted on the computation sheets.

*If the far end of a member is hinged*, rather than fixed (say end $C$ in Fig. 6.12*a*), the moment it exerts on the locked joint (Fig. 6.12*b*) is the *moment at the fixed end of a beam hinged*, rather than fixed, *at the other end*. In this case the corresponding moments are those listed in the last

two columns of Table 6.3. The rotation at the near end of a beam whose far end is hinged is obtained from Case *a* of Table 6.2 and is

$$\theta_1 = \frac{M_1}{3EK} \tag{6.18}$$

Comparison with Eq. (6.13) shows that for such a beam the rotation for a given moment is four-thirds of the rotation for the same moment $M_1$, but with other end fixed. Put in different terms, it may be said that the stiffness of a beam with far end hinged is three-fourths of that of the same beam with far end fixed. Hence, for computing distribution factors, the *stiffness of a member with far end hinged* is

$$K = 0.75\,\frac{I}{l} \tag{6.19}$$

rather than $K = I/l$ for far end fixed. For such members, of course, *no moment is carried over to the hinged end* since such a support is not capable of resisting a moment ($M_{21} = 0$ in Case *a* of Table 6.2).

So far, the moment-distribution method has been discussed only for a structure with one single joint ($A$ in Fig. 6.12). If more than one such joint is present, the same procedure is applied to one joint after another, in sequence. The method is best explained for a concrete example, such as the continuous beam of Fig. 6.14.

At first all joints of the structure are locked (Fig. 6.14*a*). In this state only the indicated fixed-end moments act at the ends of the members. Next, one joint is unlocked, such as $B$ (Fig. 6.14*b*). To establish equilibrium at $B$, the distributed moments are added as before. These, in turn, cause carry-over moments at the far ends, resulting in the indicated end moments. Now that $B$ is in equilibrium it is relocked and $C$ is released. The unbalanced moment at that joint ($M^f_{CD} + M^c_{CB}$) is balanced, as before, by the indicated distributed moments which, in turn, again result in carry-over moments at the far ends. At this stage the moments at $C$ are balanced so that it can be relocked. This completes what is known as the first distribution cycle. However, a new moment $M^c_{BC}$ has now been added at $B$ so that this joint is no longer balanced. It is therefore again unlocked and the process repeated. As will be seen from later, numerical examples, the new moments added in each cycle decrease rapidly in magnitude. For this reason the computation can be terminated when the additional moments become so small as to be negligible. Then, the sum of all moments entered in one column represents the final moment at that end of the member.

In physical terms this procedure can be interpreted as follows: The deflected shape under load, for the particular beam, is shown in Fig. 6.14*d*. The beam assumes this shape immediately upon loading, with

the final moments acting on it. In the distribution procedure this final or real state is approached in stages. In the first stage, with all joints locked, the deflected shape and the corresponding moments are very different from the actual shape of the loaded beam (Fig. 6.14*a* vs. 6.14*d*). The release of joint *B* allows the structure to come closer to its final shape (Fig. 6.14*b*); consequently the moments acting in it approach the final moments more closely. In each consecutive step (such as Fig.

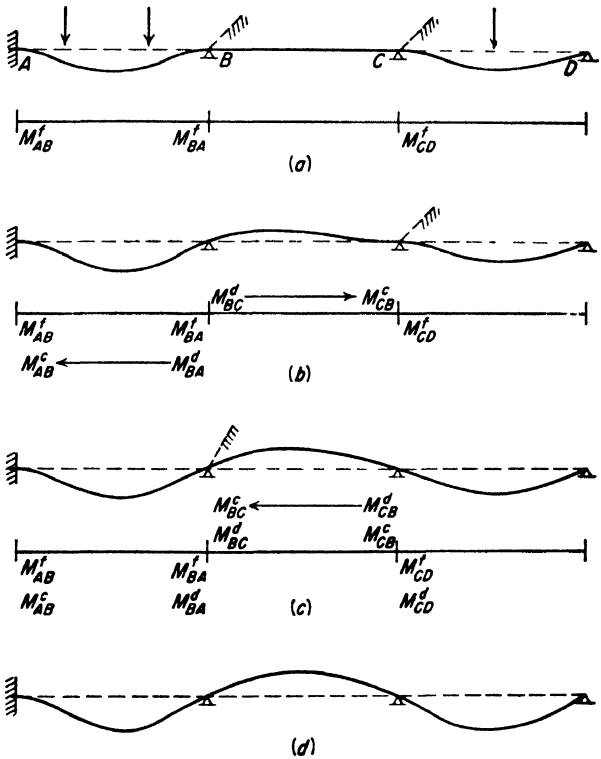



FIG. 6.14 One cycle of moment distribution on continuous beam.

6.14*c*) a still closer approach is made to the actual, deflected shape, and consequently the moments corresponding to each step differ less and less from those when all joints are unlocked, as they actually are in the structure.

The advantage of this seemingly complicated procedure is that the moments in the first stage, with all joints locked, are known directly, since they are the fixed-end moments. In each subsequent step the corresponding correction moments are computed in a simple, arithmetical manner, and the final moments are obtained by simple addition of the original fixed-end moments and all subsequent corrections. Simul-

taneous equations, as they arise in the slope-deflection method, are entirely avoided so that complicated structures can be analyzed with relative ease, even when the slope-deflection method would result in such a large number of simultaneous equations as to make analysis practically prohibitive.

*Illustrative Example.* Solve Example II, Art. 6.6, by moment distribution. With joints $B$ and $C$ locked, the stiffnesses of spans $AB$ and $BC$ are $K_{AB} = K_{BA} = I/20 = 0.05I$. Span $CD$ is hinged at $D$, so that its stiffness is $K_{CD} = (\frac{3}{4})(I/15) = 0.05I$. Since all stiffnesses happen to be equal, the distribution factors of all members at joints $B$ and $C$ are $K/\Sigma K = 0.5/1.0 = 0.5$. The fixed-end moments from left to right are given in Art. 6.6 as $\mp 100$, $\mp 50$, and $-66.7$ ft-kips, the last value having been computed for far end hinged. Figure 6.15 shows the com-

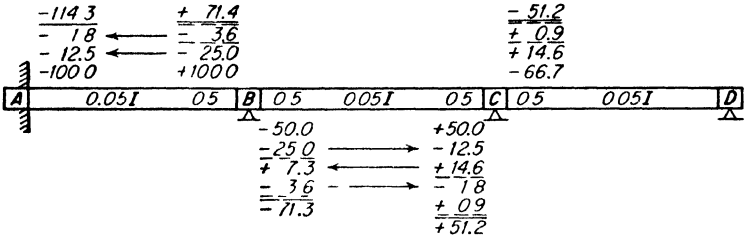



FIG. 6.15

plete computation. Joint $B$ is released first and the distributed moments $-25.0$ applied to both members. The joint, now balanced, is relocked, which is indicated by a dashed line. Moments are carried over as indicated. The unbalanced moment at $C$ is now

$$50.0 \quad 66.7 \quad 12.5 \quad -29.2$$

The joint is released and the distributed moments, 14.6 each, applied for balance, after which the joint is relocked, as indicated by the dashed line. This completes the first cycle of distribution. A moment of 7.3 is carried over to joint $B$ and the whole process repeated until the second cycle is completed, as indicated by the second dashed lines in each column. If the moments at each of the five ends are now summed up, the following final end moments obtain, from left to right: $-114.3$, $+71.4$, $-71.3$, $+51.2$, $-51.2$. It is seen that ($a$) the moments at $B$ and $C$ balance, as they should; ($b$) the moments from this two-cycle distribution deviate by only a fraction of a per cent from those obtained by slope deflection in Art. 6.6, *viz.*, $-114.5$, $+71.1$, $-71.1$, $+51.1$, and $-51.1$.

It should be noted how quickly the correction moments decrease. In this particular case two distribution cycles have resulted in more than adequate accuracy. In more complex structures three or four cycles

may be required for that purpose. It is interesting to observe that, for this example, the three-moment method results in three simultaneous equations for $M_A$, $M_B$, $M_C$, the slope-deflection method in two simultaneous equations for $\theta_A$ and $\theta_B$, while by the moment distribution the entire analysis is carried out in a numerical fashion.

For an example of simple moment distribution (without sidesway) for a more complicated frame, see part (*a*) of the illustrative example in Art. 6.8.

**6.8. Moment Distribution for Frames with Joint Displacement (Sidesway).** The method discussed in Art. 6.7 permits the determination of moments in structures whose joints, though rotated, are not linearly displaced (translated) under load. In case such linear displacements

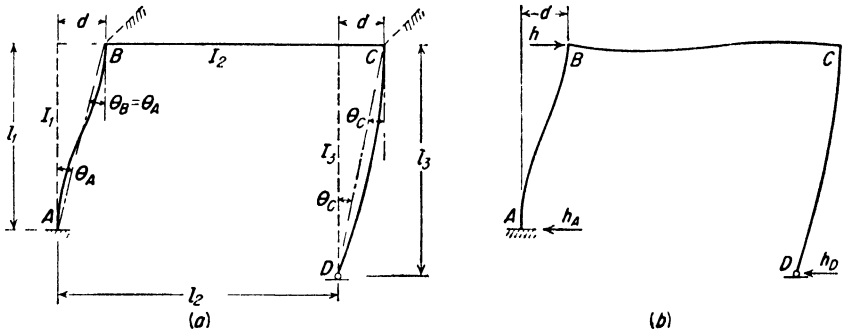



FIG. 6.16

occur, such as from horizontal wind loads or other sources, computations must be correspondingly modified to account for this influence. This process is exactly similar to that employed in Examples III and IV, Art. 6.6, except that now moments will be determined by distribution instead of by slope-deflection equations

If a horizontal force acts on a frame, it will deflect an amount $d$, as indicated in Fig. 6.16. It was shown in Example V, Art. 6.6, how the magnitude of such a deflection can be computed for a given value of the horizontal force. Even though, for any design, the horizontal force is usually known (say, the wind pressure) it is more convenient in connection with moment distribution to proceed in the reverse order:

The frame is given an arbitrary displacement $d$ of convenient magnitude. Next, the moments due to this displacement and the magnitude of the force required to produce it are computed. Let the magnitude of this force be $h$, and the corresponding moments $m$, while the actual horizontal force is $H$. Then, obviously, the actual moments are

$$M = m\frac{H}{h} \tag{6.20}$$

To compute $m$ and $h$, the joints $B$ and $C$, Fig. 6.16*a*, are first locked against rotation, and then are given an equal arbitrary displacement $d$. This causes moments $m'$ to occur at $A$, $B$, and $C$ ($D$ being hinged in this particular example). Leg $AB$, with tangents at both ends parallel, bends antisymmetrically, as in Fig. 6.6*d*. Consequently, for a given displacement the moments are obtained from Case *d* of Table 6.2:

$$m'_{AB} = m'_{BA} = 6EK\theta_A = 6K\frac{Ed}{l_1} \tag{6.21}$$

which holds for the case that *the bottom end of the column is fixed.* Column $CD$, being hinged at one end, corresponds to Fig. 6.6*a*. Consequently, *for a column with bottom end hinged*, the moment at the top is

$$m'_{CD} = 3EK\theta_C = 3K\frac{Ed}{l_3} \tag{6.22}$$

Next, the temporarily locked joints are released and the moments $m$ corresponding to the actual configuration of the frame (Fig. 6.16*b*) are computed by the usual procedure of distributing the moments $m'$. The horizontal reactions $h_A$ and $h_D$ are now determined by considering individually the equilibrium of each leg (see Fig. 6.9*b*). Hence,

$$h_A = \frac{m_{AB} + m_{BA}}{l_1} \tag{6.23}$$

and, since $D$ is hinged, so that $m_{DC} = 0$,

$$h_D = \frac{m_{CD}}{l_3} \tag{6.24}$$

Hence, the horizontal force corresponding to the arbitrary displacement $d$ is

$$h = h_A + h_D \tag{6.25}$$

Since the actual force that acts on the frame is $H$ rather than the fictitious, auxiliary force $h$, the actual moments are then computed from Eq. (6.20).

As was shown in Examples III to V, Art. 6.6, sidesway and corresponding moments are caused not only by actual, horizontal loads, but also by asymmetry of vertical loads and/or of frame configuration.

The analysis of frames with sidesway is illustrated below.

*Illustrative Example.* To compute the moments in the frame of Fig. 6.17*a*, the stiffnesses $K = I/l$ are calculated for all members except that for the hinged member $DE$, $K = 0.75I/l$. Next, the distribution factors are computed for all members at joints $B$, $C$, and $D$ and, together with the stiffnesses, are indicated on the computation sketch, Fig. 6.18*a*.

The fixed-end moments for the loaded span $BC$ may be obtained from Table 6.3.

*a.* With these data the moment distribution of Fig. 6.18*a* is carried out in the usual way for "no sidesway." The distribution was started by unlocking $B$ and then successively $C$ and $D$, the completion of the first cycle in each column of figures being indicated by the dashed line. The second cycle is always carried out in the same sequence, here $B$, $C$, $D$, its completion being indicated by the respective second dashed lines. It is seen that the corrections produced by the second cycle (moments shown at all points between first and second dashed lines) amount to less than 10 per cent of those of the first cycle. Since correction moments in successive cycles decrease in about the same proportion, a third cycle, if carried out, would contribute corrections of about 1 per cent or less of the final moments. These can be neglected, so that, in this example, a two-cycle distribution provides satisfactory accuracy.

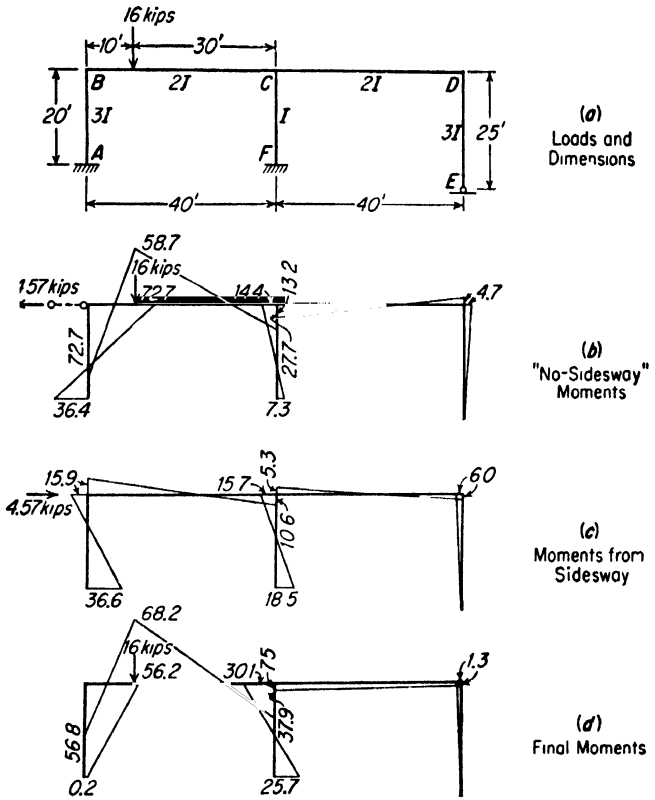
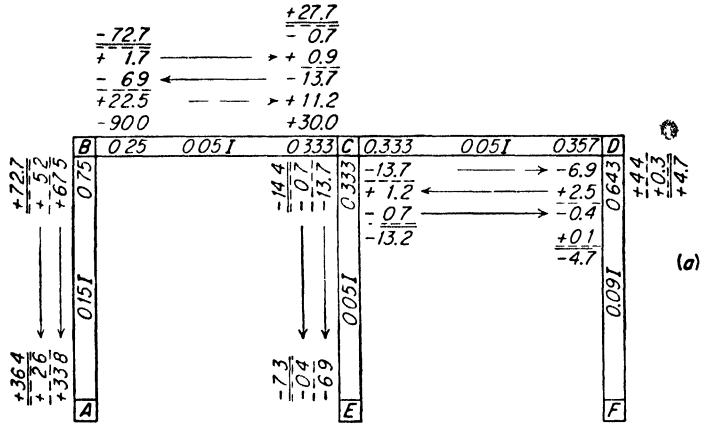



FIG. 6.17. Effect of sidesway on moments in rigid frame.

After completion of the distribution computations, each joint is checked for equilibrium, that is, $\Sigma M = 0$. It is seen that joints $B$ and $D$ check exactly. At joint $C$ there is a discrepancy in the third digit. Such discrepancies in the last digit result frequently from rounding off further



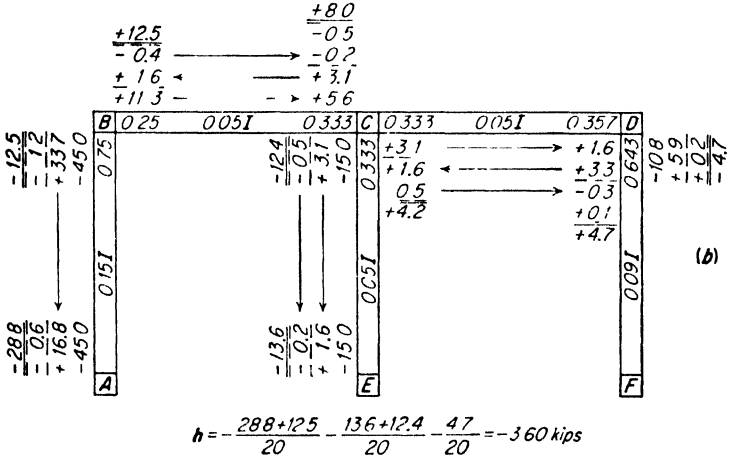




Fig. 6.18

digits not shown in the computation. If computations are carried to three digits, as they usually are, slight discrepancies in the last digits are obviously negligible.

Computing the shears at the bottom of the three legs, as shown in Fig.

6.18*a* [see Eqs. (6.23) and (6.24)], an unbalanced force of $H = 4.57$ kips is obtained. In other words, exactly as in Example IV, Art. 6.6, the frame would be in equilibrium under the moments of Fig. 6.18*a* only if an additional support were present to prevent sidesway and to supply a horizontal reaction of 4.57 kips. This fictitious state of loading and the corresponding moment diagram are shown in Fig. 6.17*b*.

*b*. It is now necessary to determine the "sidesway correction," *i.e.*, to apply a force equal and opposite to $H$. This amounts to removing the fictitious support at $D$ which permits the natural sidesway to occur. The sidesway moments caused by $H$, if superposed on those of Fig. 6.17*b*, give the correct end moments.

To compute these moments, joints $B$, $C$, and $D$ are locked and given an arbitrary horizontal displacement. This displacement is so chosen as to result in moments of convenient magnitude; here $Ed = 1000$ serves this purpose. The corresponding end moments for legs $AB$ and $CF$ are computed from Eq. (6.21), and for the hinged leg $DE$ from Eq. (6.22); for $Ed = 1000$, they are $-15.0$, $-15.0$, and $-10.8$ ft-kips, respectively. These moments are applied and distributed in the usual manner, as shown in Fig. 6.18*b*. It is seen that a two-cycle distribution is again sufficient, since a third cycle would add corrections of less than 1 per cent. The horizontal shears are computed as in (*a*) [see Eqs. (6.23) and (6.24)] and the total horizontal force $h = -3.60$ kips, corresponding to the given deflection $Ed = 1000$, is calculated as indicated in Fig. 6.18*b* [see Eq. (6.25)]. Since the moments of Fig. 6.18*b* are caused by a force of 3.60 kips, while the force required to eliminate the reaction of Fig. 6.17*b* is 4.57 kips, the actual sidesway moments are 4.57/3.60 times those of Fig. 6.18*b* [see Eq. (6.20)]. The corresponding sidesway moment diagram is shown in Fig. 6.17*c*.

*c*. To obtain the final moments, the sidesway moments of Fig. 6.17*c* are added to those for no sidesway, Fig. 6.17*b*, resulting in the final moment diagram shown in Fig. 6.17*d*. Comparison of Fig. 6.17*b* and *d* shows the very large influence sidesway may have; not only are the magnitudes of the moments changed by 50 per cent and more, but even signs are reversed in some cases, such as at points $A$ and $D$. Even though the relative influence of sidesway may be considerably smaller in many cases, the present example, as well as that of Art. 6.6, shows that it is unsafe to neglect sidesway in design except (1) if symmetry of the frame and loading forestall any sidesway tendency or (2) if the structure is so supported as to make sidesway physically impossible.

*d*. If a real, horizontal force of, say, 10 kips, such as a wind load, acts on this frame, from left to right, the moments are 10/4.57 times those of Fig. 6.17*c* or, which amounts to the same, 10/3.60 of those determined by the distribution of Fig. 6.18*b*.

## Problems

Solve Probs 2 to 4, Art 6 6, by moment distribution

**6.9. Building Frames, Vertical Loads.** Rigid frames with a small number of spans, and one or at most two stories high, occur frequently in industrial and commercial structures and in bridges Since the number of joints in such frames is relatively small, their analysis represents no particular difficulty, as is illustrated by the examples of the preceding articles

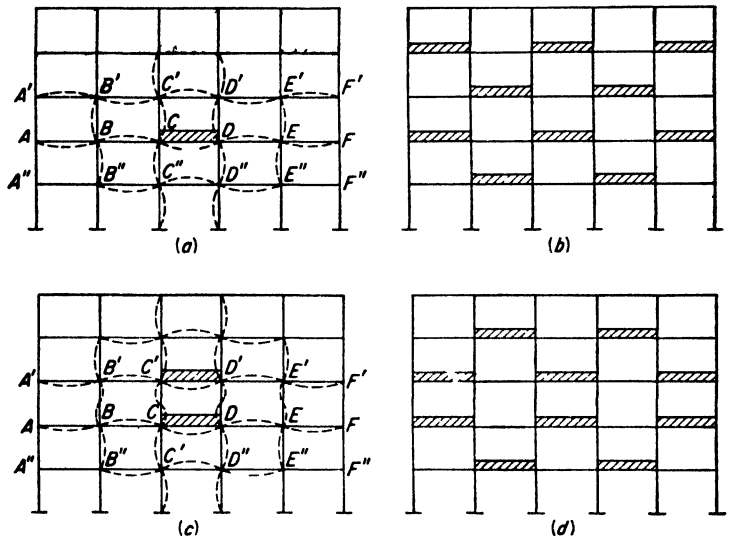



Fig 6 19

In contrast, the frames of multistory apartment or office buildings, warehouses and the like have a multitude of members and joints as is seen on the relatively small frame of Fig 6 19 Such frames are known for short as *building frames* and are composed of columns and of girders or beams These members must be designed for the largest moments, shears and, in the case of columns, axial loads which can reasonably be expected to occur during the lifetime of the building These moments, shears and axial loads are brought about in part by the weight of the structure (dead load) and in part by live load While the former is constant, live loads, such as floor loads from human occupancy, can be placed in various ways, some of which will result in larger effects than others

In Fig 6 19*a*, only span *CD* is loaded by live load The distortions of the various frame members are seen to be largest in and immediately

adjacent to the loaded span, and decrease rapidly with increasing distance from the load. Since bending moments are proportional to curvatures, the moments in more remote members are correspondingly smaller than those in, or close to, the loaded span. However, the loading of Fig. 6.19*a* does not produce the maximum possible positive moment in *CD*. In fact, if additional live load were placed on span *AB*, this span would bend down, *BC* would bend up, and *CD* itself would bend down in the same manner, although to a lesser degree, as it is bent by its own load. Hence, the positive moment in *CD* is increased if *AB* and, by the same reasoning, *EF* are loaded simultaneously. Expanding the same reasoning to the other members of the frame, it is easily seen that the "checkerboard pattern" of live load of Fig. 6.19*b* produces the largest possible positive moments not only in *CD* but in all loaded spans. Hence, two such checkerboard patterns are required to obtain the maximum positive moments in all spans.

In addition to maximum span moments it is often necessary to investigate minimum span moments. Dead load, acting as it does on all spans, usually produces only positive span moments. However, live load, placed as in Fig. 6.19*a* and, even more so, in Fig. 6.19*b*, is seen to bend the unloaded spans upward, *i.e.*, to produce negative moments in the span. If these negative live-load moments are larger than the generally positive dead-load moments, a given girder, depending on load position, may be subject at one time to positive span moments, and at another to negative span moments. It must be designed to withstand both types of moments, *i.e.*, it must be furnished with tensile steel at both top and bottom. Thus, the loading of Fig. 6.19*b*, in addition to giving maximum span moments in the loaded spans, also gives minimum span moments in the unloaded spans.

Maximum negative moments at the supports of the girders are obtained, on the other hand, if loads are placed on the two spans adjacent to the particular support and in a corresponding pattern on the more remote girders. A separate loading scheme of this type is then required for each support for which maximum negative moments are to be computed.

In each column, the largest moments occur at the top or bottom, as can easily be seen from the examples of the preceding articles. While the loading of Fig. 6.19*c* results in large moments at the ends of columns *CC'* and *DD'*, the reader can easily convince himself by a simple moment distribution that these moments are further augmented if additional loads are placed as shown in Fig. 6.19*d*.

It is seen from this brief discussion that, in order to calculate the maximum possible moments at all critical points of a building frame, live load must be placed in a great variety of different schemes. If a complete

moment distribution for the entire frame is carried out for each of these cases, the required amount of computation becomes prohibitive, even for frames with a limited number of spans and stories. The need is apparent, then, for simplified approximate methods which would allow the determination of these moments with reasonable accuracy and yet with a limited amount of computation. Although such abbreviated methods cannot result in the largest theoretically possible moments, the use of approximations is justified for a variety of reasons.

Thus, even though it is theoretically necessary to load span $B''C''$, among others, to obtain the maximum positive moment in $D'E'$ (Fig. 6.19*b*), the former is so far removed from the latter that its effect on this moment is negligibly small. Hence, the influence on any moments of loads on spans sufficiently far removed can generally be neglected. That such a procedure is legitimate is emphasized also by the fact that it is quite improbable for the exact loading schemes of Fig. 6.19*b* and *d* ever to occur during the lifetime of the structure. Considerations like these justify methods in which, to determine any particular moment, only the influence of the adjacent parts of the frame is considered.

In this connection it is also important to realize the difference in the importance of accurate moment determination for girders on the one hand and for columns on the other. In the former the required cross sections are dictated exclusively by moments and shears, and for this reason a relatively accurate determination of these quantities is called for. Columns, on the other hand, must resist the axial loads from overlying parts of the structure, in addition to the bending moments induced in them by the rigidly connected girders. Some inaccuracy in determining these moments affects, therefore, only one of the two factors (loads and moments) which determine the required cross sections and, for this reason, has smaller over-all effects than similar inaccuracies in determining girder moments. It should also be noted that the load distribution which results in maximum column moments does not at the same time result in maximum axial forces in the same columns. For instance, even though the loading of Fig. 6.19*d* results in maximum $M_{cc'}$, the axial load on that column would attain its maximum if all, instead of just a few, of the adjacent girder spans in the upper stories were loaded. While such full loading of overlying spans would result in maximum axial force, it would, conversely, cause a smaller moment $M_{cc'}$ than that resulting from Fig. 6.19*d*.

It is evident from this discussion that an approximate method of moment determination is reliable for design use if it permits the determination of girder moments with a relatively high degree of accuracy. For column moments, a cruder approximation is permissible, if such moments are used in conjunction with maximum axial loads (*i.e.*, all overlying

adjacent spans fully loaded, except for such decrease in live load with increasing number of stories as is permitted in most building codes).

Extensive investigation by means of numerous trial computations has shown that, for building frames with reasonably regular outline not involving unusual asymmetry of loading or shape, the influence of sidesway caused by vertical loads can be neglected.

In that case moments are determined with sufficient accuracy by breaking up the entire frame into simpler subframes. Each of these consists of one girder, plus the top and bottom columns framing into that particular girder. Placing the live loads on that girder in the most unfavorable manner permits sufficiently accurate determination of all girder moments and of the moments at the top ends of the bottom columns and at the bottom ends of the top columns. For this partial structure, the far ends of the columns are considered as fixed, except for such first-floor or basement columns where soil and foundation conditions more closely approach hinged ends (see Art. 6.11). Consequently, the ACI Code specifies for floor members, including girders:

(1) The live load may be considered to be applied only to the floor under consideration, and the far ends of the columns may be assumed as fixed. (2) Consideration may be limited to combinations of dead loads on all spans with full live load on two adjacent spans (for negative support moments) and with full live load on alternate spans (for positive span moments).

In regard to columns, the Code indicates:

In building frames, particular attention shall be given to the effect of unbalanced floor loads on both exterior and interior columns and of eccentric loading due to other causes. In computing moments in columns, the far ends may be considered fixed. Columns shall be designed to resist the axial forces from loads on all floors, plus the maximum bending due to loads on a single, adjacent span of the floor under consideration.

The following example illustrates the computation of moments in building frames.

*Illustrative Example.* The structure shown in Fig. 6.20 represents the second-story girder of the frame of Fig. 6.19, including the abutting top and bottom columns, isolated as just described. Dimensions and moments of inertia are shown in the main sketch, while the magnitude and character of loads is indicated at the top of the figure. Since span lengths differ by more than 20 per cent and since the ratio of live to dead load, at least for some spans, exceeds 3:1, design by approximate moment coefficients such as those of Art. 3.6 would be misleading and unsafe. Moment determination by more accurate frame analysis is, therefore, *mandatory* for this structure. Let it be required to compute all necessary

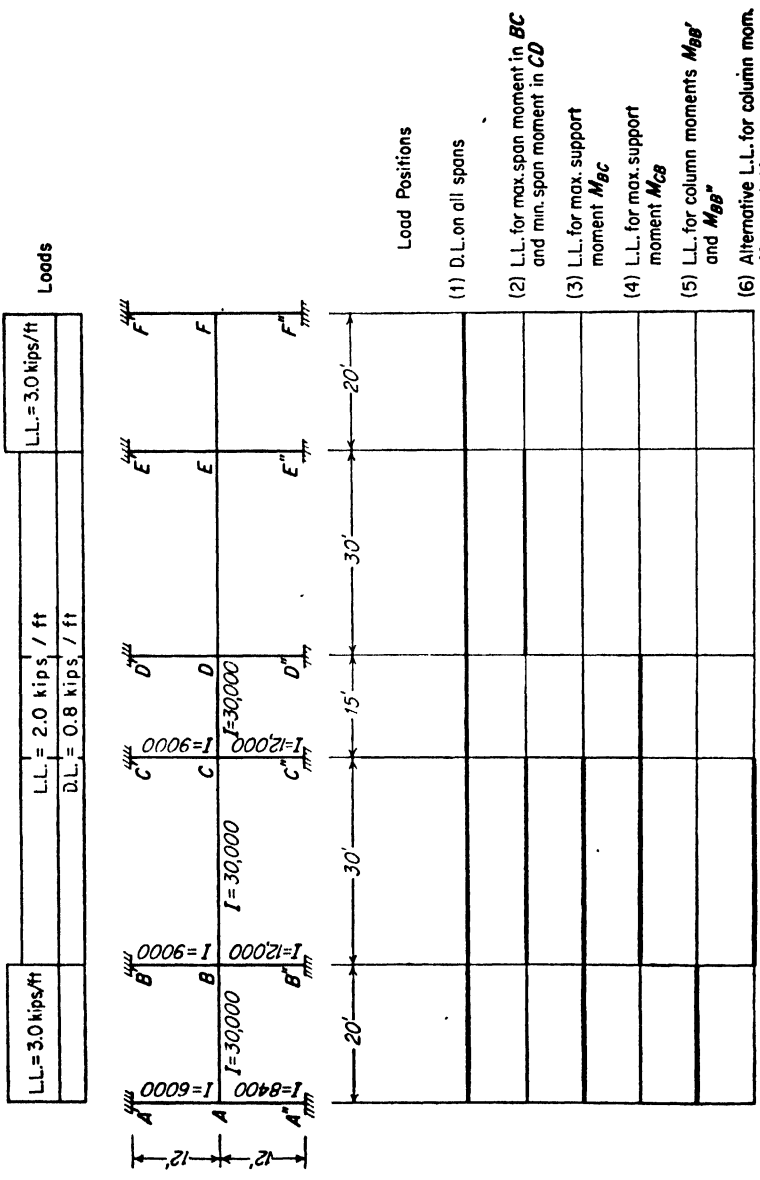



FIG. 6.20. Approximate determination of moments in building frames (vertical load).

moments for span $BC$, the minimum span moment for $CD$, and the column moments $M_{BB'}$ and $M_{BB''}$.

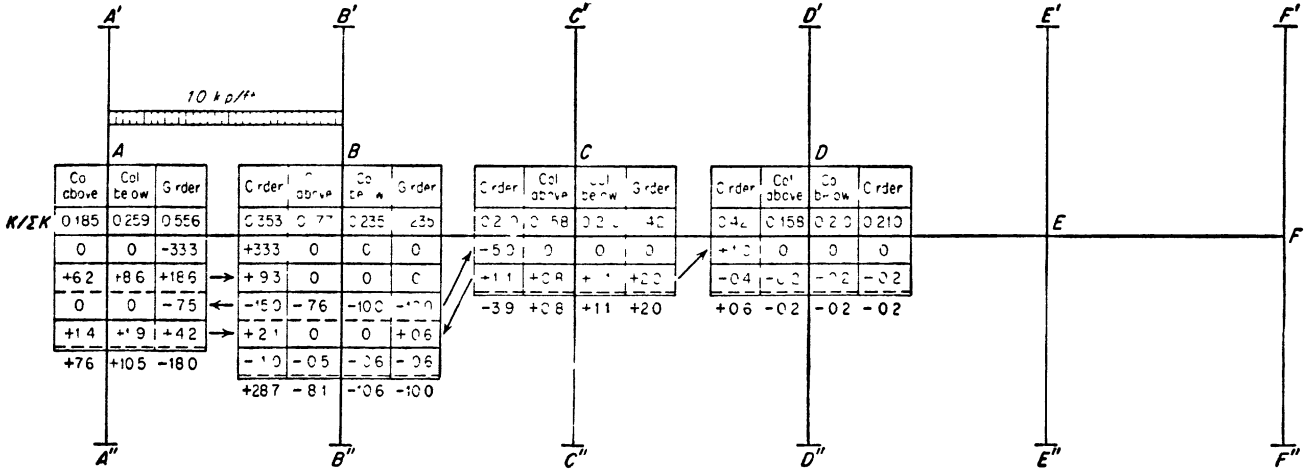
According to the preceding discussion and to the quoted provision of the ACI Code it is necessary, for this purpose, to determine moments from the six loading schemes shown at the bottom of Fig. 6.20. This would require six separate moment distributions. However, in this particular case considerable simplification can be achieved by noting that (*a*) all live and dead loads are uniformly distributed, though of varying magnitudes, and (*b*) the structure is symmetrical. To reduce computations, moments are determined separately for a uniform load of 1 kip per ft placed individually on span $AB$, $BC$, and $CD$. The moments for all six loadings of Fig. 6.20 can then be computed by simple superposition of the moments obtained from these three distributions. Simplifications such as this depend on the shape of the structure and the type of loading and should always be used when possible.

The fixed-end moments $wl^2/12$ for $w = 1$ kip per ft are 33.3, 75.0, and 18.75 ft-kips for spans $AB$, $BC$, and $CD$, respectively. The stiffnesses and distribution factors are computed in the usual manner from the given lengths and moments of inertia.

For this particular type of moment computation in building frames, a special way of recording the calculations is often found convenient. This scheme is shown in detail in Fig. 6.21*a* and is self-explanatory. It has the advantage that all figures are written in one instead of two directions. Carry-overs along the girders can be indicated by arrows in the usual manner. This would not be possible for carry-overs along columns. However, these are not required, since only moments in the column end sections adjacent to the girder are determined in this special form of distribution. Moment $M_{B'B}$, for instance, would be found, not from this calculation, but from a similar analysis of girder $A'F'$ of Fig. 6.19, with its adjacent columns.

The three distributions for the unit loads on the three spans are given in Fig. 6.21*a* to *c*. In examining these calculations, the following features are noteworthy: (*a*) It is seen from Fig. 6.21*a* and *b* that moments decrease very rapidly with increasing distance from the loaded span, amounting here to less than 2 per cent of the loaded-span moments at those joints which are twice removed from the loaded span. This reduces very considerably the amount of calculations required for single-span loadings. (*b*) Two-cycle distribution at the near joints and a single cycle at the remoter joints is seen to result here in satisfactory accuracy, although in some cases more cycles may be required.

For Fig. 6.21*a* the order of joint release was $A$, $B$, $C$, $D$, $A$, $B$, while for Fig. 6.21*b* it was $B$, $C$, $D$, $E$, $A$, $B$, $C$, $D$, $A$, further distribution on each joint being discontinued as soon as correction moments have decreased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Girder
0·185
0·259
0·556
0
0
-333
+62
+86
+186
0
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-75
+14
+19
+42
+76
+105
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+333
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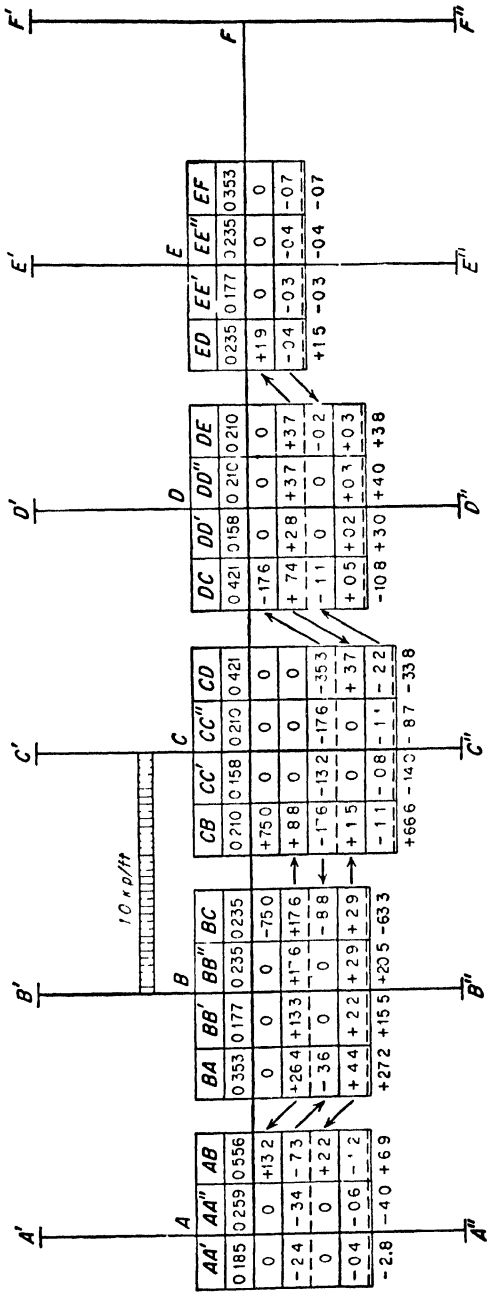
Fig. 6.21*a*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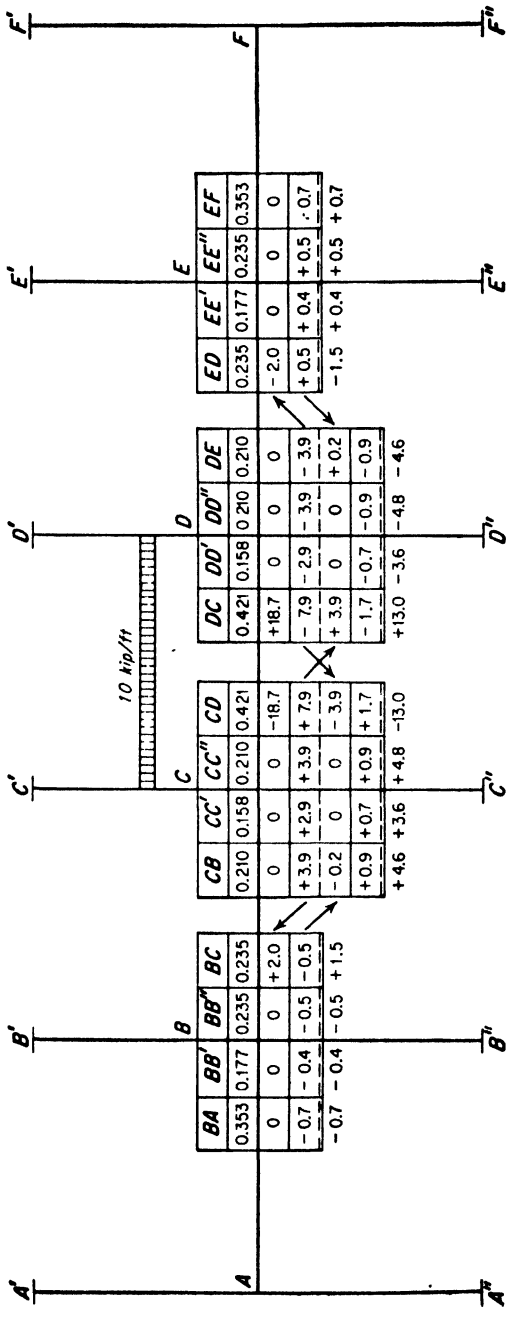
10 k p/ft
AA' 0185 | AA'' 0259 | AB 0556
0 | 0 | +13 2
-24 | -34 | -73
0 | 0 | +22
-04 | -06 | -12
-2.8 -40 +69
BA 0353 | BB' 0177 | BB'' 0235 | BC 0235
0 | 0 | 0 | -75 0
+26 4 | +13 3 | +17 6 | +17 6
-36 | 0 | 0 | -88
+44 | +22 | +29 | +29
+27 2 +15 5 +20 5 -63 3
CB 0210 | CC' 0158 | CC'' 0210 | CD 0421
+75 0 | 0 | 0 | 0
+88 | 0 | 0 | 0
-176 | -132 | -176 | -353
+15 | 0 | 0 | +37
-11 | -08 | -11 | -22
+66 6 -14 0 -87 -33 8
DC 0421 | DD' 0158 | DD'' 0210 | DE 0210
-176 | 0 | 0 | 0
+74 | +28 | +37 | +37
-11 | 0 | 0 | -02
+05 | +02 | +03 | +03
-10 8 +30 +40 +38
ED 0235 | EE' 0177 | EE'' 0235 | EF 0353
+19 | 0 | 0 | 0
-04 | -03 | -04 | -07
+15 -03 -04 -07

Fig. 6.21b


A'
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A''
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B
B''
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D'
D
D''
E'
E
E''
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F
F''
10 kip/ft
BA 0.353 | BB' 0.177 | BB'' 0.235 | BC 0.235
0 | 0 | 0 | +2.0
−0.7 | −0.4 | −0.5 | −0.5
−0.7 −0.4 −0.5 +1.5
CB 0.210 | CC' 0.158 | CC'' 0.210 | CD 0.421
0 | 0 | 0 | −18.7
+3.9 | +2.9 | +3.9 | +7.9
−0.2 | 0 | −3.9 | −3.9
+0.9 | +0.7 | +0.9 | +1.7
+4.6 +3.6 +4.8 −13.0
DC 0.421 | DD' 0.158 | DD'' 0.210 | DE 0.210
+18.7 | 0 | 0 | 0
−7.9 | −2.9 | −3.9 | −3.9
+3.9 | 0 | 0 | +0.2
−1.7 | −0.7 | −0.9 | −0.9
+13.0 −3.6 −4.8 −4.6
ED 0.235 | EE' 0.177 | EE'' 0.235 | EF 0.353
−2.0 | 0 | 0 | 0
+0.5 | +0.4 | +0.5 | +0.7
−1.5 +0.4 +0.5 +0.7

Fig. 6.21c

sufficiently. For the case of Fig. 6.21*c*, advantage was taken of symmetry by unlocking symmetrically located joints simultaneously in pairs. Thus $C$ and $D$ were simultaneously released first, next $B$ and $E$, and finally $C$ and $D$. It is seen that if computed in this manner all component moments on one side are equal and opposite to those for the symmetrically located sections on the other side. With some experience such computations in case of symmetry are carried out by showing one set of figures only. The resulting moments caused by unit loads on individual spans are shown in the table below.

| Unit load on span | Column moments | | | | | | Girder moments | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $M_{AA'}$ | $M_{AA''}$ | $M_{BB'}$ | $M_{BB''}$ | $M_{CC'}$ | $M_{CC''}$ | $M_{AB}$ | $M_{BA}$ | $M_{BC}$ | $M_{CB}$ | $M_{CD}$ |
| $AB$ | +7 6 | +10 5 | −8 1 | −10 6 | +0 8 | +1 1 | −18 0 | +28 7 | 10 0 | 3 9 | −0 6 |
| $BC$ | −2 8 | −4 0 | +15 5 | +20 5 | −14 0 | −18 7 | +6 9 | +27 2 | −63 3 | +66 6 | −33 8 |
| $CD$ | | | −0 4 | −0 5 | +3 6 | +4 8 | | −0 7 | 1 5 | +1 6 | −13 0 |
| $DE$ | | | +0 3 | +0 4 | −3 0 | −4 0 | | +0 7 | 1 5 | 3 8 | +10 8 |
| $EF$ | | | | | +0 2 | +0 2 | | | | +0 2 | +0 6 |
| All spans | +4 8 | +6 5 | +7 3 | +9 9 | −12 4 | −16 6 | −11 1 | +55 9 | 73 3 | +63 7 | 36 0 |

The moments for span loads on $DE$ and $EF$ in the table above were obtained as the equal and opposite of those for symmetrically located sections for span loads on $BC$ and $AB$, respectively.

From these moments caused by the unit loads, the actual frame moments are easily computed as follows:

*a*. Maximum girder moments at supports:

$M_{BC}$:

| | | |
|---|---|---|
| D.L. all spans: | 0.8 (−73.3) | = −58 6 |
| L.L. on $BC$: | 2.0 (−63.3) | −126 6 |
| L.L. on $AB$: | 3.0 (−10.0) | = −30 0 |
| max $M_{BC}$ | | = −215 2 ft-kips |

$M_{CB}$:

| | | |
|---|---|---|
| D.L. all spans: | 0.8 (+63.7) | = +51 0 |
| L.L. on $BC$: | 2.0 (+66.6) | = +133 2 |
| L.L. on $CD$: | 2.0 (+4.6) | = +9 2 |
| max $M_{CB}$ | | = +193 4 ft-kips |

(By the beam sign convention, both these moments are negative.)

*b.* Maximum mid-span moment in $BC$:

$M_{BC}$:

| | | |
|---|---|---|
| D.L. as in (*a*) | = | −58.6 |
| L.L. on $BC$ as in (*a*) | = | −126.6 |
| L.L. on $DE$: 2.0 (−1.5) | = | −3.0 |
| $M_{BC}$ | = | −188.2 ft-kips |

$M_{CB}$:

| | | |
|---|---|---|
| D.L. as in (*a*) | = | +51.0 |
| L.L. on $BC$ as in (*a*) | = | +133.2 |
| L.L. on $DE$: 2.0 (−3.8) | = | −7.6 |
| $M_{CB}$ | = | +176.6 |

Mid-span moment:

$$M_{l/2} = 2.8 \times \frac{30^2}{8} - \frac{188.2 + 176.6}{2}$$
$$= 132.6 \text{ ft-kips}$$

(Since the two end moments differ somewhat, the maximum span moment does not occur at mid-span but at the point of zero shear. In the present case the difference is extremely small.)

*c.* Minimum mid-span moment in $CD$:

$M_{CD} = -M_{DC}$:

| | | |
|---|---|---|
| D.L. all spans: 0.8 (−36.6) | = | −29.7 |
| L.L. on $BC$: 2.0 (−33.8) | = | −67.6 |
| L.L. on $DE$: 2.0 (+10.8) | = | +21.6 |
| $M_{CD} = -M_{DC}$ | = | −75.7 ft-kips |

Mid-span moment:

$$M_{l/2} = 0.8 \times \frac{15^2}{8} = -75.7 = -55.2 \text{ ft-kips}$$

(It is seen that the minimum mid-span moment in $CD$ is negative so that tension reinforcement is required near the top face of the girder.)

*d.* Column moments:

$M_{BB'}$:

| | | |
|---|---|---|
| D.L. all spans: 0.8 (+7.3) | = | +5.8 |
| L.L. on $BC$: 2.0 (+15.5) | = | +31.0 |
| $M_{BB'}$ | = | +36.8 ft-kips |

$M_{BB''}$:

| | | |
|---|---|---|
| D.L. all spans: 0.8 (+9.9) | = | +7.9 |
| L.L. on $BC$: 2.0 (+20.5) | = | +41.0 |
| $M_{BB''}$ | = | +48.9 ft-kips |

(Column moments of opposite sign are caused by live load on $AB$, instead of $BC$. It is quickly verified that the absolute values of the column moments at $B$ are larger if live load is placed on $BC$ rather than on $AB$. In case of doubt, moments due to live load on either side of the column must be checked; the larger of them govern design.)

Moments at other sections of columns or girder can be computed in the same manner without additional distributions

If dead and live loads are of different character so that they cannot be obtained as simple multiples of unit loads, and or if the structure is not symmetrical, it may become necessary actually to carry out separate moment distributions for each case of loading such as those illustrated in Fig. 6.20. In numerous practical cases, however, simplifications, such as those illustrated in the example, can be used to shorten computations.

**6.10. Building Frames, Horizontal Loads.** Horizontal loads on building frames are caused primarily by wind pressure. Pressures per square foot of vertical wall surface are specified in building codes and, depending on locality, exposure, etc., vary from 10 to 50 psf. In addition, earthquake shocks produce horizontal sway which results in inertia forces acting horizontally on the structure. It was found by extensive observation that these horizontal forces, rather than those induced by the simultaneous vertical motion of the ground, are chiefly responsible for structural earthquake damage. For this reason building codes in localities with high earthquake incidence require that structures be designed to resist corresponding horizontal forces. These are specified as stipulated percentages of the total load of the building above the given floor, and are of the order of 10 per cent of that load.

In older-type buildings, with massive bearing walls, wind forces are resisted by these walls acting as cantilever beams (loaded in the direction of the axis of the wall). In modern frame buildings, light curtain walls are usually employed which do not possess significant strength and stiffness so that horizontal loads must be resisted by the frame proper. With regard to earthquake forces the older-type building was found to be easily damaged because the heavy and brittle bearing walls did not possess the necessary strength and flexibility to withstand the swaying motion. Correctly designed rigid-frame buildings, on the other hand, were found to stand up very well under such stress.

These horizontal forces, by action of walls and floors acting as rigid diaphragms in their own planes, are transmitted to the joints of the frame. For wind pressures, it is customary to compute the load on a joint from the area of exterior wall surface pertaining to it, *i.e.*, from the rectangle outlined by vertical center lines between adjacent frames and by horizontal center lines between floor levels. Earthquake forces are distributed to individual joints in similar ways. These distributions are

not very exact, particularly if the external forces do not act in the planes of the frames, and also if the outline of the structure is irregular in plan and/or elevation. More accurate ways of distributing the total forces to the individual frames are discussed in the references cited later in this article. The references also indicate that the actual distribution of wind forces, in particular, is frequently very different from the assumed uniform pressure on the surface of the windward walls.

For the combination of live and dead loads plus wind load the ACI Code allows permissible stresses one-third larger than those allowed for live plus dead load alone. This is done in recognition of the fact that the coincidence of full wind force with most unfavorably distributed live load is extremely unlikely. Also, the rarity and temporary nature of maximum wind forces is likely to result in less damage than if these same forces were of great frequency and extended duration. Hence, wind stresses are treated in codes as somewhat secondary effects, in comparison with the main stresses caused by dead and live loads.

Horizontal forces on building frames produce sidesway and their effects can be accurately computed only by sidesway analysis, as discussed in previous articles. It will be realized that such an analysis, for frames with anywhere from 5 to 20 or more stories, requires very lengthy computations, even by the more rapid moment-distribution method. In most cases, however, such accurate determinations are neither required nor justified. It was pointed out that considerable uncertainty prevails regarding the magnitude as well as the distribution of wind and earthquake forces. It is useless, in analysis, to aim at an accuracy greater than that of the load data and other assumptions. In addition, the somewhat secondary effect of these rare and transitory forces also justifies the use of more approximate, analytical methods.

A considerable number of such approximate methods has been in use for many years. Although some of them result in slightly better theoretical accuracy than others, in view of the prevailing uncertainties these differences are usually not significant enough to discard the simpler methods in favor of those of slightly better theoretical accuracy. For this reason only the simplest of those methods which are in reasonable agreement with accurate analysis will be discussed here, the approach which is known as the *portal method*.[1]

In simple frames, horizontal forces result in moments distributed as shown in Figs. 6.9*c* and 6.17*c*. It is seen that in all members moments vary linearly and, except in hinged members (*DE* in Fig. 6.17), have

[1] For more information on other types of analyses, as well as on magnitude and distribution of horizontal forces, the reader is referred to "Wind Bracing in Steel Buildings," Final Report of Sub-committee 31, *Trans. ASCE*, vol. 105, p. 1713, 1940, and "Continuity in Concrete Building Frames," Portland Cement Association.

opposite signs at both ends (in the sense of the beam sign convention). It is further seen that in these members the points of zero moment (inflection points, see Fig. 6.9*a*) are located reasonably close to the mid-point of each member. In hinged members, of course, zero moment obtains only at the hinge. At the inflection points, with zero moments, only shear and axial forces act on the cross section. For symmetrical portals, such as that of Fig. 6.9, the shear forces are the same in both legs and are equal, each, to half of the external horizontal load, as is easily checked from the information on that figure. If one of the legs is more rigid than the other (larger value of $K$), it would require a larger horizontal force to displace it horizontally by the same amount as the more flexible leg. Consequently the portion of the total shear resisted by the stiffer column is larger than that of the more flexible column. This is easily checked by computing and comparing shears in $AB$ and $CF$ of Fig. 6.17*c*.

In building frames, moments and forces in the girders and columns of each individual story are distributed in substantially the same manner as just discussed for single-story frames. The portal method of computing approximate moments, shears, and axial forces from horizontal loads is, therefore, based on the following three simple propositions:

1. The total horizontal shear in all columns of a given story is equal and opposite to the sum of all horizontal loads acting above that story.
2. The horizontal shear in both exterior columns is the same; the horizontal shear in each interior column is twice that of an exterior column.
3. The inflection points of all members, columns or girders, are located midway between joints.

Although the last of these propositions is commonly applied to all columns, including those of the bottom floor, the authors prefer to deal with the latter separately, depending on conditions of foundation (see Art. 6.11). If the actual conditions are such as practically to prevent rotation (foundation on rock, massive pile foundations, etc.), the inflection points of the bottom columns are above mid-point (see Figs. 6.9*c* and 6.17*c*) and may be assumed at a distance of $2h/3$ from the bottom. If little resistance is offered to rotation, such as for relatively small footings on compressible soil, the inflection point is located closer to the bottom and may be assumed at a distance of $h/3$ from the bottom or even lower. (In ideal hinges the inflection point is at the hinge, *i.e.*, at the very bottom.) Since shears and corresponding moments are largest in the bottom story, a judicious evaluation of foundation conditions as they affect the location of inflection points is of considerable importance.

The first of the three cited propositions follows from the requirement that horizontal forces be in equilibrium at any level. The second takes account of the fact that in building frames interior columns are generally

*more rigid than exterior ones because (a) the larger axial loads require larger cross sections and (b) exterior columns are restrained from joint rotation only by one abutting girder,* while interior columns are so restrained by two such members. The third proposition is very nearly true because, except for the top and bottom columns and, to a minor degree, for the exterior girders, each member in a building frame is restrained about equally at both ends. For this reason it deflects under horizontal loads in an antisymmetrical manner, with the inflection point at mid-length.

The actual computations according to this method are extremely simple. At an inflection point, with $M = 0$, only shears and axial forces are transmitted from one half of the member to the other. An inflection point is, therefore, equivalent to a hinge, which also transmits only forces, but not moments. Once column shears are determined from propositions 1 and 2, and inflection points (hinges) located from proposition 3, all moments, shears, and forces are simply computed by statics. The process is illustrated in Fig. 6.22*a*.

For illustration, consider joints $A$ and $B$. The total shear in the second story is $3 + 6 = 9$ kips. According to proposition 2, the shear in each exterior column is $9/6 = 1.5$ kips, and in each interior column $2 \times 1.5 = 3.0$ kips. The shears in the other floors are obtained in the same manner, and act at the hinges as shown. Considering the equilibrium of the rigid structure between hinges $a$, $b$, and $c$, the column moments, 3.0 and 9.0, respectively, are obtained directly by multiplying the shears by their lever arms, 6 ft. The girder moment at $C$, to produce equilibrium, is equal and opposite to the sum of the column moments. The shear in the girder is obtained by recognizing that its moment (*i.e.*, shear times half of girder span) must be equal to the girder moment at $C$. Hence this shear is $12.0/10 = 1.2$ kips. The moment at the other end, $D$, is equal to that at $C$, since the inflection point is at mid-span. At $D$, column moments are computed in the same manner from the known column shears and lever arms. The sum of the two girder moments, to produce equilibrium, must be equal and opposite to the sum of the two column moments, from which the girder moment to the right of $C$ is $18.0 + 6.0 - 12.0 = 12.0$. Axial forces in the columns also follow from statics. Thus, considering the rigid body $aEd$, a vertical shear of 0.3 kip is seen to act upward at $d$. To equilibrate it, a tensile force of $-0.3$ kip is required in the column $CE$. In the rigid body $abc$ an upward shear of 1.2 kips at $b$ is added to the previous upward tension of 0.3 kip at $a$. To equilibrate these two forces, a tension force of $-1.5$ kips is required in column $AC$. If the equilibrium of all other partial structures between hinges is considered in a similar manner, all moments, forces, and shears are rapidly determined.

*In the present case relatively flexible foundations were assumed and* the location of the lowermost inflection points was estimated to be at $h/3$ from the bottom. The general character of the resulting moment distribution is shown in Fig 6 22*b*

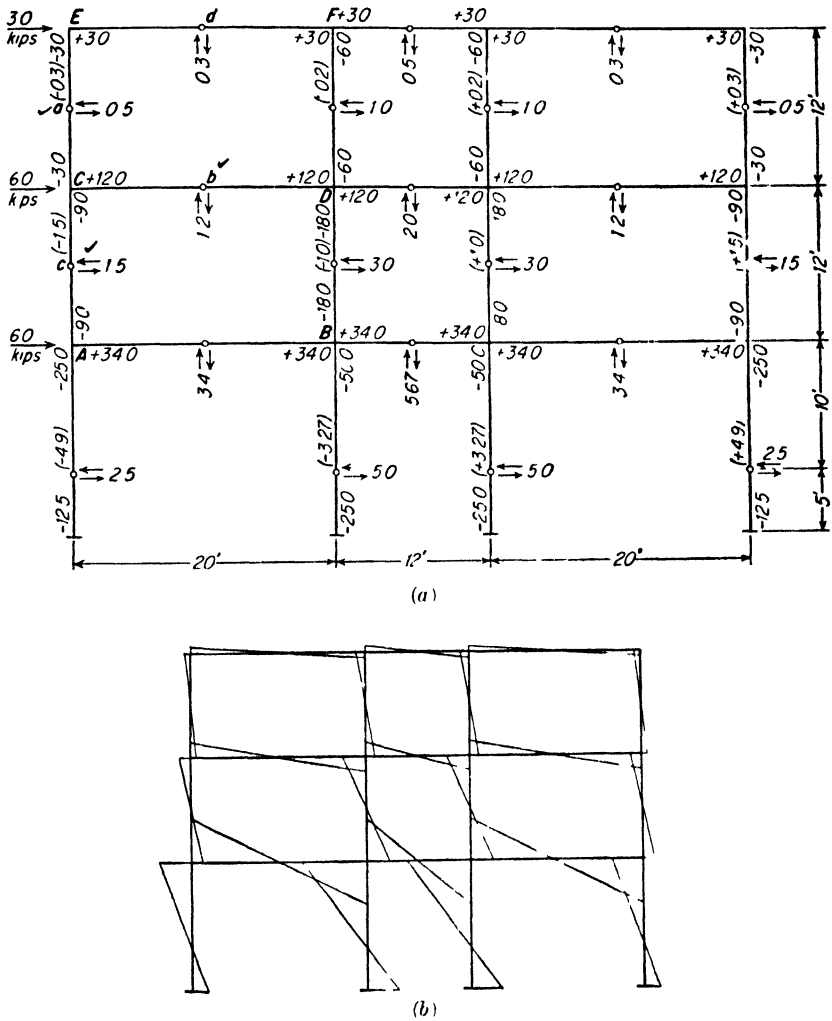



FIG. 6.22. Portal method for determining moments from wind load in building frame

**6.11. Effective Spans, Moments of Inertia, Support Conditions.** *Effective Spans.* In structural analysis, structures are usually represented by simple line diagrams Actually, the depths of beams or girders and the widths of columns (in the plane of the frame) amount to sizable

fractions of the respective lengths of these members. Their clear or free lengths are therefore considerably smaller than the respective center-line distances between joints. It must then be decided whether, in the analysis, center-line distances or clear lengths should be used. The usual methods of analysis, such as those discussed, assume that each member has a given flexibility over its entire length. Actually, the block of concrete formed at a joint by the members there intersecting is extremely rigid in both directions, contrary to the assumptions of analysis. It is customary, nevertheless, to consider the lengths of the members to be given by center-to-center distances between joints. Although this is not strictly exact, it is closer to the truth than the alternative of using clear lengths. Indeed, if these were employed in a frame of several bays and stories, the analyzed frame might easily be several yards shorter and lower than the actual structure so that the necessary similitude between analysis and structure would be lost.

However, even though joint moments are determined on the basis of center-line distances, one has still to decide at what location the end moments shall be computed for which girder and column sections are dimensioned. A girder intersecting a column obviously has a very great depth along the center line of the latter and, physically, assumes its design depth only from the column face outward. Since the diagram of negative moments is usually quite steep in the neighborhood of supports, the theoretical moment at the joint is frequently considerably larger than that at the face of the column. If the former were used for dimensioning, an unnecessarily large section would obtain. For this reason the ACI Code specifies that "moments at faces of supports may be used for design of beams and girders." The same situation arises for columns. However, for these (*a*) the moment curve is generally not as steep as in girders so that the difference between the theoretical joint moment and that at the face of the girder is relatively smaller, and (*b*) in view of the simultaneous axial force a moderate change in moment affects but slightly the required cross section. For this reason theoretical moments at joints, without reduction, are frequently used for designing columns, although the use of moments computed at the face of the girders may be advisable in unusual cases.

*Moments of Inertia.* The design of flexural members is based on the cracked section, *i.e.*, on the supposition that the part of the concrete which is in tension is ineffective. It would seem, therefore, that moments of inertia, for the purpose of computing stiffnesses $K$, should be determined in the same manner, *i.e.*, for the transformed section (see Art. 3.12), discounting the concrete in tension. This, however, would not give a measure of the true stiffness of the member. Indeed, even though the concrete is capable of resisting only a moderate amount of tension,

hairline cracks from this cause form only in limited portions of the length of the beam (due to high moments or shears) and even in these portions they are rather widely spaced. Between cracks the member acts essentially as if it were monolithic. Only at the cracks proper and possibly in their immediate neighborhood is the effective moment of inertia that of the transformed section, with concrete in tension discounted, while between cracks the full section is effective. Since the sum of the widths of all hairline cracks in a beam is extremely small as compared to the span, the actual bending rigidity is much more closely characterized by the moment of inertia of the full section. For this reason, in the analysis of continuous structures, moments of inertia are mostly computed for the full concrete section. The contribution of reinforcement is usually neglected, which compensates to some extent for the neglect of the influence of cracks.

For T-shaped members allowance must be made for the effect of the flange. Although the question as to the exact amount of this effect is not settled, it seems reasonable to assume the same effective width of flanges for computing moments of inertia as are considered in stress computations (see Art. 3.39). For continuous T beams the flanges are also effective in regions of negative moments where the flange is in tension. As was indicated above, there is little difference between concrete in compression and tension in regard to its effect on the flexural stiffness of the member.

For columns, likewise, the moment of inertia is computed for the full, unreduced concrete section, neglecting the influence of reinforcement.

*Conditions of Support.* Beams and frames are supported at various points, usually by other structures which, in the end, transmit the loads to the underlying soil. Thus, columns either rest on individual footings, foundations, piles, etc., or they may be supported by basement walls. In buildings, columns are often placed only in the interior, while the outer ends of the girders are supported by brick or concrete walls. In bridge frames, such as that of Fig. 6.11, the outer ends of the girders often rest on retaining walls or abutments. In all these cases it is important to decide whether these supports should be regarded as fixed or hinged. Their actual condition is often between these two extremes, but it is difficult in frame analysis to account for such intermediate restraints. For this reason fixed or hinged supports are usually assumed, depending on which of the two conditions is more closely approached in the structure.

Moments are frequently affected to a considerable degree by the choice made. Thus, in a portal frame under vertical load, fixity at the base results in larger girder and column moments at the knees, but in smaller span moments in the girder. This is so because the stiffness of a fixed-ended member is four-thirds of the stiffness of a hinged member; there-

*fore the girder is subject to larger end restraints by the columns if the latter are fixed. For horizontal loads, hinged conditions at the bottom* of columns result in larger moments than for a fixed base, since, in the former case, the point of inflection is at the bottom end, instead of at about mid-height, and correspondingly increases the lever arm of the shear forces. An assumption of support conditions must, therefore, be made which is at the same time realistic and safe, *i.e.*, conservative.

For columns supported on footings which in turn rest on compressible soil, a hinge is generally assumed, since such soils offer but little resistance to rotation of the footing. This is particularly true for footings of moderate horizontal dimensions. If the footings rest on solid rock or on a cluster of piles driven into highly compacted soil or to rock, complete fixity is often approached very closely and should be assumed.

Outer ends of beams resting on brick walls should be assumed as hinged. On the other hand, if such ends are cast monolithically with concrete walls and are anchored into them, they may be assumed as fixed if the moment of inertia of the walls (center to center between girders) is very large as compared to that of the girder. Otherwise, the walls may be treated as part of the frame by regarding them as columns with a moment of inertia computed in the above manner.

**6.12. Preliminary Design.** The analysis of moments and forces in continuous structures requires the prior knowledge of cross-sectional dimensions so that $I$ and $K$ can be computed before analysis is begun. The problem of detailed design, on the other hand, is the exact opposite: the determination, from known forces and moments, of the required cross sections to ensure strength, safety, and serviceability. To attain this objective, it is therefore necessary to proceed in two steps: (1) A preliminary estimate of approximate sizes of members must be made, based on experience and rule of thumb so that $I$ and $K$ for all members can be computed. (2) An accurate moment analysis is then carried out to check the assumed sizes, to make such adjustments and changes as may be required, and to obtain the complete information on moments and forces which is necessary for designing the details of reinforcement.

The moments and shears obtained in step 2 are accurate only if the final sizes of members are the same as those used for computing $I$ and $K$, that is, those estimated in step 1. Consequently, if the analysis shows that the originally assumed dimensions were considerably in error, it may become necessary to repeat step 2 with revised dimensions. This shows the practical importance of making the best possible preliminary design before beginning an elaborate analysis.

On the other hand, excessive accuracy in this connection is illusory and superfluous for various reasons. For one, the effective moments of inertia of reinforced concrete sections can be ascertained only approximately

(see Art. 6.11). Furthermore, to carry out moment analyses, a number of approximate, simplifying assumptions are made (conditions of restraint, subdividing of the structure into smaller frames, etc.) which result in approximate, rather than accurate, moments and shears. No purpose is served in obtaining an apparent accuracy of analysis better than that of the assumptions on which it is based. Finally, and most important, reinforced concrete frameworks, in view of the long-time plasticity of the material (see Sec. 12), have the very fortunate property of adapting themselves to limited amounts of local overstressing without undue consequences. A beam overstressed in the span will sag, imperceptibly in most cases, and thereby transfer part of the excess moments to the supports; a column overstressed by an excessive bending moment at one end will deform slightly and bring into play its strength at the other end or that of other abutting members to resist the moment.

In view of these considerations the authors lean toward the opinion that a moment analysis need not be revised unless actual relative stiffnesses deviate from those assumed in the analysis by more than about 30 per cent. It is noted in this connection that what matters are only the ratios of the stiffnesses (as in the distribution factors), rather than their absolute values. If, in the preliminary analysis, the size of all members was underestimated by the same percentage, the results of the analysis would not be affected by this error.

In view of the unlimited variety in the shapes of reinforced concrete frameworks, it is not possible to give any reliable rules on the preliminary determination of cross-sectional dimensions. The following hints may be helpful, even though the experienced designer will modify them according to the special conditions of his particular structure.

In *building frames*, the sizes of *girders* are usually governed by the negative moments and the shears at the supports, where their effective section is rectangular no matter whether or not they are shaped as T beams. The shears to be used in preliminary design can usually be taken as the simple beam reactions and the moments as the fixed-end moments of the particular span. Alternatively, the approximate moment and shear coefficients of Art. 3.6 may often be used for preliminary design, with due adaptation if loads are other than uniformly distributed. Moments in *columns* are generally considerably smaller than in girders, so that their sizes are primarily governed by the axial loads they have to carry. To determine these sizes preliminarily, it is best to compute the sections which would be required if axial load alone were present, and then to increase them slightly to provide for the additional influence of moments. The additional areas so required may vary from about 5 per cent for interior columns in the lower floors to possibly 25 per cent or more for exterior columns in the upper floors. In deciding on these esti-

mated increases, the following should be considered: Moments are larger in exterior than in interior columns since, in the latter, dead-load moments from adjacent spans will largely balance, in contrast to exterior columns. In addition, the influence of moments, as compared to that of axial loads, is larger in upper-floor than in lower-floor columns, since the moments proper are usually of about the same magnitude, while, of course, the axial loads are larger in the latter than in the former.

For *single-story* frames preliminary design depends even more on judgment and experience than in building frames. As an example rather than a rule, girders may be dimensioned for simple beam shears and for moments equal to almost the full fixed-end moment at interior supports, and somewhat smaller fractions thereof at exterior ones. Interior columns may be predesigned for the axial load plus a moment equal to some fraction of the difference of fixed-end moments of adjacent spans, while exterior columns, in addition to their axial load, have to resist moments about equal to the exterior support moments of the adjacent girder. Slender columns with heavy girders result in relatively large span moments for the girders and in relatively small column moments. Sidesway generally results in larger column moments than would occur in the same structure if sidesway were prevented. Judicious consideration of factors such as these will enable a designer with some experience to produce a reasonably accurate preliminary design.

**6.13. Summary.** The methods of analysis discussed in this section furnish the tools for the design of the most frequent types of continuous reinforced concrete structures. They are limited, however, to structures composed of straight, prismatic members and in which all members are either in line, such as in continuous beams, or meet at right angles. For conditions other than these, modifications of the analytical method must be employed. Their discussion is beyond the scope of this text. Cases of this sort are not infrequent. Thus, haunched girders and tapering columns are frequently used in bridges or single-story frames. In gable frames, members are joined at a variety of angles, and curved members are frequently used in spanning large halls. For such cases the reader is referred to special texts on analysis of indeterminate structures. A number of publications issued by the Portland Cement Association and available from them on request will be found useful. Among these are "Moment Distribution Applied to Continuous Structures," "One-story Concrete Frames Analyzed by Moment Distribution," "Continuity in Concrete Building Frames," and "Handbook of Frame Constants."

# SECTION 7

# FOOTINGS

**7.1. Types and Function of Substructures.** The substructure, or foundation, is that part of a structure which is usually placed below the surface of the ground and which transmits the load to the underlying soil or rock. All soils compress noticeably when loaded and cause the supported structure to settle. The two essential requirements in the design of foundations are that the total settlement of the structure shall be limited to a tolerably small amount and that differential settlement of the various parts of the structure shall be eliminated as nearly as possible. With respect to possible structural damage, the elimination of differential settlement, *i.e.*, different amounts of settlement within the same structure, is even more important than limitations of uniform overall settlement.

To limit settlements in the indicated manner it is necessary (1) to transmit the load of the structure to a soil stratum of sufficient strength and (2) to spread the load over a sufficiently large area of that stratum to minimize bearing pressure. If adequate soil is not found immediately below the structure, it becomes necessary to use deep foundations such as piles or caissons to transmit the load to deeper, firmer layers. If satisfactory soil directly underlies the structure, it is merely necessary to spread the load, by footings or other means. Such substructures are known as spread foundations and it is mainly this type which will be discussed. Information on the more special types of deep foundations can be found in texts on foundations engineering.

**7.2. Bearing Capacity of Soils.** Soils consist of products of rock disintegration which are of various sizes, from microscopic clay and silt particles through sand and gravel to substantial boulders. These variously sized particles occur in an unlimited variety of mixtures whose compositions together with their water content determine the compressibility, strength, and, thereby, bearing capacity of each individual soil. To limit the amount of settlement, it is necessary to select a bearing pressure under the footings small enough to prevent excessive compression of the soil mass. The great variety of soils makes it impossible to give

generally valid prescriptions for the selection of this bearing pressure. In built-up areas with reasonably uniform soil conditions, previous experience often provides a satisfactory guide. On virgin sites, soil-loading tests are conducted, often supplemented by laboratory tests. As examples of average magnitudes of permissible bearing values, the stipulations of the Boston Building Code of 1944 and of the New York City Code of 1948 are summarized in Table 7.1.

**TABLE 7.1. Allowable Bearing Pressures, Tons per Square Foot**

| Description | Boston | New York |
|---|---|---|
| Massive bedrock and thoroughly cemented conglomerates | 100 | 60 |
| Sound, laminated rocks (slate, schist) | 35 | 40 |
| Sound shale | 10 | |
| Residual deposits of shattered bedrock | 10 | |
| Soft rock | | 8 |
| Hardpan | 10 | 12 |
| Compact gravel and sand-gravel mixtures | 5 | 10 |
| Loose gravel and sand-gravel mixtures, coarse compact sand | 4 | 6 |
| Coarse, loose sand fine, compact sand | 3 | 4 |
| Very compact sand-silt soils | | 6 |
| Fine, loose sand | 1 | 2 |
| Medium compact sand-silt soils | | 2 |
| Hard clay | 6 | 5 |
| Compact sand-clay soils | | 3 |
| Medium clay | 4 | 1 5 |
| Soft clay | 1 | 1 |

The appreciable differences in some of the values given by the two codes are no reflection on the soundness of these documents. On the contrary, they illustrate the fact that a mere description, such as "medium clay," is not a sufficient criterion for selecting bearing pressures unless it is supplemented either by test information or by local experience with the particular soil in question.

If a soft stratum underlies a firmer soil, large settlements may be caused in the former, even though the bearing pressure on the firm upper stratum may not be excessive. In order to safeguard against such settlements, it is necessary (*a*) to investigate by borings to a sufficient depth the nature of the deeper strata and (*b*) to ensure that the pressure on deeper, weak soils be not greater than specified. For the latter purpose the Boston Code suggests that the pressure on the surface of a deeper layer can be computed by assuming it to spread laterally and downward from the edges of the footing at an angle of 60° with the horizontal.

The judicious evaluation of the bearing capacity of soils and its effect on the correct design of substructures is one of the most difficult and

responsible problems of the structural engineer. Experience and sound judgment are supplemented by the relatively new science of soils mechanics, on which special texts should be consulted

**7.3. Proportioning of Footing Areas.** It was pointed out that adequate foundation design must secure tolerably small general settlement of the structure and must eliminate differential settlement as nearly as possible. If the amount of settlement of all footings in a large building is the same, no structural harm will occur since the whole building will settle uniformly and parallel. To be sure, if such uniform settlement is excessive, it becomes objectionable in view of misalignments of floor and street levels, possible leakages or breaks in pipes and other conduits, etc Normal, general settlement varies from $\frac{1}{4}$ in. to 2 in. and more, depending on soil and other conditions. The safe bearing pressure, discussed in Art. 7.2, is to be selected in such a manner as to limit the over-all settlement to such tolerable amounts.

On the other hand, if differential settlement occurs in the various parts of a building, floors will warp, doors will stick, and cracks will form in walls and partitions. Even more serious damage may occur to the structural framework, actually impairing its carrying capacity. Fortunately, the two main structural materials, steel and reinforced concrete, possess a certain amount of plasticity, or ductility, which enables such structures to adapt themselves without serious damage to moderate amounts of differential settlements.

*Maximum Design Load on Footings.* The live load used in the design of footings is not necessarily the sum of the full live loads on all contributory areas of all floors. In most structures, especially for commercial, office, or residential occupancy, it is extremely improbable that full live load will ever exist on all floors simultaneously. For this reason, in computing design loads for columns and footings, some reduction of the full load on all floors is permitted, depending primarily on the number of stories supported by the given column or footing.

As an example, the New York City Building Code provides that the following live loads shall be taken for the design of columns and footings: 100 per cent of the roof load, 85 per cent of the load of the top floor, 80 per cent for the next lower floor, with a successive reduction of 5 per cent for each additional floor, up to a total reduction of 50 per cent. Such reductions are permitted in buildings of normal occupancy; in storage structures a reduction of only 15 per cent is permitted. The Boston Code is somewhat more liberal. It allows a reduction of 15, 20, 30, 40, and 50 per cent of the total live load for members supporting two, three, four, five, or six or more stories, respectively. The load determined in this manner will be called the maximum, or full design, live load for the footing.

*Design for Uniform Settlement.* If it were certain that the full design load, live plus dead, as just defined, would actually, simultaneously, and permanently occur on all footings, the discussed limitations of general and differential settlement would be ensured by proportioning the areas for full design load in such a manner that the pressure under all footings is equal and less than the allowable bearing pressure. Actually, matters are not as simple as this even in this ideal case, as will be shown later on. The difficulty in providing for uniform settlement is caused by two factors. (1) In most soils except for some coarse-grained types, such as coarse sands and gravels, settlement does not take place immediately upon loading, but is a gradual, continuing process which extends over months and even years. As a consequence, in soils with significant amounts of fine material, such as clay and silt, the total settlement depends on the long-time average, rather than on the short-time peak load. (2) While the dead load $D$ is known and constant, the live load $L$ is relatively uncertain and varies in time. The long-time average load, therefore, consists of the dead load plus a certain fraction $\Delta L$ of the maximum live load, generally one-third to two-thirds, depending on type of occupancy. It may be assumed that in soils containing fine-grained components the total settlement is approximately proportional to the bearing pressure under this average load $D + \Delta L$. (Reservations to this statement are discussed at the end of this article.) Hence, in order to limit general settlement and to minimize differential settlement, footing areas are so proportioned that (*a*) for full design load $D + L$ the safe bearing pressure is not exceeded under any footing, and (*b*) the bearing pressure under the average load $D + \Delta L$ is the same under all footings.

Such proportions are determined in the following manner. Dead and live loads are tabulated for all footings, including the estimated weights of the footings proper. Next, the percentages of total load which the live loads constitute are computed. These percentages differ and are generally larger for interior than for exterior footings. The "critical footing" is that footing with the greatest percentage of live load. Let $A_c$, $D_c$, and $L_c$ be the area, dead, and live load of the critical footing, while $A$, $D$, and $L$ stand for the same quantities for any footing other than the critical, and $B$ is the allowable bearing pressure of the soils. The required area of the critical footing is then found from

$$A_c = \frac{D_c + L_c}{B} \tag{7.1}$$

The time-average bearing pressure of the critical footing $B_c$ under dead load $D_c$ plus partial live load $\Delta L_c$ is

$$B_c = \frac{D_c + \Delta L_c}{A_c} \tag{7.2}$$

The area of all other footings is then found from

$$A = \frac{D + \Delta L}{B_c} \tag{7.3}$$

It is easily checked that if the footings are proportioned in this manner, the maximum pressure under the critical footing at full load will be equal to the allowable value $B$, the maximum pressure under full load for any other footing will be less than $B$, and the time-average pressure under partial live load will be the same and equal to $B_c$ for all footings.

*Illustrative Example.* Four typical footings in a given building support dead and probable live loads as tabulated below. If the maximum safe bearing power of the soil on which these footings are to rest is 5000 psf, what must be the area of each of these footings in order that uniform settlement may be expected, when the footings are loaded with the full dead load plus one-third of the live load?

| Footing | Dead load, lb | Live load, lb | Live load, per cent | D.L. + ⅓ L., lb |
|---|---|---|---|---|
| $A$ | 500,000 | 300,000 | 37 | 600,000 |
| $B$ | 400,000 | 210,000 | 34 | 470,000 |
| $C$ | 400,000 | 300,000 | 42 | 500,000 |
| $D$ | 500,000 | 180,000 | 26 | 560,000 |

The critical footing is $C$, and the required area is

$$\text{Footing } C = \frac{400{,}000 + 300{,}000}{5000} = 140 \text{ sq ft}$$

A footing 12 ft square is selected, the area of which is 144 sq ft.

The unit pressure $B_c$ under this footing when it is loaded with the full dead load plus one-third of the probable live load is

$$\frac{500{,}000}{144} = 3470 \text{ psf}$$

The required areas of the other footings are, therefore,

$$\text{Footing } A = \frac{600{,}000}{3470} = 173 \text{ sq ft (13 ft-3 in. square)}$$

$$\text{Footing } B = \frac{470{,}000}{3470} = 135 \text{ sq ft (11 ft-9 in. square)}$$

$$\text{Footing } D = \frac{560{,}000}{3470} = 161 \text{ sq ft (12 ft-9 in. square)}$$

*Size Effect.* The above method for minimizing differential settlement, which is widely used, rests on the assumption that the amount of settle-

ment depends only on the magnitude of the unit pressure under the footing. Actually this is not so. If a large and a small footing are placed on the same soils and so loaded that their unit pressures are equal, the larger footing will settle more than the smaller one. The reason for this is easily understood. Figure 7.1 represents the approximate distribution of vertical pressure in the soil under a circular footing. The pressures shown are those that act in the line of the axis of the footing. Pressures at any given level decrease, of course, with increasing distance from the axis. It is seen that these vertical soil pressures dissipate and decrease rather quickly with increasing depth. At a depth equal to the diameter of the footing, the pressure is only about 30 per cent of the maximum value, and it becomes entirely negligible at a depth equal to

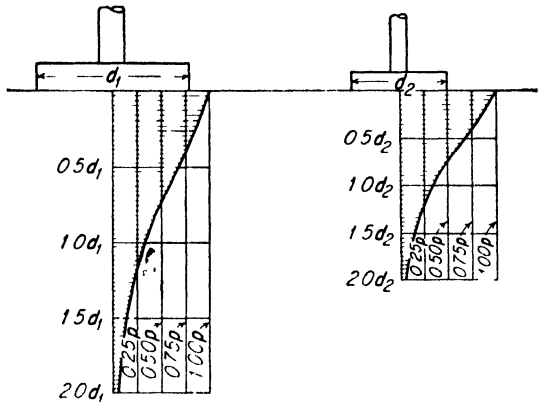



FIG. 7.1 Pressure distribution in soil under two footings with equal bearing pressure $p$ but with different diameters $d_1 = 1.6d_2$.

about two diameters. (Actual details of pressure distributions depend on shape of footing and type of soils. Figure 7.1, therefore, should be understood in a schematical rather than literal sense.) It can be said that each unit volume of soil under the footing will compress roughly proportional to the pressure acting on it. Hence, most of the settlement will occur in the soil immediately underlying the footing, say, to a depth of 1 to $1\frac{1}{2}$ diameters. The average pressure in this length, multiplied by that length, will be about proportional to the total settlement. This length in turn is proportional to the size of the footing, as is shown schematically in Fig. 7.1. Hence, under a large footing a thicker layer of soil is compressed than under a small footing, resulting in a larger settlement for the larger footing, even though the bearing pressures are the same.

This size effect is not taken into account in the method of proportioning discussed earlier in this article. It can be neglected if the areas of

the various footings of the same building do not differ substantially. On the other hand, if column loads and corresponding footing areas vary greatly, attention must be paid to this influence. A method of proportioning which accounts for the size effect can be found in Art. 3.18 of "Substructure Analysis and Design," by Paul Andersen, Ronald Press, 1948.

**7.4. Types of Spread Foundations.** Footings generally can be divided into wall and column footings. The horizontal outlines of the most common types are given in Fig. 7.2. A wall footing is simply a strip of reinforced concrete, wider than the wall, to distribute its pressure. Single-column footings are usually square, sometimes rectangular, and represent the simplest and most economical type. Their use under exterior columns meets with difficulties if property rights prevent the use of footings projecting beyond the exterior walls. In this case combined footings or strap footings are used which enable one to design a footing which will not project beyond the wall column. Combined footings under two or more columns are also used under closely spaced, heavily loaded interior columns where single footings, if they were provided, would completely or nearly merge.

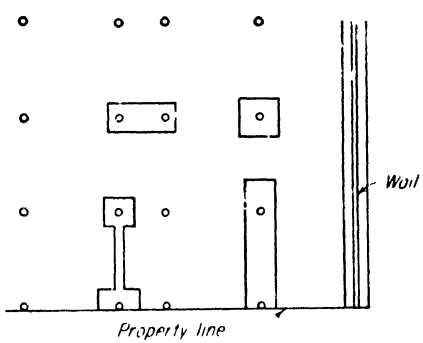


FIG. 7.2. Types of footings.

Such individual or combined column footings are the most frequent types of spread foundations on soils of reasonable bearing capacity. If the soil is weak and/or column loads are great, the required footing areas become so large as to be uneconomical. In this case, unless a deep foundation is called for by soil conditions, a mat or raft foundation is resorted to. This consists of a solid reinforced concrete slab which extends under the entire building and which, consequently, distributes the load of the structure over the maximum available area. Such foundations, in view of their own rigidity, also minimize differential settlement. They consist, in their simplest form, of a concrete slab reinforced in both directions. A form which provides more rigidity and, at the same time, is often more economical consists of an inverted beam-and-girder floor. Girders are located in the column lines in one direction, with beams in the other, mostly at closer intervals. If the columns are arranged in a square pattern, girders are equally spaced in both directions, and the slab is provided with two-way reinforcement. Inverted flat slabs, with capitals at the bottom of the columns, are also used for mat foundations.

**7.5. Factors Affecting the Design of Concrete Footings.** In ordinary constructions the load on a wall or column is transmitted vertically to the footing, which, in turn, is supported by the upward pressure of the soil on which it rests. If the load is symmetrical with respect to the bearing area, the bearing pressure is assumed to be uniformly distributed It is known that this is only approximately true. Under footings resting on coarse-grained soils the pressure is larger at the center of the footing and decreases toward the perimeter. This is so because the individual grains in such soils are somewhat mobile so that the soil located close to the perimeter can shift very slightly outward in the direction of lower soil stresses. In contrast, in clay soils pressures are higher near the edge than at the center of the footing in view of the fact that in such soils the load produces a shear resistance around the perimeter which adds to the vertical, upward pressure. It is customary to disregard these nonuniformities (*a*) because their numerical amount is uncertain and highly variable, depending on type of soils, and (*b*) because the influence of these irregularities on the magnitudes of bending moments and shearing forces in the footing is relatively small.

In order to avoid tilting of footings, the pressures under them should be symmetrically distributed, which implies that columns should be placed at the center of single footings. For the same reason, for combined footings, the resultant of the column loads should strike at the center of the footing area. If this is not done, uneven pressure distribution will result in uneven settlement, with a corresponding tilting of the footing toward the side of higher soil pressure and possible damage to the superstructure. Only on quite incompressible soils (rock, highly compacted gravels) may exceptions be made to this rule.

Footings must be located below frost depth in order to protect them from the heaving action which frost causes in some soils and which can be strong enough to lift footings even under multistory buildings. This depth amounts to 4 to 5 ft and more in northern climates.

The accurate determination of stresses, particularly in single-column footings, is not practical since they represent relatively massive blocks which cantilever from the column in all four directions. Under uniform upward pressure they deform in a bowl shape, a fact which would greatly complicate an accurate stress analysis. For this reason present procedures for the design of such footings are based almost entirely on the results of two extensive, experimental investigations, both carried out at the University of Illinois.[1] The results of the latter investigation are not yet completely reflected in the latest edition of the ACI Code.

[1] A. N. Talbot, "Reinforced Concrete Wall and Column Footings," *Univ. Ill. Eng. Exp. Sta. Bull.* 67, 1913; F. E. Richart, "Reinforced Concrete Wall and Column Footings," *J. ACI*, vol. 20, pp. 97, 237, 1948.

**7.6. Reinforced Concrete Wall Footings.** The simple principles of beam action apply to wall footing with only minor modifications. Figure 7.3 shows a wall footing with the forces acting on it. If bending moments were computed from these forces, the maximum moment would be found to occur at the middle of the width. Actually, the very large rigidity of the wall modifies this situation, and the tests quoted in Art. 7.5 show that for footings under concrete walls it is satisfactory to compute the moment at the face of the wall (section 1-1). Tension cracks in these tests formed at the locations shown in Fig. 7.3, *i.e.*, under the face of the wall rather than in the middle. For footings supporting masonry walls, the maximum moment is computed midway between the middle and the face of the wall, since masonry is generally less rigid than concrete. The maximum bending moment in footings under concrete walls is, therefore, given by

$$M = {}^1\!/_8 p(l - a)^2 \qquad (7.4)$$

The calculation of bond stresses is based on the shear for the same section, *i.e.*,

$$V_b = {}^1\!/_2 p(l - a) \qquad (7.5)$$

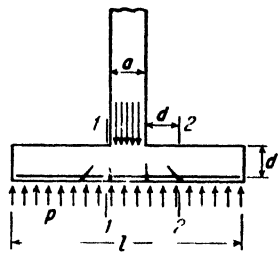


Fig. 7.3 Wall footing

and the allowable bond stress is the same as for beams, *i.e.*, $0.10f'_c$ for deformed bars and $0.045f'_c$ for plain bars. In previous codes, hooks were required for all footing bars. Richart's tests have shown that for deformed bars which satisfy the ASTM A 305 Specifications, hooks in footing bars are not necessary in order to develop the full strength of the bar. For this reason the 1951 edition of the ACI Code does not require hooks in footings, except if plain bars are used.

Although formal application of the beam theory would indicate that the maximum vertical shear stresses also occur at the face of the wall, the portion of the footing slab below and directly adjacent to the wall is primarily stressed in vertical compression. These compression stresses neutralize to a large degree the diagonal tension stresses caused by the vertical shear. In consequence, diagonal tension is largest, not at the face of the wall, but at some distance from it. Shear or diagonal tension cracks in the quoted tests formed, accordingly, in the manner shown in Fig. 7.3, and propagated upward and inward at about 45° toward the intersection of the wall surface with the top of the footing. Accordingly, for determining shear stresses, the vertical shear force is computed on section 2-2, located at a distance $d$ from the face of the wall. Thus,

$$V_s = p\left(\frac{l - a}{2} - d\right) \qquad (7.6)$$

It is generally not economical to use web reinforcement in footings. Hence, the allowable shear stress in section 2-2 is $0.03f'_c$ but not more than 75 psi, the values permitted by the ACI Code for concrete in footings.

**7.7. Design of a Wall Footing.** A 16-in. wall supports a total load of 23,100 lb per ft. The allowable bearing pressure of the soil is 2 tons per sq ft. Design a footing for this wall, using 2500-psi concrete and intermediate-grade steel.

Assuming the weight of the footing to be 900 lb per ft, the required width is 24,000/4,000 = 6 ft. The net upward pressure caused by the wall load alone (see Art. 7.8 for explanation) is 23,100/6 = 3850 lb per ft. This results in a moment at the critical sections 1-1 of

$$M = \tfrac{1}{8} \times 3850 \times (6 - 1.33)^2 \times 12 = 126{,}000 \text{ in.-lb per ft}$$

For the given grades of material, $K = 196$ and $j = 0.866$. The effective depth required for bending is

$$d = \sqrt{\frac{126{,}000}{196 \times 12}} = 7.3 \text{ in.}$$

Assuming temporarily $d = 7.5$ in., the shear force to be used in computing shear stresses at a distance $d$ from the wall is

$$V_s = 3850[\tfrac{1}{2}(6 - 1.33) - 7.5/12] = 6550 \text{ lb per ft}$$

and the effective depth required for shear

$$d = \frac{6550}{12 \times \tfrac{7}{8} \times 75} = 8.3 \text{ in.}$$

It is seen that shear governs in this case. A depth of $d = 8.5$ in. is selected. The ACI Code calls for a 3-in. clear insulation for footing bars, which results in a total depth of 12 in. The weight of the footing is then $6 \times 150 = 900$ lb per ft, as assumed.

The required area of reinforcement is

$$A_s = \frac{126{,}000}{20{,}000 \times 0.866 \times 8.5} = 0.856 \text{ sq in. per ft}$$

The shear force to be used in bond computations is

$$V_b = \tfrac{1}{2} \times 3850(6 - 1.33) = 9000 \text{ lb per ft}$$

and the required perimeter

$$\Sigma_0 = \frac{9000}{250 \times \tfrac{7}{8} \times 8.5} = 4.73 \text{ sq in. per ft}$$

Number 6 bars 6 in. center to center furnish $A_s = 0.88$ sq in. per ft and $\Sigma_0 = 4.7$ sq in. per ft. The required shrinkage reinforcement is 0.002

$\times$ 12 $\times$ 8.5 = 0.2 sq in. per ft of width of footing; No. 5 bars spaced 12 in. on centers provide 0.31 sq in. per ft. They have the added function of spacing the main reinforcement while the concrete is being poured. In addition, they help the footing to bridge accidental weak spots in the soil. For this reason it is good practice to select these bars somewhat in excess of the minimum requirements for shrinkage reinforcement.

**7.8. Single-column Footings, General Information.** In plan, single-column footings are usually square. Rectangular footings are used if space restrictions dictate this choice, or if the supported columns are of strongly elongated rectangular cross section. In the simplest form, they consist of a single slab, Fig. 7.4*a*. Another type is that of Fig. 7.4*b*, where a pedestal or cap is interposed between the column and the footing slab; the pedestal or cap provides for a more favorable transfer of load and in many cases is also required in order to provide the necessary length for dowels. This form is also known as a stepped footing. All

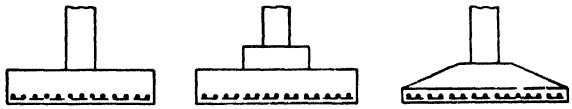



FIG. 7.4. Types of single-column footings.

parts of a stepped footing must be poured in a single pour, in order to provide monolithic action. Sometimes sloped footings like those in Fig. 7.4*c* are used. They require less concrete than stepped footings, but the additional labor necessary to produce the sloping surfaces (formwork, etc.) usually makes stepped footings more economical.

Single-column footings represent, as it were, cantilevers projecting out from the column in both directions, and loaded upward by the soil pressure. Corresponding tension stresses are caused in both these directions at the bottom surface. Such footings are therefore reinforced by two layers of steel, perpendicular to each other and parallel to the edges.

The required bearing area is obtained by dividing the total load, including the weight of the footing, by the selected bearing pressure. Weights of footings, at this stage, must be estimated and amount usually to 4 to 8 per cent of the column load, the former value applying to the stronger types of soils.

In computing bending moments and shears, only that part of the upward pressure is considered which is caused by the column load. The weight of the footing proper does not cause moments or shears, just as, obviously, no moments or shears are present in a book lying flat on a table.

The correct determination of bending, shear, and bond stresses in sin-

gle-column footings has long been a matter of some dispute and even today cannot be regarded as definitely settled. For this reason, in the following articles, information will be given not only on the design method as presently specified in the ACI Code but also on those points where this method is either confirmed by or somewhat at variance with the results of the quoted tests. In addition, previous design procedures will be briefly mentioned since they are still specified in some building codes and in foreign countries.

**7.9. Bending Moments and Reinforcement of Single Footings.** Earlier design rules and codes provided that the moment about the section *ab*, Fig. 7.5*a*, be determined by passing the imaginary cuts *ac* and *bd* through the footing and computing the moment about *ab* of the total upward pressure on the trapezoid *abcd*. Later it was recognized that this reasoning is inaccurate, since, in actuality, the critical section does

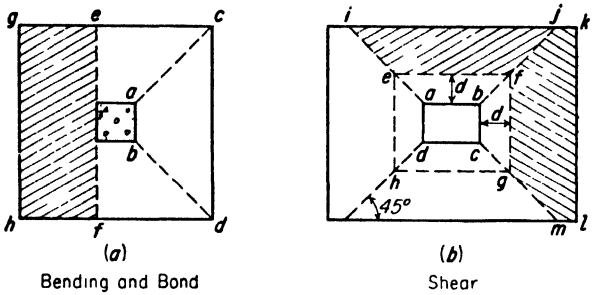



FIG. 7.5. Critical sections in single-column footings.

not end at *a* or *b* but extends without interruption to the edges. Consequently, since 1940, codes provide that the *critical section* for determining moments shall be located at the edge of the column and extend over the entire width of the footing (that is, *ef* in Fig. 7.5*a*). The design moment is that produced about *ef* by the upward pressure on the area *efgh*. This corresponds to the basic sense of a bending moment as being the sum of the moments of all forces located to one side of the section, and is fully born out by the very careful stress measurements made by F. E. Richart. The present ACI Code provides that the reinforcement perpendicular to *ef* shall be required to resist only 85 per cent of this bending moment. The tests by Richart do not confirm this assumption and show that the bending moment which is resisted by the reinforcement is equal to 100 per cent of the above value.

For footings supporting steel columns, the section *ef* is located, not at the edge of the steel base plate, but halfway between the edge of the column and that of the base plate.

In footings with pedestals, the width resisting compression in section *ef* is that of the pedestal; the corresponding depth is the sum of the

thicknesses of pedestal and footing. A further section parallel to *cf* is passed at the edge of the pedestal and the moment on it determined in the same manner, to check concrete and steel stresses at that location, where the depth is that of the footing only.

For footings with relatively small pedestals, the latter are often discounted in moment and shear computation, and bending is checked at the face of the column for a section *cf*, with width and depth equal to that of the footing proper.

In square footings, the reinforcement is uniformly distributed over the width of the footing, in each of the two layers, *i.e.*, the spacing of the bars is constant. The moments for which the two layers are designed are the same. However, the effective depth $d$ for the upper layer is less by one diameter than that of the lower layer. Consequently, the required $A_s$ is larger for the upper layer. Instead of using different spacings or different bar diameters in each of the two layers, it is customary to determine $A_s$ for the upper layer and to use the same arrangement of reinforcement for the lower layer.

In rectangular footings, the reinforcement in the long direction is again uniformly distributed over the pertinent (shorter) width. In locating the bars in the short direction, one has to consider that the support provided to the footing by the column is concentrated near the middle. Consequently, the curvature of the footing is sharpest, *i.e.*, the moment per foot largest, immediately under the column, and it decreases in the long direction with increasing distance from the column. For this reason a larger steel area per longitudinal foot is needed in the central portion than near the far ends of the footing. The ACI Code provides, therefore, that:

> In the case of the reinforcement in the short direction, the portion [of the reinforcement] determined by Eq. (7.7) shall be uniformly distributed across a band-width ($B$) centered with respect to the column and having a width equal to the length of the short side of the footing. The remainder of the reinforcement shall be uniformly distributed in the outer portion of the footing.

$$\frac{\text{Reinforcement in band-width } (B)}{\text{Total reinforcement in short direct}} = \frac{2}{(S+1)} \tag{7.7}$$

In formula (7.7), $S$ is the ratio of the long side to the short side of the footing.

**7.10. Shear and Bond in Single Footings.** The shear force $V_b$ which is used in determining *bond stresses* is calculated for the same critical section *cf* as the bending moment and is equal to the total net upward pressure on area *cfgh* of Fig. 7.5*a*. Consistent with its moment requirements, the ACI Code provides that for calculating bond stresses only $0.85V_b$ need be considered. This proposition is not confirmed by Richart's tests. The allowable bond stress for deformed bars in two-

way footings is $u = 0.08f'_c$, as compared to $0.10f'_c$ in beams. (This difference is often explained by the argument that bond is weakened in two-way footings at the numerous points where bars cross. Also, the tension perpendicular to one set of bars caused by the bending moment in the other direction tends to pull the concrete away from the bars, which may weaken bond strength.)

Until 1951, hooks were required on all footing bars, whether deformed or not. Richart's tests have shown that in column as in wall footings hooks do not contribute to the bond resistance of bars deformed in accordance with the requirements of ASTM A 305. For this reason, beginning with its 1951 edition, the ACI Code no longer calls for hooks on footing bars.

With regard to designing footings for *shear*, older specifications often provided for checking the "punching shear." That is, it was thought that in footings of insufficient depth the column would punch through the slab along its perimeter *abcd* (Fig. 7.5*b*). The quoted tests have shown, however, that failure in such cases occurs as shown on Fig. 7.6. That is, a truncated pyramid, with sides sloping at about 45°, is punched out of the slab under the column. As was discussed for wall footings in Art. 7.6, the portion of the slab under and directly adjacent to the column is essentially vertically compressed. The diagonal tension stresses caused by shear are reduced by this compression and attain their largest values only at some distance from the column. The tests give no entirely clear information on the exact value of that distance, which is probably equal to or somewhat less than the effective depth of the footing.

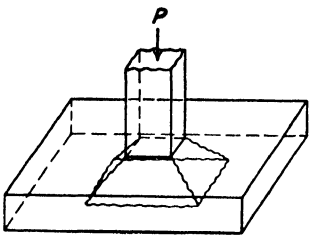


Fig. 7.6. Shear failure in single footing.

The ACI Code provides that in column as in wall footings the critical sections for shear shall be located at a distance $d$ from the faces of the column, and their lengths shall be determined by 45° lines drawn from the corners of the column. Hence, the critical sections for shear are *ef*, *fg*, *gh*, and *he* in Fig. 7.5*b*. They are seen to coincide approximately with the outline of the base of the truncated pyramid of Fig. 7.6. The shear force $V_s$ to be used in determining shear stresses is equal to the total net upward pressure on the bearing areas contributory to the respective critical sections. Hence, the shear on section *ef* is the total pressure on area *efji*, while for section *fg* the corresponding area is *fjklmg* (Fig. 7.5*b*).

It is not customary to provide shear reinforcement. Consequently, the permissible shear stress is that on concrete alone, $v_c = 0.03f'_c$. In many cases it is shear rather than bending which determines the depth of the footings.

*For round or octagonal columns,* the preceding rules for locating critical sections are not directly applicable, since, in this case, the "face of the column" needs special definition. The ACI Code specifies that for this purpose the "face" of the column shall be taken as the side of a square having the same area as that of the column.

**7.11. Transfer of Stress at Base of Column.** The compressive stress in the longitudinal reinforcement at the base of a reinforced concrete column is transferred to the pedestal or footing by means of dowels. There should be at least one dowel for each column bar and the total area of the dowels must be at least equal to the total area of the column bars. The dowels have to extend into the pedestal or footing the length necessary to transfer to the concrete, by bond, their full working value, *i.e.*, a force equal to the bar area times the allowable steel stress. The dowels must extend up into the column a distance equal to that required for lapping of longitudinal column bars. This latter length is specified as 20 bar diameters for concretes with $f'_c = 3000$ psi or more, and 26.7 bar diameters for concrete of smaller cylinder strength. For plain bars, twice the above values must be used.

When a column rests on a footing or pedestal, it transfers its load only to a part of the total area of the supporting member. The adjacent concrete, which is not under direct load, provides lateral support to the loaded part of the concrete, somewhat analogous to the function of spirals in columns. The permissible bearing stress on such partially loaded areas is, therefore, larger than that permitted in bearing over the full area. The ACI Code specifies the allowable bearing pressure on the full area (such as on the base area of a pedestal) as $0.25f'_c$, and on a partially loaded area (such as the top area of pedestal or footing) as $0.375f'_c$ if the loaded portion is one third or less of the full area. The allowable stress on an area larger than one third but less than the full area is obtained by linear interpolation between these two values. If the stress on the bottom surface of a pedestal is larger than $0.25f'_c$, the pedestal must be designed as a reinforced concrete column.

**7.12. Plain Concrete Footings.** For steel and timber columns with relatively light loads and for masonry piers and pilasters, plain unreinforced concrete footings are often economical. The ACI Code provides that the width and depth of such footings shall be such that the tension stress in the concrete shall not exceed $0.03f'_c$ and the shear stress $0.02f'_c$. The critical sections for bending and shear are determined in the same manner as for reinforced concrete footings.

To keep shear and, particularly, bending stresses within these limits for plain concrete, the depth of such footings must be made considerably larger than with reinforcement. Plain concrete footings are either stepped or sloped. The ratio of depth to width to satisfy the code

quirements depends on load, bearing capacity, and grade of concrete. As a rule of thumb it can be said that for stepped footings the ratio of projection to depth of any step usually does not exceed 1:2, while the sides of sloped footings subtend an angle with the vertical not greater than 30°.

**7.13. Design of a Square Footing.** A column 24 in. square and reinforced with twelve No. 8 hard-grade steel bars supports a total load of 400,000 lb. Design a square footing, using 2500-psi concrete and intermediate-grade steel. The safe soil pressure is 5000 psf.

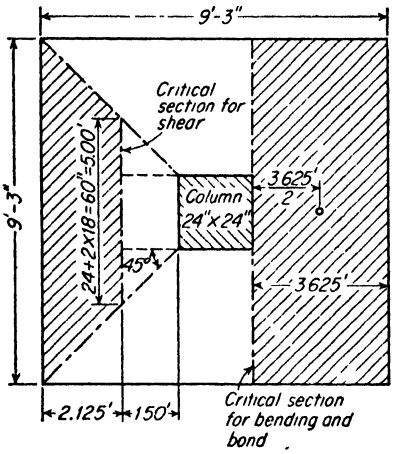



FIG. 7.7

Assuming the weight of the footing as 6 per cent of the column load, or 24,000 lb, the required bearing area is 424,000/5000 = 84.8 sq ft. A base 9 ft-3 in. square is selected, furnishing 85.5 sq ft. The net upward pressure due to the column load only is 400,000/85.5 = 4680 psf.

The depth of footings is mostly governed by shear stresses; but in order to locate the critical section for shear, the depth must be known or assumed. An approximate value is obtained by first computing the depth required for bending.

The maximum moment (see Fig. 7.7) is

$$M = 4680 \times 9.25 \times \frac{(3.625)^2}{2} \times 12 = 3{,}420{,}000 \text{ in.-lb}$$

With $K = 196$,

$$d = \sqrt{\frac{3{,}420{,}000}{9.25 \times 12 \times 196}} = 12.6 \text{ in.}$$

Since the depth required for shear is usually greater, $d = 18$ in. will be assumed for a trial. The critical section for shear (Fig. 7.7) is then 18 in. from the face of the column, and its width is

$$24 + 2 \times 18 = 60 \text{ in.} = 5.0 \text{ ft}$$

The shear on this section is

$$V_s = \frac{5.0 + 9.25}{2} \times 2.125 \times 4680 = 71{,}000 \text{ lb}$$

With an allowable shear stress $v = 75$ psi and assuming $j = 0.9$, the effective depth required for shear is

$$d = \frac{71{,}000}{60 \times 75 \times 0.9} = 17.5 \text{ in.}$$

The assumed value of $d = 18$ in. is seen to be adequate. (If a revision of the initial assumption were necessary, the width as well as the location of the critical section and consequently also $V_s$ would change.)

The computed value of $d$ refers to the upper layer of bars. Allowing 4 in. for insulation plus the distance center to center between top and bottom layer of bars, the total height of the footing is 22 in. and its weight 23,600, very close to the assumed value.

According to the 1951 ACI Code, the moment to be used for computing the steel area is $0.85 \times 3{,}420{,}000 = 2{,}910{,}000$ in.-lb and

$$A_s = \frac{2{,}910{,}000}{20{,}000 \times 0.9 \times 18} = 9.0 \text{ sq in.}$$

The shear force on the critical section for bond, the same as for bending, is 85 per cent of the net upward pressure on the area in front of that section or

$$V_b = 0.85 \times 9.25 \times 3.625 \times 1680 = 134{,}000 \text{ lb}$$

The allowable bond stress is $u = 0.08 \times 2500 = 200$ psi so that the required total perimeter

$$\Sigma_0 = \frac{134{,}000}{200 \times 0.9 \times 18} = 41.2 \text{ in.}$$

Sixteen No. 7 bars provide $A_s = 9.62$ sq in. and $\Sigma_0 = 44.0$ in. (If the reduction factor 0.85 were not applied to $M$ and $V_b$, as seems indicated by Richart's test results, the required $A_s$ and $\Sigma_0$ would be larger than the above values in the ratio 1.0/0.85, and nineteen No. 7 bars would have to be used.)

Dowels are placed in the footing, one for each column bar and equal to them in size, to transfer the stress from the column reinforcement into the footing. According to Art. 7.11, they should extend into the column 26.7 bar diameters, or, for No. 8 dowels, 2 ft-3 in. The full working value (bar force when stress is equal to allowable) of one column bar is $0.8 \times 20{,}000 \times 0.785 = 12{,}600$ lb, which is also the working value of the dowel. With a permissible bond stress $u = 0.10 \times 2500 = 250$ psi, the length by which the dowels have to extend down into the footing to transfer their forces to the concrete by bond is

$$l = \frac{12{,}600}{250 \times 3.142} = 16 \text{ in.}$$

where 3.142 is the perimeter of one No. 8 bar. This length can be accommodated within the 19 in. available in the footing (22 in. minus 3 in. for insulation) so that no pedestal is required.

*The bearing stress 400,000/24 × 24 = 692 psi is less than the allowable value of 0.375 × 2500 = 938 psi. Complete details are shown in* Fig. 7.8. For design examples of square footings supporting round columns, and for rectangular footings, see Arts. 8.53 and 8.54.

**7.14. Multiple-column Footings.** It is sometimes desirable and quite frequently necessary to support more than one column on the same footing. A multiple-column footing is necessary when two or more heavily loaded columns are so close to each other that there is not sufficient space for individual footings. They are also necessary when the face of an exterior column coincides with, or is close to, the property line so that a single footing centered under the column would project beyond that line. If an eccentric, single footing were used, this would result in unequal bearing-pressure distribution, with the possibility of tilting of the footing and consequent bending in the column. In such cases a combined footing, supporting the exterior and the adjacent interior column, can be so proportioned that the centroid of the footing area coincides with the resultant of the column loads producing an even pressure distribution and uniform settlement without tilting.

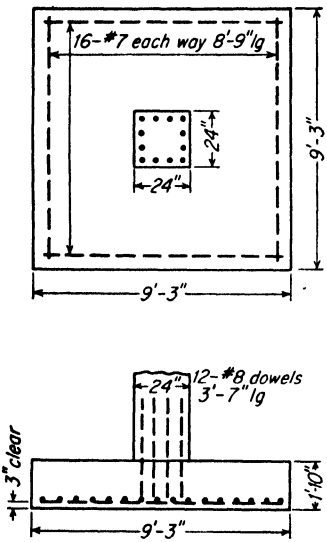


FIG. 7.8

A combined footing may be rectangular in plan or trapezoidal. The form is so chosen as to make the centroid and the resultant coincide. Hence, rectangular footings are suitable for interior columns, or for exterior columns when the exterior column has the lighter load and the footing may be extended beyond the interior column as far as necessary. The trapezoidal shape is required if column loads are unequal and if, for any reason, the footing cannot extend any appreciable distance beyond the heavier column.

Another expedient that is being used if an independent footing cannot be centered under an exterior column is to place the exterior-column footing eccentrically and to connect it with the nearest interior-column footing by a beam or strap poured monolithically with both footings. This beam, which is counterweighted by the interior-column load, resists the tendency of the eccentric exterior footing to overturn and equalizes the pressure under it. Such constructions are known as strap, or cantilever, or connected footings.

With respect to combined footings, the ACI Code (1951) states that

"the committee is not prepared at this time to make recommendations for combined footings." Proper judgment, however, can provide a reliable guide for a safe design. In locating critical sections, it will be recalled that the peculiarities of single-footing design stem from the fact that they represent deep blocklike structures as distinct from the more common beam and slab constructions in which the depth is small as compared to the span. It is logical, therefore, in combined footings to apply the single-footing procedures only where the action involved is of the same type as in these single footings, but to use beam-design procedures where the structure performs more in the manner of beams of ordinary proportions.

It follows that in the transverse direction, in which combined footings cantilever in much the same manner as single footings, the critical sections will be located as in the latter. In the direction of the line connecting the two columns, on the other hand, combined footings are essentially ordinary beams, with or without overhangs at one or both ends. Design sections are, therefore, located as in ordinary beam design (for flexure at points of maximum moment, for shear at the face of the columns, or of transverse beams, if such are used, etc.).

In regard to allowable bond stress it will be recalled that $0.10f'_c$ is permitted in beams, whereas $0.08f'_c$ is allowed in two-way footings. Hence, in combined footings, the latter value should be used where reinforcement is provided of the same type as in single footings, *i.e.*, in two layers at right angles, $0.10f'_c$ is permissible, on the other hand, where ordinary beam action is involved, except that, as in beams, $0.07f'_c$ must be used for "top bars." The same reasoning applies to the selection of allowable shear stresses.

**7.15. Design of a Combined Footing Supporting Two Interior Columns.** Two columns, each 24 in. square and each carrying 310,000 lb, are spaced 10 ft-0 in., as shown in Fig. 7.9. Design a rectangular combined footing for an allowable soil pressure of 4000 psf, using a 3000-psi concrete and intermediate-grade steel.

The footing proper consists of a slab spanning longitudinally and projecting beyond the columns in that direction. Since the slab, transversely, is wider than the columns, a transverse beam is provided under each column to distribute the load and spread it over the entire width of the slab. Assuming the weight of the footing to be

$$0.06 \times 2 \times 310{,}000 = 37{,}000 \text{ lb}$$

the required bearing area is

$$\frac{2 \times 310{,}000 + 37{,}000}{4000} = 164.3 \text{ sq ft}$$

The proper selection of width and length is governed by several factors. If the footing is too long, the moments in the portions projecting beyond the columns longitudinally will be excessive. If the footing is too short, the span moments between columns will be uneconomically large. Also, in this case the width will become quite large, which would call for excessively heavy transverse beams. A few approximate trial designs are necessary to secure the most economical proportions, considering excavation, concrete, and steel. In the present case the size finally selected is 18 ft-9 in. by 8 ft-9 in., providing an area of 164.1 sq ft.

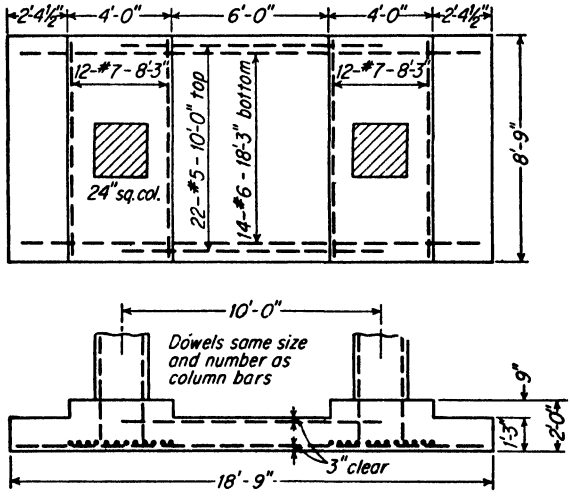



FIG. 7.9

*Design in Transverse Direction.* The load per linear foot of each of the transverse beams is 310,000/8.75 = 35,400 lb per ft. As in single footings, the moment will be computed at the edge of the column and is

$$M = 35{,}400 \times \frac{(3.375)^2}{2} \times 12 = 2{,}420{,}000 \text{ in.-lb}$$

The width of the transverse beam is selected by experience, usually from one to two times the width of the column. With a width $b = 48$ in. the required effective depth is

$$d = \sqrt{\frac{2{,}420{,}000}{235 \times 48}} = 14.7 \text{ in.}$$

This depth is insufficient for shear. Since the transverse beams are quite short and deep, the critical section for shear is assumed as in single footings, *i.e.*, at a distance $d$ from the face of the column. After several trial assumptions, $d = 20$ in. appears satisfactory. Then, at 20 in. from the

edge of the column, the critical shear force is

$$V_s = 35{,}400(3.375 - 2\%_{12}) = 60{,}500 \text{ lb}$$

and the shear stress

$$v = \frac{60{,}500}{48 \times 0.9 \times 20} = 70 \text{ psi}$$

Since the critical section has been selected by the procedure for single footings, to be consistent the allowable shear must also be that for such footing, *i.e.*, 75 psi. The depth is seen to be satisfactory. (A value of $j = 0.9$ is used throughout this computation because the reinforcement ratio $p$ is usually quite low in such footings, of the order of 0.005, which, from Table 6, Appendix D, gives a $j$ varying very little from 0.9.) A depth smaller than 20 in. would have required the use of web reinforcement, which is generally avoided in footings.

The required steel area is

$$A_s = \frac{2{,}120{,}000}{20{,}000 \times 0.9 \times 20} = 6.73 \text{ sq in.}$$

The critical section for bond is at the edge of the column where the shear force is

$$V_b = 35{,}400 \times 3.375 = 119{,}500$$

Since the reinforcement of the transverse beams is laid on top of that of the longitudinal slab (see Fig. 7.9), the bar arrangement is the same as in single footings and the permissible bond stress therefore is 240 psi. Hence,

$$\Sigma_0 = \frac{119{,}500}{240 \times 0.9 \times 20} = 27.7 \text{ in.}$$

Twelve No. 7 bars are selected, which provide $A_s = 7.22$ sq in. and $\Sigma_0 = 33$ in.

*Design in Longitudinal Direction.* The load per linear foot of the longitudinal slab, which spans between transverse beams, is

$$(2 \times 310{,}000)/18.75 = 33{,}050 \text{ lb per ft}$$

The positive moment at the outer edge of the transverse beam is

$$M = 33{,}050 \times \frac{(2.375)^2}{2} \times 12 = 1{,}120{,}000 \text{ in.-lb}$$

while the negative moment midway between columns is

$$M = \left(33{,}050 \times \frac{(9.375)^2}{2} - 310{,}000 \times 5\right) 12 = -1{,}200{,}000 \text{ in.-lb}$$

The latter is seen to govern, and the required effective depth becomes

$$d = \sqrt{\frac{1{,}200{,}000}{235 \times 8.75 \times 12}} = 7.0 \text{ in.}$$

In the longitudinal direction the slab represents a beam of great width, but with a span-depth ratio of customary magnitude, *i.e.*, the depth is a small fraction of the span. For this reason shear and bond stresses are computed as in beams rather than as in footings, and allowable stresses are taken correspondingly.

The shear force at the inner face of the transverse beam is

$$V = 33{,}050(2.375 + 1.0) - 310{,}000 = -99{,}000 \text{ lb}$$

and at the outer face

$$V = 33{,}050 \times 2.375 = 78{,}500 \text{ lb}$$

The former is seen to govern and, with an allowable shear stress of 90 psi, the required depth becomes

$$d = \frac{99{,}000}{90 \times 8.75 \times 12 \times 0.9} = 11.6 \text{ in.}$$

A depth of $d = 11.5$ is selected, which results in $v = 91$ psi, only about 1 per cent larger than the allowable value. With a total insulation of 3.5 in. from the center of bars to the bottom of the slab, the total slab thickness is $11.5 + 3.5 = 15$ in.

The steel area required midway between columns is

$$A_s = \frac{1{,}200{,}000}{20{,}000 \times 0.9 \times 11.5} = 5.8 \text{ sq in.}$$

This reinforcement, near the top face of the slab, is required theoretically only to the point where the negative moment becomes zero, *i.e.*, the inflection point. It is at this point that the shear force is largest anywhere along the required length of these bars, and hence the bond stress must be checked at that point. To find the inflection point, the moment at any section is expressed in term of the distance $x$ from one end of the slab, *i.e.*,

$$M = 33{,}050\frac{x^2}{2} - 310{,}000(x - 1.375)$$

To find the point of zero moment, the above expression is equated to zero, and the resulting quadratic equation is solved for $x$. One obtains, in this manner, $x = 6.60$ ft. The shear force at this point, which is just inside the inner edge of the transverse beam, is

$$V = 33{,}050 \times 6.6 - 310{,}000 = -92{,}000 \text{ lb}$$

For *top bars* the Code allows a bond stress of 210 psi so that the required perimeter is

$$\Sigma_0 = \frac{92{,}000}{210 \times 0.9 \times 11.5} = 42.3 \text{ in.}$$

Twenty-two No. 5 bars are selected, which furnish $A_s = 6.64$ sq in. and $\Sigma_0 = 43.2$ in. (It is seen that $A_s$ is considerably in excess of the required 5.8 sq in. Selecting a larger number of smaller bars would have led to some saving of steel. It would have resulted, however, in an inconveniently small spacing of bars and the saving probably would have been offset by the greater labor cost in handling and placing the larger number of bars.) These bars are placed 3.5 in. from the top of the slab and are extended to the column center lines.

At the outer edge of the transverse beam, the moment was found to be 1,120,000 in.-lb so that the required bottom steel at this section is

$$A_s = \frac{1{,}120{,}000}{20{,}000 \times 0.9 \times 11.5} = 5.4 \text{ sq in.}$$

The shear force at the same section was found to be 78,500 lb and the perimeter required for bond, with $u = 240$ psi for bottom bars, is

$$\Sigma_0 = \frac{78{,}500}{240 \times 0.9 \times 11.5} = 31.6 \text{ in.}$$

At the outer edge of the column,

$$M = 33{,}050 \times \frac{(3.375)^2}{2} \times 12 = 2{,}250{,}000 \text{ in.-lb}$$

Since the longitudinal steel is laid below the transverse bars, the effective depth at this section is about $\frac{3}{4}$ in. greater than that of the transverse beams, that is, $d = 20.75$ in. Computation shows that this is more than twice the required depth for the above moment. The required bottom steel area at this section is

$$A_s = \frac{2{,}250{,}000}{20{,}000 \times 0.9 \times 20.75} = 6.02 \text{ sq in.}$$

At the same section

$$V = 33{,}050 \times 3.375 = 111{,}500 \text{ lb}$$

$$\Sigma_0 = \frac{111{,}500}{240 \times 0.9 \times 20.75} = 24.8 \text{ in.}$$

These computations show that for the bottom longitudinal bars the required area is governed by the section at the outer edge of the column ($A_s = 6.02$ sq in.), while the required perimeter is critical at the outer

edge of the transverse beam ($\Sigma_0 = 31.6$ in.). Fourteen No. 6 bars are selected, which furnish $A_s = 6.19$ sq in. and $\Sigma_0 = 38.5$ in. These bottom bars should extend from each end of the footing to a section at least 12 diameters beyond the inflection point. This would leave about 4 ft midway between columns where bottom steel would not be required. Instead, the bars will be detailed to extend the full length of the footing. This simplifies placing and also safeguards against the possibility that uneven settlement due to soil or load irregularities may cause a different distribution of moments, with inflection points closer to the center than computed. If no steel were provided, tension cracks would then result. In view of the uncertainties inherent in most foundation design, most engineers prefer this method of placing at least some reinforcement in portions of foundation beams where computation does not indicate its necessity, but where such steel may become necessary because of such irregularities. It is for this same reason that the top bars were detailed to extend from center to center of columns. (In contrast, there is no conceivable distribution of loads or reactions which could cause tension, and require steel, in the top of the outer cantilevered portions of the longitudinal slab.)

The transfer of stress at the base of the column is handled in the same manner as in single footings and will not be discussed in detail. The weight of the footing as designed is 37,900 lb, which is sufficiently close to the assumed value of 37,000 lb.

Figure 7.9 shows complete details. In designing the reinforcement for this footing, the aim was maximum economy, which results from using the smallest number of bars compatible with area and perimeter requirements. This method, in this case, led to the use of three different bar diameters within the same footing. This is not objectionable on larger constructions, particularly if the footings are repetitive. For smaller structures, it may be more advantageous to use uniform bar diameters as far as possible, even at some sacrifice in steel and labor cost, to avoid extra charges for small quantities of various diameters and the problem of separate storage on the site of a great variety of various bar sizes.

**7.16. Design of a Combined Footing Supporting One Exterior and One Interior Column.** An exterior column, 24 by 18 in. with a total load of 200,000 lb, and an interior column, 24 by 24 in. with a total load of 300,000 lb, are to be supported on a combined rectangular footing whose exterior end cannot protrude beyond the outer face of the exterior column (see Fig. 7.2). The distance center to center of columns is 18 ft-0 in. The allowable soil pressure is 4000 psf. A 3000-psi concrete and intermediate-grade steel are to be used.

It is difficult to estimate the weight of footings of this type. The bearing area, as in single footings, depends only on the column loads and the

bearing pressure. The depth, however, is often governed by the longitudinal bending moment, which increases rapidly with increasing distance between columns. Hence, for equal column loads and bearing pressures, larger column spacings result in heavier footings. As a crude rule, 6 to 15 per cent of the combined column loads can be taken as the approximate weight of such footings. In the present case, since the column spacing is relatively large and the bearing capacity rather low, the footing weight will be assumed as 12 per cent of the column loads, or 60,000 lb. The required bearing area is then $560{,}000/4000 = 140$ sq ft.

In order to secure uniform soil pressure, the centroid of the bearing area must coincide with the resultant of the column loads. The latter is located at a distance of $300{,}000 \times 18/500{,}000 = 10.8$ ft from the center of the exterior column. The length of the footing must be

$$2(10.8 + 0.75) = 23.1 \text{ ft}$$

A length of 23 ft-3 in. is selected. The required width is then

$$140/23.25 = 6.0 \text{ ft}$$

(see Fig. 7.11).

The method of design is analogous to that of the symmetrical footing of Art. 7.15. That is, longitudinally it represents an upward-loaded beam spanning between columns and cantilevering beyond the interior column. Since this beam is considerably wider than the columns, the column loads are distributed crosswise by transverse beams, one under each column. In the example of Art. 7.15, the relatively large width of the footing resulted in large moments and shears in these transverse beams so that their required depth was found to be larger than that of the footing slab. Hence, the effective width, for compression and shear, of these transverse beams was simply equal to the real width of the portion projecting above the slab. In the present relatively narrow and long footing it will be found that the required minimum depth for the transverse beams is smaller than is required for the footing in the longitudinal direction. These "beams," therefore, are not really distinct members but merely represent transverse strips in the main body of the footing, so reinforced that they are capable of resisting the transverse bending moments and the corresponding shears. It then becomes a question to decide how large the effective width of this transverse "beam" can be assumed to be. Obviously, the strip directly under the column does not deflect independently and is strengthened by the adjacent parts of the footing. The effective width of the transverse beams is, therefore, evidently larger than that of the column, but cannot be taken as very much larger than the latter since the column load must be able to distribute itself reasonably uniformly over the entire width of the beam. In the

absence of definite rules for this case, or research results on which to base such rules, the authors recommend that the effective width of such integral, transverse beams be determined in the following manner. The effective width shall be equal to that of the column plus, on either side of the column, a strip of width equal to half of that of the column, or to half the depth of the footing, whichever is smaller. (This rule assumes that the load from the column will spread at an angle of about 30° with the vertical, a rather conservative assumption.)

*Design in Longitudinal Direction.* The net upward pressure per longitudinal foot is 500,000/23.25 = 21,500 lb per ft. The maximum negative moment between the columns occurs at the section of zero shear.

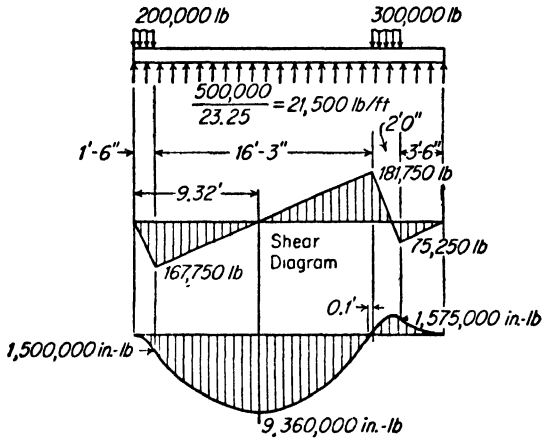



FIG. 7.10

Let $x$ be the distance from the outer edge of the exterior column to this section. Then (see Fig. 7.10)

$$V = 21{,}500x - 200{,}000 = 0$$

results in $x = 9.32$ ft. The moment at this section is

$$M = \left[21{,}500 \times \frac{(9.32)^2}{2} - 200{,}000(9.32 - 0.75)\right] 12 = -9{,}360{,}000 \text{ in.-lb}$$

The moment at the right edge of the interior column is

$$M = 21{,}500 \times \frac{(2.75)^2}{2} \times 12 = 1{,}575{,}000 \text{ in.-lb}$$

and the details of the moment diagram are as shown on Fig. 7.10.

For the given grades of steel and concrete, $K = 235$, and the depth required for bending is

$$d = \sqrt{\frac{9{,}360{,}000}{235 \times 72}} = 23.5 \text{ in.}$$

This computation shows that the depth of the footing is at least equal to the width of the interior column so that the width of the transverse beam under the interior column can now be established by the previously discussed rules to be $24 + 2 \times 12 = 48$ in. The shear will be checked

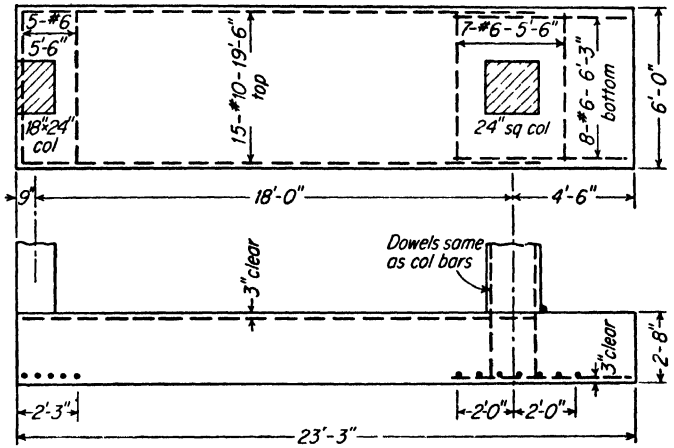



FIG. 7.11

at the interior "edge" of this transverse beam. From the shear diagram of Fig. 7.10 it is seen that at this point the shear is

$$V_i = 181{,}750 - 1 \times 21{,}500 = 160{,}250 \text{ lb}$$

By the same rule the width of the exterior transverse beam is

$$18 + 9 = 27 \text{ in.}$$

and the shear at the interior "edge" of that beam is

$$V_e = 167{,}750 - 0.75 \times 21{,}500 = 151{,}400 \text{ lb}$$

Since, longitudinally, the footing acts as a beam of normal span-depth ratio, if no web reinforcement is to be used, the allowable shear stress is that permitted for ordinary beams, *i.e.*, 90 psi, and the depth required for shear is

$$d = \frac{160{,}250}{90 \times 72 \times \frac{7}{8}} = 28.3 \text{ in.}$$

With $d = 28.5$ in. and 3.5 in. insulation from the center of bars to the concrete surface, the total thickness of the footing is 32 in. and its weight 55,700 lb, which agrees closely with the assumed value.

For the negative moment in the span between the columns, the required steel area is

$$A_s = \frac{9{,}360{,}000}{20{,}000 \times \frac{7}{8} \times 28.5} = 18.8 \text{ sq in.}$$

The critical sections for bond, for these bars, are at the inner edge of the exterior column and at the point of inflection (see Fig. 7.10), which is about 0.1 ft from the edge of the interior column. It is seen from the shear diagram that the latter section governs in this design, and

$$V_b = 181{,}750 - 0.1 \times 21{,}500 = 179{,}600 \text{ lb}$$

For top bars, the allowable bond stress is 210 psi and

$$\Sigma_0 = \frac{179{,}600}{210 \times \frac{7}{8} \times 28.5} = 34.3 \text{ in.}$$

Fifteen No. 10 bars are selected, with a spacing of about $4\frac{1}{4}$ in.

For the portion of the longitudinal beam which cantilevers beyond the interior column,

$$A_s = \frac{1{,}575{,}000}{20{,}000 \times \frac{7}{8} \times 28.5} = 3.15 \text{ sq in.}$$

The critical section for bond is at the edge of the column, where

$$V = 21{,}500 \times 3.5 = 75{,}250 \text{ lb}$$

Since these bars are at the bottom of the footing and in contact with the transverse bars, the allowable bond stress is the same as in single footings, *i.e.*, 240 psi. Hence,

$$\Sigma_0 = \frac{75{,}300}{240 \times \frac{7}{8} \times 28.5} = 12.6 \text{ in.}$$

Eight No. 6 bars are selected.

*Design of Transverse Beam under Interior Column.* The net upward load per linear foot of the transverse beam is 300,000/6 = 50,000 lb per ft. The moment at the edge of the interior column is

$$M = 50{,}000 \times \frac{2^2}{2} \times 12 = 1{,}200{,}000 \text{ in.-lb}$$

Since the effective width of the transverse beam was determined as 48 in., the required depth is

$$d = \sqrt{\frac{1{,}200{,}000}{48 \times 235}} = 10.3 \text{ in.}$$

Since the transverse bars are placed on top of the longitudinal bars (see Fig. 7.11), the actual value of $d$ furnished is, to the nearest quarter inch,

$28.5 - 0.75 = 27.75$ in. Then

$$A_s = \frac{1{,}200{,}000}{20{,}000 \times 0.9 \times 27.75} = 2.40 \text{ sq in.}$$

At the edge of the column, $V = 50{,}000 \times 2 = 100{,}000$ lb, and

$$\Sigma_0 = \frac{100{,}000}{240 \times 0.9 \times 27.75} = 16.7 \text{ in.}$$

Seven No. 6 bars are selected and placed within the 48-in. effective width of the transverse beam, at a spacing of about 8 in.

Since the transverse beam is quite deep in relation to its span, *i.e.*, dimensionally close to single footings, the critical section for shear can be taken as in such footings, *i.e.*, a distance $d$ from the column. Since this section falls outside the footing, no investigation for shear is called for.

(It will be noticed that in the longitudinal design $j = \frac{7}{8}$ was used and in the transverse design $j = 0.9$. From Table 5, Appendix D, it is seen that the former value applies closely for steel ratios not too much below that for balanced reinforcement. This is the case in the longitudinal direction. The transverse beam, in view of its excessively large $d$, has a very small steel ratio, in which case $j = 0.9$ is a closer approximation. A stress check for the designed dimensions and reinforcements will show the design on the basis of these approximate $j$ values to be quite close.)

The design of the transverse beam under the exterior column is in no way different from that under the interior column, except that the effective width is 27 in., as previously determined. The details of the calculations are not shown. It will be easily checked that five No. 6 bars, placed within the 27 in. effective width, satisfy all requirements. Design details are shown in Fig. 7.11.

**7.17. Design of a Strap Footing.** In a strap or connected footing, the exterior footing is placed eccentrically under its column, in order not to project beyond the property line. Such a position would result in an uneven distribution of the bearing pressure and tipping of the footing. To counteract this tendency, the footing is connected by a beam or strap to the nearest interior footing.

The footing areas are so proportioned that the pressure under each of them is uniform and is the same under both footings. To achieve the latter, it is necessary, as in other combined footings, that the centroid of the combined area of the two footings coincide with the resultant of the column loads. The strap proper is generally constructed in such a manner that it will not bear directly on the soil.

To illustrate this design, the columns of Art. 7.16 will be so supported. Connected footings for larger column distances generally require less concrete than rectangular footings. Therefore the combined weight will be

*assumed to be 0.08(300,000 + 200,000) = 40,000 lb. The total required bearing area is 340,000/4000 = 135 sq ft.*

*In Art. 7.16 the resultant of the column loads was found to be located* 10.8 ft from the center of the exterior column, or 11.55 ft from the exterior face of that column. To distribute the two footing areas so that their total will add up to the required 135 sq ft and their centroid will be located at 11.55 ft from the outer edge requires several trial computations. The arrangement shown in Fig. 7.13 was determined by such trials and provides exactly 135 sq ft of area. Its centroid is located from the outer edge at a distance $\bar{x} = (72 \times 18.75 + 63 \times 3.0)/135 = 11.40$ ft, which only insignificantly differs from 11.55 ft.

To compute the moments and shears in the various parts of the connected footings, it is first necessary to determine the intensity and location of all external forces and reactions. These are shown in Fig. 7.12.

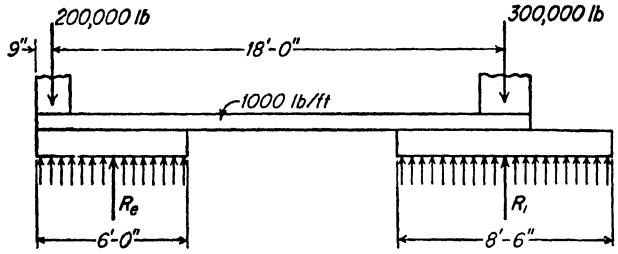



FIG. 7.12

Since the footing areas are so arranged that the pressure on each of them is uniform, the resultants of these net pressures, $R_e$ and $R_i$, act at the centers of the respective footings. Assuming the weight of the strap to be 1000 lb per ft, $R_e$ is found by equating to zero the moments about the center line of the interior column, *i.e.*,

$$200{,}000 \times 18.0 + 1000 \times \frac{(18.75)^2}{2} - R_e \times 15.75 = 0$$

and hence $R_e = 240{,}000$ lb. The other reaction $R_i$ is then found by equating to zero the sum of all vertical forces, *i.e.*,

$$200{,}000 + 300{,}000 + 1000 \times 18.75 - 240{,}000 - R_i = 0$$

hence $R_i = 278{,}750$ lb.

*Design of Strap.* The maximum moment in the strap occurs at the section at which the shear force is zero. Designating by $x$ the distance of this section from the exterior edge, and noting that the uniform upward load from the exterior footing, per foot of strap, is 240,000/6 = 40,000 lb per ft,

$$V = -200{,}000 + 40{,}000x - 1000x = 0$$

from which $x = 5.13$ ft. The moment at this section is

$$M = \left[-200{,}000(5.13 - 0.75) - 1000 \times \frac{(5.13)^2}{2} + 40{,}000 \times \frac{(5.13)^2}{2}\right] \times 12 = -4{,}360{,}000 \text{ in.-lb}$$

For the strap a width $b = 28$ in. is selected. With a column width of 24 in. this permits simple placing of beam and column bars where they intersect and allows the column forms to be supported on the top surface of the strap. Then

$$d = \sqrt{\frac{4{,}360{,}000}{235 \times 28}} = 25.8 \text{ in.}$$

With an effective depth of 26.5 in. and, for insulation, 3.5 in from the center of bars to the bottom face, the total depth of the strap becomes 30 in. The required steel area is

$$A_s = \frac{4{,}360{,}000}{20{,}000 \times 0.866 \times 26.5} = 9.5 \text{ sq in.}$$

The shear at the face of the column

$$V = -200{,}000 - 1000 \times 1.5 + 40{,}000 \times 1.5 = -141{,}500 \text{ lb}$$

requires, for top bars, a total perimeter of

$$\Sigma_0 = \frac{141{,}500}{210 \times 0.866 \times 26.5} = 29.4 \text{ in.}$$

Ten No. 9 bars furnish $A_s = 10$ sq in. and $\Sigma_0 = 35.4$ in. They can be placed in one layer with clear distances slightly larger than the bar diameter, which is permissible by the Code. (Some designers may feel that even though the formal spacing requirements are satisfied, a solid layer of top bars with clear distances between them of about 1⅛ in. may seriously interfere with placing the concrete. In this case bars can be arranged in two layers of five bars each, with a clear vertical distance of 1 in., which would call for a total depth of 31 in. for the strap.)

The moment in the strap decreases toward the interior column and is zero at the center line of that column. Hence, part of the strap reinforcement can be discontinued where no longer needed. Half of that reinforcement will be terminated where the moment in the strap is half the maximum moment, that is, $M = 4{,}360{,}000/2 = 2{,}180{,}000$ in.-lb. The net downward load on the strap at the interior column is

$$300{,}000 - 278{,}750 = 21{,}250 \text{ lb}$$

If $x$ is the distance from the center line of the interior column to the

*point where the moment has the above value, one obtains*

$$21{,}250x + 1000\left(\frac{x^2}{2}\right) = \frac{2{,}180{,}000}{12}$$

from which $x = 7.35$ ft. The distance of this point from the outer face of the wall is then 18 ft-9 in. − 7 ft-4 in. = 11 ft-5 in. Bars are extended

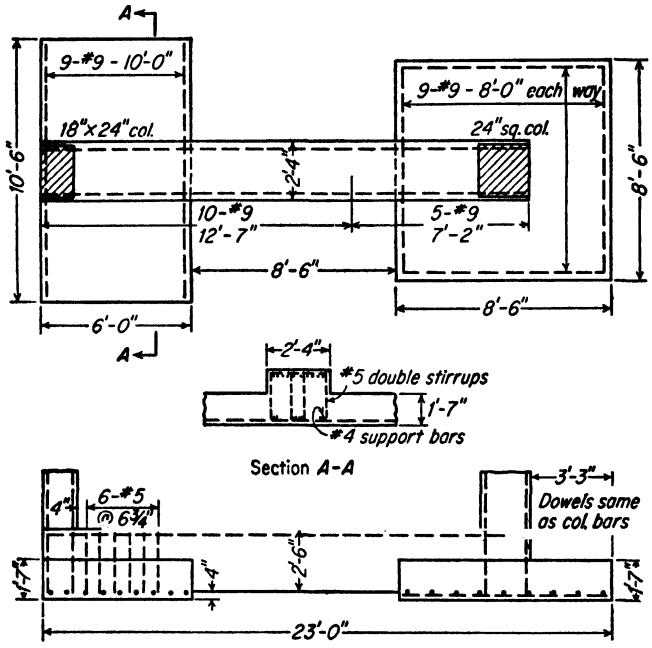



FIG. 7.13

12 diameters beyond the point where they are no longer needed. Hence, half of the No. 9 bars can be discontinued at a distance of

$$11 \text{ ft-5 in.} + 12 \times \frac{1.156}{12} = 12 \text{ ft-7 in.}$$

from the outer face of the wall column, as shown in Fig. 7.13.

The maximum shear stress is

$$v = \frac{141{,}500}{28 \times 0.866 \times 26.5} = 220 \text{ psi}$$

which exceeds $v = 90$ psi allowed for the concrete. Web reinforcement is, therefore, called for. To determine the length over which stirrups are needed, the shear carried by the concrete is determined to be

$$V_c = 90 \times 28 \times 0.866 \times 26.5 = 57{,}900 \text{ lb}$$

The distance $x$ from the outer face to the point where the concrete alone is capable of carrying the shear is found from

$$V = -200{,}000 + 40{,}000x - 1000x = -57{,}900$$

from which $x = 3.65$ ft.

The required spacing of No. 5 double-looped stirrups at the inner edge of the exterior column is

$$s = \frac{4 \times 0.307 \times 20{,}000 \times 0.866 \times 26.5}{141{,}500 - 57{,}900} = 6\tfrac{3}{4} \text{ in.}$$

Hence, over a distance from the inner edge of the column of

$$3.65 - 1.5 = 2.11 \text{ ft}$$

six stirrups are required at the above spacing.

*Design of Exterior Footing.* The exterior footing performs exactly like a wall footing of 6 ft length. Even though the column is located at its edge, the balancing action of the strap is such as to transmit the total reaction $R_e = 240{,}000$ lb uniformly over the 6-ft length, thus resulting in the desired, uniform soil pressure. The net upward pressure is

$$\frac{240{,}000}{(10.5 \times 6)} = 3810 \text{ psf}$$

The distance from the face of the strap to the edge of the footing is $5.25 - {}^{14}\!/_{12} = 4.09$ ft so that the moment at the strap

$$M = 6 \times 3810 \times \frac{(4.09)^2}{2} \times 12 = 2{,}300{,}000 \text{ in.-lb}$$

The required effective depth is

$$d = \sqrt{\frac{2{,}300{,}000}{235 \times 72}} = 11.7 \text{ in.}$$

Computation shows that this depth is not sufficient for shear. For $d = 15$ in., the critical section for shear is located at a distance from the strap equal to $4.09 - {}^{15}\!/_{12} = 2.84$ ft, so that the shear force at that section becomes

$$V_s = 3810 \times 6 \times 2.84 = 64{,}900 \text{ lb}$$

The shear stress

$$v = \frac{64{,}900}{72 \times {}^{7}\!/_{8} \times 15} = 68.7 \text{ psi}$$

is smaller than the permissible value of 75 psi for concrete alone.

The required steel area is

$$A_s = \frac{2{,}300{,}000}{20{,}000 \times {}^{7}\!/_{8} \times 15} = 8.76 \text{ sq in.}$$

*At the critical section for bond, i.e., at the face of the strap,*

$$V_b = 3810 \times 6 \times 4.09 = 93{,}500 \text{ lb}$$

and

$$\Sigma_0 = \frac{93{,}500}{300 \times \frac{7}{8} \times 15} = 23.8 \text{ in.}$$

Nine No. 9 bars furnish $A_s = 9.00$ sq in. and $\Sigma_0 = 31.8$ in.

The *interior footing* is designed as a simple, two-way square footing for an effective column load of $R_i = 278{,}750$ lb, disregarding the slightly modifying influence of the strap. The procedure is exactly the same as that of Art. 7.13. Computations are not shown here. Dimensions and reinforcement are given in Fig. 7.13.

The weight of this strap footing is 44,700 lb, sufficiently close to the assumed value of 40,000 lb. It is interesting to compare this value with the weight of the combined footing designed in Art. 7.16 for the same situation. This weight was 55,700 lb, or about 25 per cent more than that of the strap footing. The actual saving in using the latter will be somewhat smaller than 25 per cent, since the steel ratio in the strap footing is larger than that in the combined footing, and since somewhat more complex formwork is required for the former. Even so, the saving is substantial and would be even larger for greater distances between the two columns.

Complete details are shown on Fig. 7.13. The bottom of the strap has been located 4 in. above the bottom of the footings. This permits the bottom planks of the strap forms to be placed on wedges, so that they can be withdrawn after the concrete has hardened. This prevents direct bearing of the strap on soil. Some designers recommend that additional longitudinal reinforcement be placed at the bottom of the strap to guard against moments of opposite sign which may conceivably occur because of irregular settlement.

**7.18. Footings on Piles.** If soil conditions require the use of piles, these are usually driven in clusters, one to each column, and the load is transferred from the column to the piles by a footing. The design of such pile-supported footings is not much different from those resting directly on soil, except for some special features.

The size of the footing is determined by the required number of piles and by the spacing between them, which usually, for friction piles, is not less than 3 ft for concrete piles or 2.5 ft for timber or H piles. End-bearing piles are sometimes driven at closer distances. The top of the piles must be securely embedded in the footing. For this purpose the bottom of the footing is located not less than 6 in. below the top of the piles, and the distance from the center of outside piles to the edge of the footing is not

made less than 1.5 ft. Reinforcement is located at a distance of 3 in. above the tops of the piles.

Occasionally individual piles are not driven to the design depth, with the result that their tops are above the planned elevation. Under no circumstances should the footing reinforcement in such cases be bent around the protruding pile. If this were done, the smaller depth of concrete left above the top of such a pile and the complete absence of reinforcement above it would make it possible for the pile to punch through the footing. For this reason such piles must be cut to the required elevation before the footing is poured.

In designing the footing, the load from the column is assumed to be uniformly distributed to all piles in the cluster. For this purpose piles must be arranged symmetrically about the axis of the column. The net load per pile, *i.e.*, the column load divided by the number of piles, is then assumed to act as an upward load on the footing, concentrated at the center of the pile.

The critical sections for moment and bond are the same as for footings resting on soil, and the ACI Code allows the same coefficient of 0.85 to be applied in computing tensile and bond stresses.

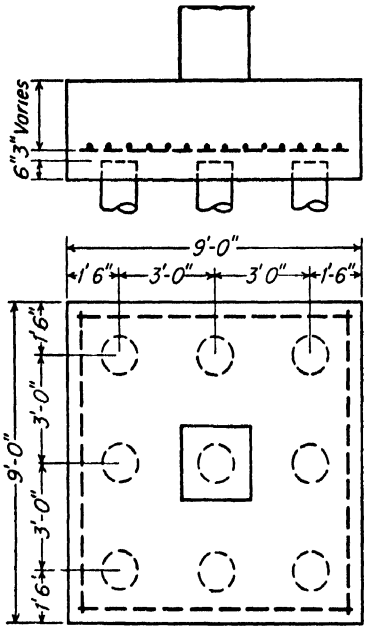
A somewhat more conservative assumption than in soil-supported footings is used in locating the critical sections for shear. These sections are placed at a distance of $d/2$ (rather than $d$) from the faces of the column or the pedestal, as the case may be.

While for bending computations the pile reactions are assumed to be concentrated at the pile centers, in computing shear forces (both for bond and for shear calculations) account is taken of the fact that these reactions are actually distributed over the bearing area of the pile. The fact that shear is usually the critical feature which determines the depth of a footing is the reason for the greater refinement in this determination. It is obviously unrealistic to assume that if the center of a pile is located 1 in. inside the critical section it will not produce any shear on that section, while, if it is located 1 in. outside, its full value will contribute to the shear. Correspondingly, the ACI Code stipulates:

> In computing the external shear, the entire reaction from any pile whose center is located six inches or more outside the section shall be assumed as producing shear on the section; the reaction from any pile whose center is located six inches or more inside the section shall be assumed as producing no shear on the section. For intermediate positions of the pile center, the portion of the pile reaction to be assumed as producing shear on the section shall be based on straight line interpolation between full value at six inches outside the section and zero value at six inches inside the section.

Actual pile positions frequently differ from assigned locations. On occasion this happens even with careful construction supervision and

workmanship, particularly in irregular soils where boulders and other obstructions may deflect piles during driving. Such off-location piles may produce moments and shears significantly in excess of the computed values. By the above-quoted rule, unfavorable pile displacements as




TYPICAL PILE CAP

Fig 7 14. Typical pile cap.

small as 1 ft may change the shear on the section by an amount equal to the total pile reaction Pile footings for soils in which such displacements are likely to occur should, therefore, be designed conservatively, that is, with unit stresses considerably below the stipulated values. By this means a strength reserve is created which is available in case actual moments and shears exceed the values computed for perfect pile location. Details of a typical footing on piles are shown in Fig 7.14.

# SECTION 8

# REINFORCED CONCRETE BUILDINGS

**8.1. Adaptability of Reinforced Concrete.** Reinforced concrete has gradually become one of the leading building materials of the present day, chiefly because of its durability, its fire-resisting qualities, its adaptability to various types of design, and its pleasing architectural appearance. When used with any other type of construction, as for example, the floors in a steel frame structure, or by itself in a building all of whose constituent structural parts are of reinforced concrete, its suitability is well recognized.

In determining the type of structure to be used for any particular building, usually the two most important considerations are the time required before the building may be occupied, and the relative economy of the selected type as compared with the other available structures. While the actual erection of a steel frame building may be completed in considerably less time than a reinforced concrete building, in many cases the length of time necessary for the fabrication of the steel will result in the lapse of a longer period of time from the letting of the contract to the completion of the structure than in the case of reinforced concrete construction.

Steel frame structures in which no attempt is made to encase the steel may be lower in first cost than those of reinforced concrete. If, however, an attempt is made to have the steel structure as fireproof as the reinforced concrete structure, the ratio of relative first costs may be reversed. This is especially true of certain types of buildings in which long spans and heavy loads exist. A real comparison between buildings of different materials should be made only after a consideration of the first cost and subsequent annual expenditures. Experience has shown that, for many types of structures, the real cost of reinforced concrete construction compares favorably with that of other materials.

Another factor favorable to reinforced concrete is that, in times of materials shortages, reinforcing steel and cement may usually be obtained long after structural steel has disappeared from the open market.

**8.2. Floor and Roof Loads.** The minimum live loads for which the floors and the roof of any building should be designed are always specified in the building code that governs the site of the construction.

The ranges of minimum live-load values in pounds per square foot of floor or roof area, as contained in several typical building codes, are given in Table 8.1.

**TABLE 8.1. Recommended Minimum Live Loads for Buildings**

| | | | |
|---|---|---|---|
| Apartments | 40 | Roofs, flat | 20 40 |
| Auditoriums and theaters: | | School buildings | |
| With fixed seats | 50 100 | Classrooms | 40 60 |
| Without fixed seats | 100 | Corridors, public spaces | 100 |
| Dwellings ... | 40 | Garages: | |
| Hospitals..... | 40 | All types of vehicles | 100 175 |
| Hotels: | | Passenger cars only | 75 125 |
| Rooms . | 40 | Store buildings | |
| Corridors, lobbies, dining rooms | 100 | Retail | 75 125 |
| | | Wholesale | 100 125 |
| Manufacturing buildings | | Warehouses | |
| Light manufacturing | 75 125 | Light storage | 75 150 |
| Heavy manufacturing | 125 200 | Heavy storage | 200 250 |
| Office buildings: | | | |
| Office space | 50 60 | | |
| Corridors, public spaces | 100 125 | | |

The specified minimum live loads cannot always be used. The type of occupancy should be considered, and the probable loads should be computed as accurately as possible. Warehouses for heavy storage may be designed for loads as high as 500 psf or more, unusually heavy operations in manufacturing buildings may require a large increase in the 200-lb maximum specified above, and special provision must be made for all definitely located heavy concentrated loads.

Some building codes provide for occasional probable concentrated loads, such as the weight of a heavy safe, by requiring that in addition to being adequate to support the specified minimum uniform live load, the floor system shall also be capable of supporting a single concentrated load of 2000 lb (or more) on an area $2\frac{1}{2}$ ft square when this load is placed in any position on the floor. The concentrated load and the uniform load are, however, not assumed to act simultaneously.

In all cases the dead weight of the floor must be included in the total design load. When plastered ceilings are specified, as is frequently the case where ribbed floor construction is used, an additional allowance of from 10 to 15 psf is made for the lath and plaster. In certain types of buildings, particularly office buildings, where all partitions are not definitely located on the plans, an extra allowance of from 10 to 20 psf of floor area is frequently made for the weight of these partitions.

There are still many areas of this country where no building codes are in force. In such localities the choice of live loads and working stresses depends largely on the judgment of the designing engineer. He may be guided by the requirements of codes for other parts of the same general region or by codes such as those of The National Board of Fire Underwriters or of The Building Officials Conference of America.

## REINFORCED CONCRETE FLOORS AND ROOFS

**8.3. Types of Floor Systems.** The types of concrete floor systems are so numerous as to defy concise classification. In steel construction the designer is usually limited to the utilization of structural plates and shapes which have been standardized in form and size by the relatively few producers in the field. In reinforced concrete, on the other hand, the engineer has much more control over the form of the structural components of a building. In addition, many small producers of reinforced concrete structural elements and construction accessories can compete profitably in this field since plant and equipment requirements are not excessive, as they frequently are in steel work. This has resulted in the development of a few more or less standard methods of concrete floor construction and many special or proprietary methods. Only a few of the more commonly known types can be mentioned in a text of this type.[1]

In general, the commonly used reinforced concrete floor systems can be classified as follows:

1. One-way reinforcing systems (in which the main reinforcement in each structural element runs in one direction)
   *a.* Solid-slab beam-and-girder floors
   *b.* One-way ribbed floors: concrete joists with steel-pan, clay, or concrete-tile fillers
   *c.* Steel-joist floors (commonly used with steel frames)
   *d.* Precast systems:
      (1) Precast slab or deck
      (2) Precast beams and girders
   *e.* Concrete slab reinforced with steel deck
2. Two-way reinforcing systems (in which the main reinforcement in at least one of the structural elements runs in two directions)
   *a.* Two-way solid slabs with beam supports

[1] For illustrations of several other types of floor systems, see Ramsey and Sleeper, "Architectural Graphic Standards," 4th ed., John Wiley & Sons, Inc., New York, 1951. See also manufacturers' catalogues in Sweet's File, F. W. Dodge Corp., New York.

*b.* Two-way ribbed slabs with tile or steel-pan fillers and with beam supports

*c.* Flat-slab floors

Each of these types is described in the following articles.

**8.4. Solid-slab Beam-and-girder Floor.** A solid-slab beam-and-girder floor consists of a series of parallel beams supported at their extremities by girders which in turn frame into concrete columns placed at more or less regular intervals over the entire floor area. This framework is covered by a one-way reinforced concrete slab, the load from which is transmitted first to the beams and thence to the girders and columns. The beams are usually spaced so that they come at the mid-points, at the

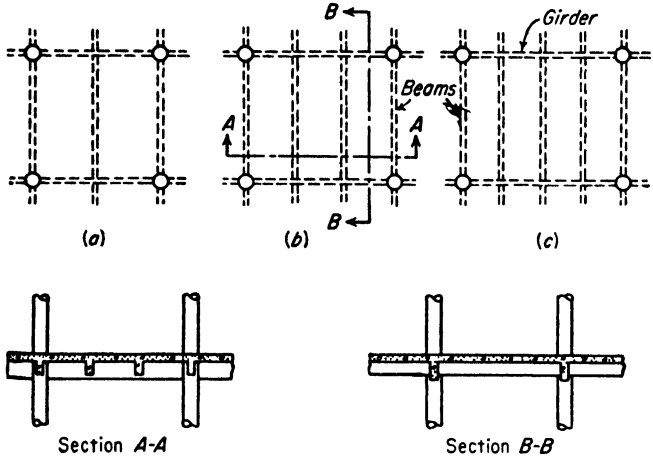



FIG. 8.1. Framing of beam-and-girder floors.

third points, or at the quarter points of the girders, as shown in Fig. 8.1. The arrangement of beams and spacing of columns should be determined by economical and practical considerations. These will be affected by the use to which the building is to be put, the size and shape of the ground area, and the load which must be carried. A comparison of a number of trial designs and estimates should be made if the size of the building warrants, and the most satisfactory arrangement selected. As the slabs, beams, and girders are built monolithically, the beams and girders are designed as T beams and advantage is taken of continuity.

Beam-and-girder floors, as they are usually called, are adapted to any loads and to any spans that might be encountered in ordinary building construction. The normal maximum spread in live-load values is from 40 to 400 psf, and the normal range in column spacings is from 16 to 32 ft. A complete design of a typical beam-and-girder floor panel is given in Arts. 8.26 to 8.29.

In normal beam-and-girder construction the depth of a beam will be about twice its stem width. For light loads, however, it may be economical to omit the intermediate beams and deep column-line girders entirely and to have the one-way slab supported by wide, shallow beams which are centered on the column lines and which frame directly into the columns. The wide beams keep the effective span of the slab from becoming excessive, the number of framing members may be considerably reduced, and, perhaps most important, shallow beams permit a reduction in the over-all height of the building. This type of framing is frequently called slab-band construction. It has been used often in multistory apartment and office buildings. In Fig. 8.2 are shown comparative sections through a conventional and slab-band type of floor such as might be used in a wing of an apartment building.

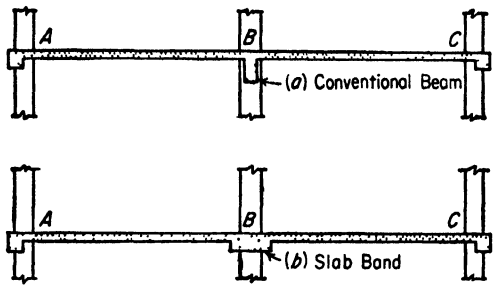



FIG. 8.2. Comparison of sections through conventional and slab-band floors.

One-way reinforced concrete slabs are also frequently used with structural-steel frame buildings. Figure 8.3 shows one method of using wire hangers for supporting formwork for the structural slab and beam fireproofing from a steel floor beam. The usual maximum spacing of beams in such construction is about 6 ft.

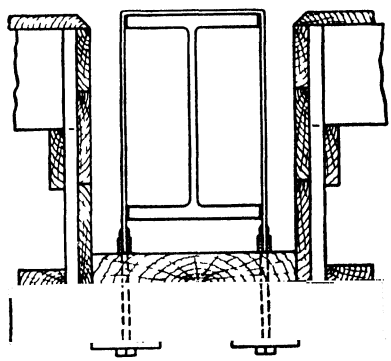


FIG. 8.3. Formwork suspended from steel floor beam.

**8.5. One-way Ribbed Floors.** A ribbed floor consists of a series of small, closely spaced reinforced concrete T beams framing into beams or girders which in turn frame into the supporting columns. The T beams (called joists or ribs) are formed by placing rows of fillers in what would otherwise be a solid slab. The fillers may be special steel pans, hollow clay-tile or lightweight concrete-tile blocks, or ordinary wood forms. The girders which support the joists are usually built as regular T beams.

Since the strength of concrete in tension is small, and is commonly neglected in design, elimination of much of the tension concrete in a slab by the use of fillers results in a saving of weight with little alteration in the structural characteristics of the slab. Ribbed floors are economical for buildings such as apartment houses, hotels, and hospitals, where the live loads are fairly small and the spans comparatively long. They are not suitable for heavy construction such as in warehouses, printing plants, and heavy manufacturing buildings.

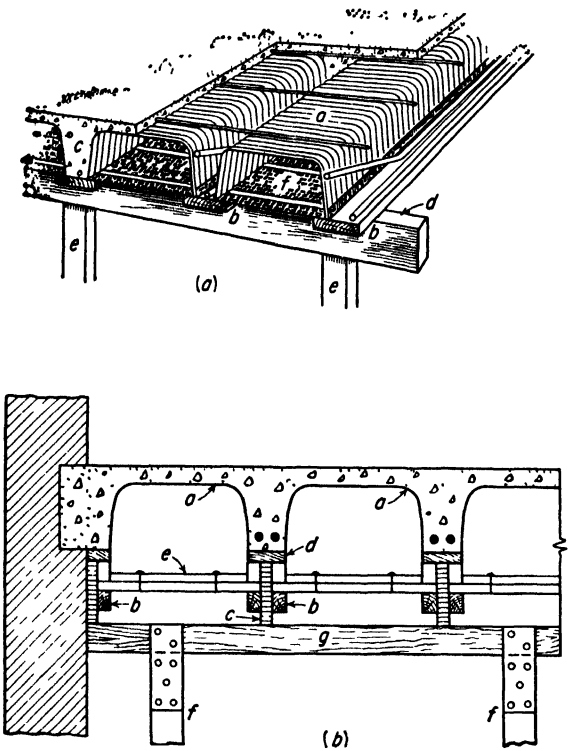



FIG. 8.4. Details of ribbed floors with steel tiles.

**8.6. Ribbed Floors with Steel-pan Fillers.** Steel-pan forms are made by many manufacturers. The steel cores usually are available in lengths of 36 in. and depths of 6, 8, 10, 12, or 14 in. Other sizes are made but these are the most common. They are tapered in cross section, as shown in Fig. 8.4*a*. The most common width at the bottom is 20 in., although other widths between 10 and 30 in. are also obtainable. In some systems the individual core lengths *a* (Fig. 8.4*a*) rest on the edges of soffit boards *b* which serve as the bottom forms for the ribs or joists *c*, and which in turn are supported by transverse timber joists *d* and vertical posts *e*.

In other systems the steel tiles *a* (Fig. 8.4*b*) are supported by ledger strips *b*, which are fastened to the sides of timber joists *c*, laid on edge under the soffit boards *d*. With the latter system one depth of tile can be used for various depths of joists. In Fig. 8.4*b*, the bottoms of the tiles are held in the correct position by the separators *e*, which are made of two pieces of 1 by 2 strips nailed together. Some tiles are furnished with collapsible steel spreaders which pass through small holes in the sides of the tile and rest on the soffit boards, providing the proper depth and width. The timber joists *c* rest on transverse joists *g*, and these are supported by the vertical posts *f*.

Concrete is placed on the soffit boards in the space between the rows of steel cores, and above the cores to form a slab 2 in. or more thick.[1] After the concrete has hardened, the steel cores are generally removed, although some types are designed to be left in place permanently.

With permanent tiles, metal lath *f* (Fig. 8.4*a*) is placed under the tiles before they are put in position and anchored to the concrete joists by suitable metal clips which project up into the joists. The entire ceiling is plastered after the formwork is removed. With removable tiles, the ceiling may be plastered by fastening metal lath to the concrete after the tiles are removed, or the joists may be left exposed. The joists are usually made 5 in. wide at the bottom, so that the distance center to center of joists is ordinarily 25 in.

The reinforcement of the joists consists usually of two bars, one bent and one straight. The slab is reinforced, primarily for temperature and shrinkage stresses, with wire mesh or small bars placed at right angles to the joists; the area of the reinforcement is usually about 0.25 per cent of the cross-sectional area of the slab.[2]

Steel cores with the sides tapered in the direction of the length are available for use at the ends of the rows in designs where the shearing stresses are relatively large. The width of tapered cores at the small end is 4 in. less than at the large end, so that the width of the joist near the support is increased accordingly. The increased width of the joist reduces the shearing stresses in the concrete. A typical construction showing tapered end lengths in position is illustrated in Fig. 8.5.

The girders which support the joists are rectangular beams or T beams with a maximum flange thickness equal to the total floor thickness, as shown in Fig. 8.6*a*. The rows of tiles are stopped at the proper distance

[1] The ACI Code specifies a minimum slab thickness of 2 in., but not less than one-twelfth of the clear distance between joists.

[2] The ACI Code specifies that such slabs be reinforced at right angles to the joists with at least the amount of reinforcement required for flexure, but in no case shall the reinforcement be less than that required for normal shrinkage and temperature reinforcement (see Art. 3.17).

from the stem of the girder. End caps are used at each end of each row of tile, as shown in Fig. 8.5, to hold the girder concrete in position.

Tile pieces which are tapered in depth are also available. These tiles are usually about 3 in. less in depth at one end than at the other, so that if one length of such tile is used at each end of each row, a slab from 5 to 6 in. thick at the edge of the girder is obtained, as shown in Fig. 8.6*b*.

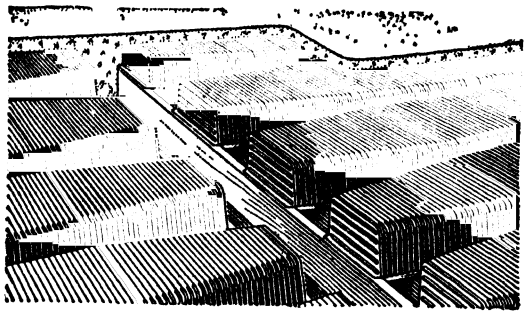


Fig. 8.5. Steel tiles with tapered ends

This thickness is often sufficient to furnish an adequate flange for the girder. A complete design of a typical ribbed floor panel with steel-tile fillers is given in Arts. 8.30 to 8.34.

**8.7. Ribbed Floors with Clay-tile or Concrete-tile Fillers.** Figure 8.7 shows a ribbed floor in which structural clay-tile blocks are used as fillers. The blocks are usually 12 in. square and can be obtained in

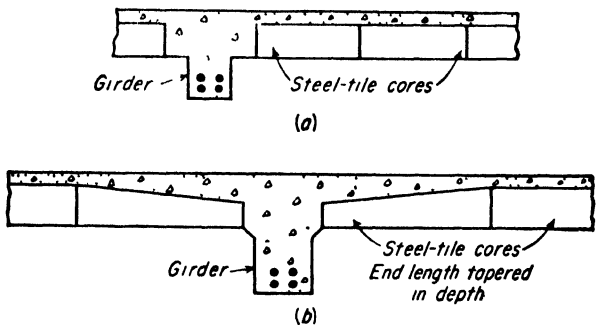



Fig. 8.6. Details of girders for steel-tile floors.

thickness varying from 3 to 12 in. Other sizes as well as specially formed blocks are also manufactured for use as form fillers. The blocks are placed flatwise and end to end, in rows between and at right angles to the girder forms. They are supported either by a solid decking similar to that used for regular concrete slabs, or on 2 by 8 planks called soffit boards, which are centered under the space between the rows of tile. The

decking or soffit boards are supported on light shoring from the floor below. The usual clear distance between rows is 4 in., thus making the distance center to center of rows 16 in. Concrete is poured on the decking or soffit boards to fill the space between the rows of tiles and also to cover the tiles to a depth[1] of 2 or 2½ in. The resulting construction consists of a series of concrete T beams, or joists, with tile fillers under the slab in the spaces between the stems of the joists. The joists frame into, and are supported by, the girders, and these in turn are supported by the columns. When the concrete has hardened, the forms are removed, but the tile fillers remain in place, securely anchored to the concrete by the projections on the surfaces of the tiles. The entire ceiling is then plastered. The arrangement of reinforcement is similar to that used in ribbed floors with steel-pan fillers.

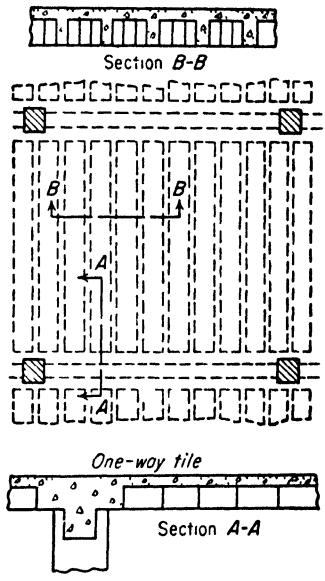



FIG. 8.7. Details of one-way tile floor.

Hollow concrete-tile fillers come in many shapes and sizes, the most common being 8 by 16 in. in plan with depths of 4, 6, or 8 in. They are usually made with portland cement and a lightweight aggregate of cinders, slag, burned clay, or mineralized organic substance. Concrete-tile fillers are used in the same manner as clay-tile fillers in floor construction.

The design of a typical ribbed floor panel with concrete- or clay-tile fillers is outlined in Arts. 8.35 to 8.37.

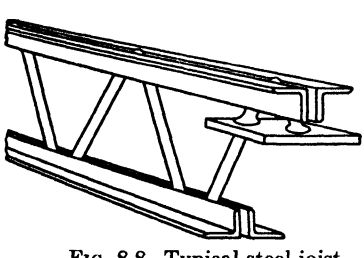


FIG. 8.8. Typical steel joist.

**8.8. Steel-joist Floors.** A steel-joist floor consists of a series of closely spaced, parallel, shallow joists or trusses of the Pratt, Warren, or double-Warren type, supported at the ends on steel or concrete beams, or on masonry walls, and covered with a thin concrete slab.

A common type of steel joist is shown in Fig. 8.8. The truss is composed of angle chords and a continuous bar web, assembled by the high-

[1] The ACI Code permits a minimum slab thickness of 1½ in., but not less than one-twelfth the clear distance between joists. Normally, a 2-in. minimum slab thickness is used.

pressure electric welding method. The welds are designed to develop the full strength of the web members.

Steel joists are manufactured in depths of 8, 10, 12, 14, and 16 in., and are carried in stock in a range of depths and lengths to meet all usual conditions of load and span. The span limits vary from 4 to 32 ft.[1] The joists are spaced from 12 to 30 in. on centers, depending on the load and span, and are covered with a 2- to 3-in. concrete slab poured on metal rib lath which rests on and is fastened to the upper surfaces of the joists.

The joists are braced transversely by some form of rigid diagonal bridging placed between adjacent joists, at intervals of from 5 to 8 ft.

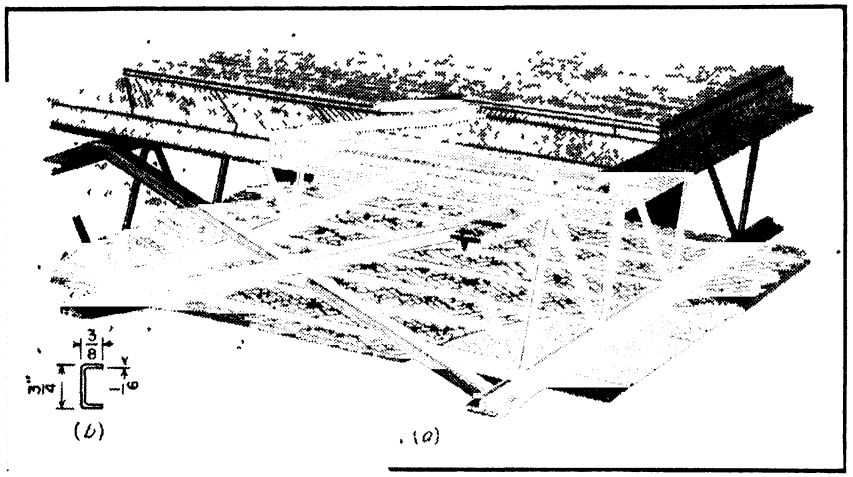



FIG. 8.9. Steel-joist floor construction.

One type of bridging is shown in position in Fig. 8.9. The bridging members are light $\frac{3}{16}$-in. channels. The channels are fastened to the chords of the trusses by extending one leg of the channel and wrapping it tightly around the chord. Bridging is quite often tack-welded to the joist.

The concrete slab which rests on the joists acts as a one-way slab spanning between joists. The metal rib lath on which it is placed (see Fig. 8.9) serves as formwork for the fresh concrete and reinforcement for the hardened slab.

Steel-joist floors are economical for light occupancies, but they are unsuitable for heavy or vibrating loads. They are easily installed and

[1] Long-span joists are also available, primarily for roofs, with depths up to 32 in., and for spans up to 64 ft. Roofs supported by these long-span joists have wood plank decks with built-up roofing, light-gauge steel decks, or concrete decks. The spacing of the joists is limited by the safe span of the deck. Long-span joists are bridged with cross bracing of not less than $1\frac{1}{4} \times 1\frac{1}{4} \times \frac{1}{8}$ in. angles, spaced not more than about 10 ft apart.

the cost of construction is low, because no forms are required for the concrete slab. Ceiling lath for plaster is attached directly to the lower chords of the trusses. Since the webs of the joists are open, all plumbing and service pipes may be concealed in the construction without interfering with the structural members. Further information on the design

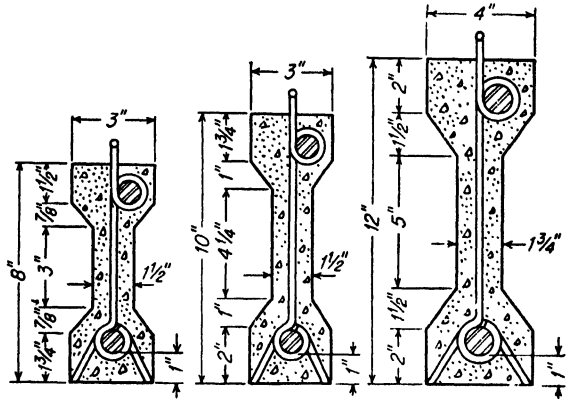



FIG. 8.10. Cross sections of typical precast concrete joists.

of joists and the details of joist construction is given in the publications of the Steel Joist Institute, a trade association.

**8.9. Precast Concrete Floor Systems.** During the last two decades, and particularly since the Second World War, a great amount of work has been done on the development of precast concrete structural units. The two distinct types of precast units currently manufactured are the joist

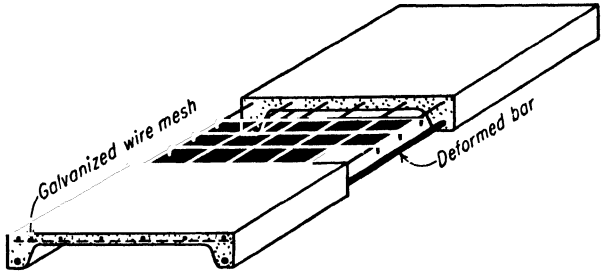



FIG. 8.11. Typical deck-type precast concrete unit.

type, an example of which is shown in Fig. 8.10, and the slab or deck type shown in Fig. 8.11.

Precast concrete units are used in a variety of ways. Examples are precast roof decks used on steel frame buildings, and precast beams supporting cast-in-place or precast slabs. Precast members are designed by standard reinforced concrete theory. In some cases departure from the

usual requirements for concrete cover around bars may be justified when the units are made by carefully controlled factory methods.[1] The principal advantages claimed for precast members are those of standardization and mass production: close control over design and manufacture, multiple use of formwork, elimination of expensive on-the-job forms, and speed of erection of the structure.

Most precast elements are designed for relatively light floor loading or for roof loads. Recently, however, heavy prestressed, precast floor beams and girders have been used in industrial structures.

**8.10. Steel Deck Floors with Concrete Slab.** In recent years considerable use has been made of light-gauge steel deck covered with a concrete slab for floors in steel frame office and apartment buildings. Two examples of this type of construction are shown in Fig. 8.12. Figure 8.12*a* shows a steel deck manufactured by the Detroit Steel Products Company covered with a poured slab. The deck acts as formwork for the fresh concrete and reinforcement for the hardened slab. Figure 8.12*b* shows the laying of a corrugated steel deck manufactured by the Granco Steel Products Company. Transverse wires welded to the ridges of the sheet provide bond between the sheet and the concrete slab, as well as serving as shrinkage and temperature reinforcement for the slab. The corrugated steel sheet serves as positive reinforcement for the slab and ordinary deformed bars are used for negative reinforcement over the supports.

Several other companies make steel deck of similar types. Manufacturers' recommendations should be consulted for loading and span limitations. These are all essentially one-way reinforced slabs.

The chief advantages claimed for this type of construction are that it is fast and that it eliminates expensive formwork.

**8.11. Two-way Solid Slabs with Beam Supports.** Solid concrete slabs supported by beams on all four sides and with main reinforcement running in two directions have been discussed in detail in Arts. 3.19 to 3.22. Floors of this type are suitable for intermediate and heavy loads on spans up to about 30 ft. The range of usefulness of two-way slabs corresponds generally to that of flat slabs. The latter are often preferred because of the complete elimination of beams in a flat-slab system. In many concrete buildings designed for other types of reinforcing systems there occur isolated floor panels which are ideally suited to two-directional reinforcement. An example of this is the floor or roof of a penthouse one bay wide in each direction.

**8.12. Two-way Ribbed Slab with Tile or Steel-pan Fillers.** As in one-way slab systems, the dead weight of two-way slabs can be reduced con-

[1] See ACI 711-46, Minimum Standard Requirements for Precast Concrete Floor Units.

siderably by the use of filler blocks of lightweight concrete or by using removable steel-pan forms which are square in plan and impart a waffle-like appearance to the underside of the slab. The Slagblok system shown in Fig. 8.13 is one type of ribbed two-way floor slab. Fillers of slag-aggregate concrete 16 by 16 in plan and from 4½ to 10 in. deep are

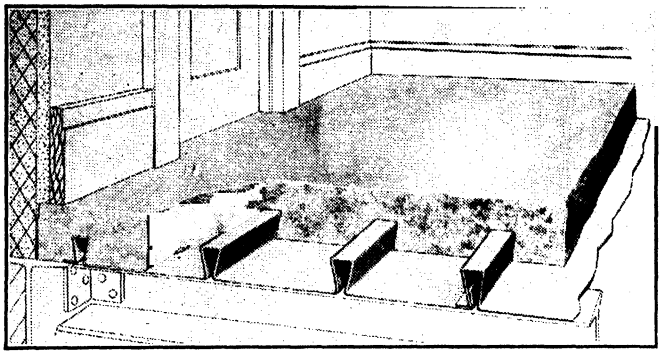


FIG. 8.12*a*. Concrete slab with steel deck reinforcing (Detroit Steel Products Co. Holorib).

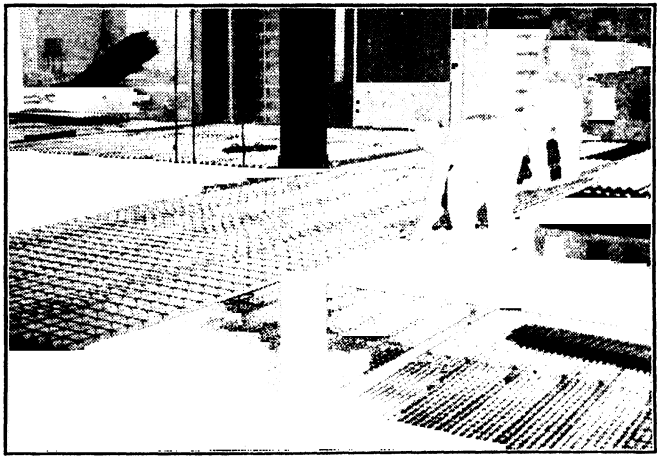


FIG. 8.12*b*. Placing corrugated sheet (Cofar panel) reinforcing for floor slab.

placed on the forms in a grid pattern with room between fillers for concrete joists running in two directions. In the illustration the supporting beams are steel, but concrete beams and columns can be used just as well. A concrete slab or topping may be poured over the blocks. Floors of this type may be designed as ordinary two-way slabs. It is usually necessary to make a ribbed slab somewhat thicker than a comparable solid slab but an over-all weight saving can be accomplished.

***8.13. Flat-slab Floors.*** *A flat-slab floor consists of a reinforced concrete slab supported directly on concrete columns without the aid of beams or girders.* The transition between the slab and the column is usually made through thickened portions of the slab called drop panels and enlarged sections at the tops of columns called capitals (see Figs. 8.14 and 8.25). This type is discussed in detail in Arts. 8.38 to 8.45. In general, flat-slab construction is economical for live loads of 100 psf or more and for spans up to about 30 ft. For lighter loads, such as are used in apartment houses, hotels, and office buildings, some form of ribbed-floor construction will usually be cheaper than a flat-slab floor,

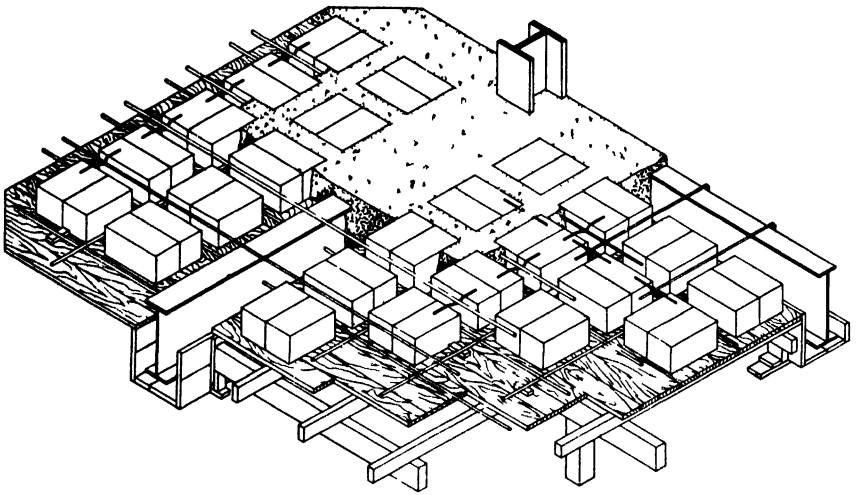


FIG. 8.13. Two-way slab using block fillers (Republic Slagblok).

although in recent years flat slabs have been used economically for a wide range of loading. For spans longer than about 30 ft, beams and girders are desirable because of the greater stiffness which can be secured with them.

Ribbed flat slabs have proved very successful in a number of applications. One type, the Grid System, uses removable dome-shaped steel pans which form concrete ribs spanning in two directions. Figure 8.14 shows a flat-slab building using the Grid System. The design procedure for this type of floor is the same as that for the ordinary flat slab. The same type of form may also be used for ribbed two-way slabs.

Flat-plate floors flat slabs with drop panels and column capitals omitted--have been developed in recent years for use in buildings with relatively light loading. The floor slab is simply a plate of uniform thickness supported directly by the columns. The only beams used in this type of construction are those at the exterior walls and around large openings

in the slab. This system, an example of which is shown in Fig. 8.15, has proved very successful in a number of apartment-house projects. Design procedure is similar to that for ordinary flat slabs, but the lack of capitals and drop panels and the use of thin slabs require that special attention be given to many of the details of analysis, design, and construction.

**8.14. Floor Surfaces.** Many types of wearing surfaces are used on concrete floors. Concrete topping $\frac{3}{4}$ to 2 in. thick is frequently used. Such topping may be either monolithic or bonded. Monolithic finishes should be placed within 45 min after the base slab is poured; bonded finishes are placed on a fully hardened base slab. In order to obtain an

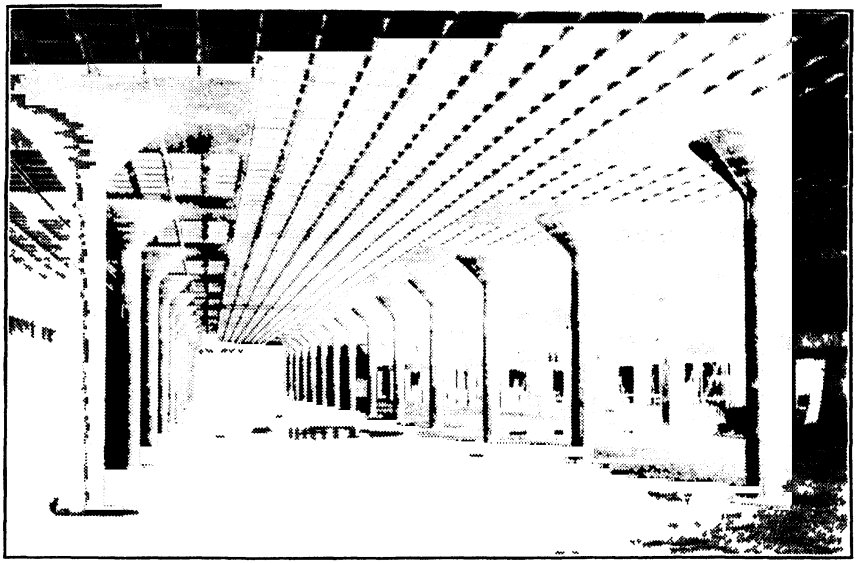


FIG. 8.14 Grid System flat-slab building

effective bond in the latter case, the base slab should be rough, and it should be thoroughly wetted and coated with a neat cement paste before placing the finish. Mix proportions vary and quite often an artificial hardening agent is added.[1]

A wooden floor may be provided for if desired, by embedding beveled nailing strips or "sleepers," usually 2 by 4's laid flat, in a layer of cinder concrete on top of the main slab, as shown in Fig. 8.16. A spacing of 16 in. for the sleepers has been found satisfactory for the ordinary floor.

Small vitrified-clay flat tiles embedded in a 2-in. layer of portland-cement mortar also provide a satisfactory floor surface. Asphalt or linoleum tile placed over a smooth cement base is also frequently used.

[1] See Portland Cement Association Bulletins, "Surface Treatments for Concrete Floors" and "Specifications for Heavy-duty Floor Concrete Floor Finish," for more complete coverage of this important topic.

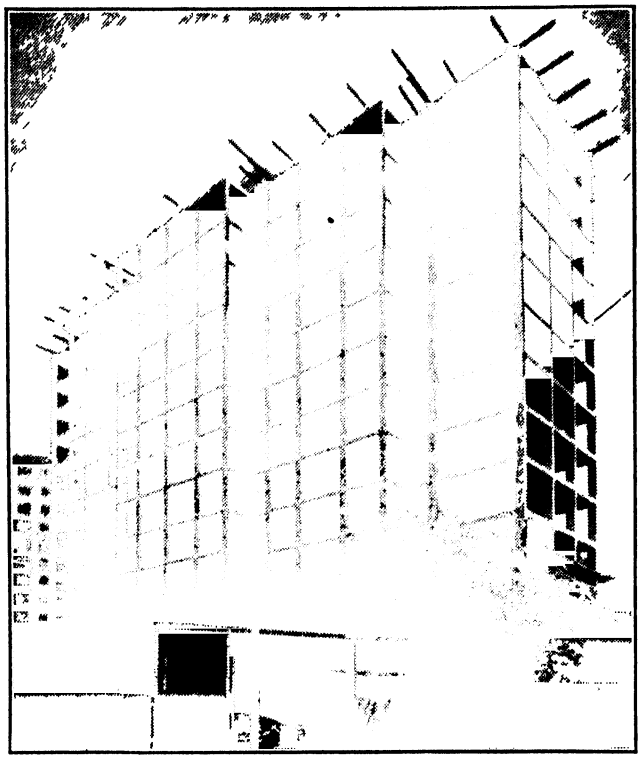


FIG. 8.15 Flat-plate apartment building

The durability of any wearing surface depends in a large measure upon the method of placing the surface. If not properly constructed, the concrete finish might spall, the wood floor "dry-rot," the tiles curl, and the linoleum crack. In order to ensure the maximum degree of serviceability

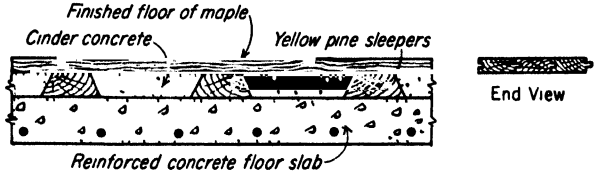



FIG. 8.16 Details of a wooden floor on a concrete slab

from a given type of floor surface, a special study of methods which have proved successful for laying that particular type of floor should be made.

**8.15. Concrete Roofs.** The design of roofs is similar to the design of floors. In addition to the structural requirements, however, the roofs must be impervious to the passage of water, provide for adequate drainage, and furnish protection against condensation.

In order to provide adequate drainage, the roof slab may be pitched slightly, or a filling of some light material such as cinder concrete, covered with a suitable roofing material, may be used, the thickness of the filling being varied so as to give the required slope to the roof surface. The amount of slope required for drainage will depend upon the smoothness of the exposed surface. A value of $\frac{1}{8}$ in. per ft might be used with a surface of hard tile. Felt and gravel roofs should have a pitch of at least $\frac{1}{4}$ in. per ft. Some form of flashing is required along the parapets to prevent the drainage from seeping into the building at the edges of the roof slab.

Condensation may best be guarded against by proper ventilation and insulation. The form of insulation to be used will depend upon the particular class of building under consideration.

Imperviousness may best be provided for by the application of some form of separate roof covering, such as a combination of felt and gravel in alternate layers cemented together and to the slab by means of coal-tar pitch or asphalt, vitrified tile embedded in asphalt, or any of the commercial types of built-up roofings. Tin, corrugated iron or copper roofings are sometimes placed on reinforced concrete buildings but are usually more expensive and less permanent than other types of coverings. If it is not desirable to use any separate roof covering, the main slab may be made reasonably waterproof by the methods mentioned in Sec. 1. Such procedure is not recommended except for structures where absolute imperviousness is not essential, because of the difficulty of preventing entirely the formation of shrinkage cracks and the attending seepage.

## WALLS AND PARTITIONS

**8.16. Panel and Curtain Walls.** As a general rule, the exterior walls of a reinforced concrete building are supported at each floor by the skeleton framework of the building, their only function being to enclose the building completely. Such walls are called panel walls. They may be made of concrete, brick, concrete blocks, tile blocks, or hollow masonry blocks faced with brick or stone. The thickness will vary according to the material, the type of construction, and the building requirements governing the particular locality where the construction takes place.

The New York City Code (revised to 1949) requires that panel walls in skeleton frame construction shall be at least 8 in. thick if such walls are constructed of solid masonry (brick or stone), at least 12 in. thick if of hollow masonry (*i.e.*, hollow concrete blocks or terra-cotta tiles), or at least 10 in. thick if the wall is a combination of hollow and solid masonry, constructed with at least 4 in. of solid masonry on the exterior face. The maximum height between supports of panel walls constructed of the minimum thicknesses specified above shall be 13 ft. When such walls exceed

13 ft in height, they shall be increased 2 in. in thickness for each 6½ ft, or fraction thereof, of height in excess of 13 ft.

The pressure of the wind is practically the only load that need be considered in determining the theoretical thickness of a reinforced concrete panel wall. Designed as a slab supported on four sides for a wind pressure of 20 psf, the walls in buildings of ordinary proportions need only be from 3 to 4 in. in thickness. This is too thin to permit of practical and economical construction and to ensure complete protection against seepage and condensation. Most building codes require a minimum thickness of 8 in. for panel walls of reinforced concrete.[1] The amount of steel necessary is usually governed by the necessity of guarding against the formation of cracks caused by expansion and contraction due to temperature changes. Small bars running horizontally and vertically are placed near each face of the wall and spaced not more than 12 or 18 in. center to center in both directions. On account of the probability of greater temperature variation on the exposed face, more steel should be placed near that face than on the inside unless the lateral pressure requirements govern these amounts.[2] These bars should extend into the columns and wall beams if the walls are poured at the same time as these members. If, as is usually the case, the walls are poured after the columns and beams, anchorage should be provided for by means of dowels projecting from the latter units.[3] It is good practice in such cases to mortise the wall into the columns and wall beams by leaving grooves in the latter members when they are poured. The grooves may be formed by nailing wooden strips on the inside of the forms.

Where a small amount of window area is inserted in a panel wall, the reinforcement may remain as for a solid wall, but additional bars should be placed near all edges of the openings. Where light is essential, as in

[1] The ACI specifies a minimum thickness of 5 in., but not less than one-thirtieth of the distance between supporting or enclosing members.

[2] The ACI requires that the reinforcement in each direction (vertical and horizontal) shall have an area at least equal to 0.0025 times the cross-sectional area of the wall if bars are used, or 0.0018 times the area if electrically welded wire fabric is used, with wires not less than No. 10 W. and M. gauge. Walls more than 10 in. in thickness shall have the reinforcement for each direction placed in two layers parallel with the faces of the wall. One layer, consisting of not less than one-half and not more than two-thirds of the total required area, shall be placed not less than 2 in. and not more than one-third the thickness of the wall from the exterior surface. The other layer, comprising the remainder of the required reinforcement, shall be placed not less than ¾ in. and not more than one-third the thickness of the wall from the interior surface. Bars, if used, shall not be less than ⅜ in. in size, nor shall they be spaced more than 18 in. on centers.

[3] The ACI requires that concrete walls shall be anchored to the columns, floors, pilasters, or buttresses with dowels at least ⅜ in. in size, spaced not more than 12 in. on centers, for each layer of wall reinforcement.

a factory, practically the entire wall panel may be enclosed by windows or filled with glass blocks as in Fig. 8.29, construction then consisting of a wall beam or lintel at each floor and a spandrel between the wall beam and the window sill. Sometimes this spandrel is made of reinforced concrete and constructed as a part of the wall beam. In other instances the concrete spandrel is considered independent of the wall beam, and is reinforced only for temperature stresses. The advantage in using independent spandrels lies in the fact that they may be placed after the structural framework is completed and greater care can be taken in the finishing than would be possible if they were part of the load-bearing skeleton of the structure. The spandrel may also be made of brick or other suitable material.

Where a panel is made up primarily of window space, the New York City Code permits a minimum thickness of 8 in. for the portion of the wall below the window sill and above the window head, whether these portions are made of brick, hollow masonry blocks, or reinforced concrete. The portion of a panel wall between the window sill and the support of the panel wall is called an apron wall in that Code, and the portion above the window head is called a spandrel wall.

Curtain walls are nonbearing enclosure walls built between columns, but not supported at each story by the frame of the building. They are used less frequently than either panel or bearing walls. Minimum thickness requirements for curtain walls are found in most building codes.

**8.17. Bearing Walls.** A bearing wall may be defined as one which carries any vertical load in addition to its own weight. Such walls may be constructed of stone masonry, brick, hollow building blocks, or reinforced concrete. Occasional projections or pilasters add to the general appearance and strength of the wall. In small reinforced concrete commercial buildings and residences the bearing-wall type of construction may be used with economy and expediency. In larger commercial and manufacturing buildings where the element of time is an important factor, the delay necessary for the erection of the bearing wall and the attending increased cost of construction often dictate the use of some other arrangement.

Bearing walls may be of either single or double thickness, the advantage of the latter type being that the air space between the walls renders the interior of the building less liable to temperature variations, and makes the wall itself more nearly moistureproof. On account of the greater gross thickness of the double wall, such construction reduces the available floor space. This feature is often sufficient in itself to warrant the selection of the solid wall unless the factors of condensation and temperature are of great importance. Hollow wall construction is usually limited to a total height of about 40 ft.

The thickness of bearing walls varies with the height. The New York City Building Code requires that, for bearing walls made of solid masonry units (brick, stone, sand-lime or concrete brick, etc.) and not more than 75 ft in height, the thickness of the uppermost 55 ft of height shall be at least 12 in. and the thickness below this shall be at least 16 in., except that for walls in buildings of one, two, and three stories the top-story thickness in each case may be 8 in. In computing wall thicknesses, a maximum story height of 13 ft shall be assumed. For bearing walls of reinforced concrete, the thickness shall be at least one twenty-fifth of the unsupported height, but not less than 8 in. Reinforcement in concrete bearing walls shall be at least equal to the reinforcement specified in Art. 8.16 for curtain or panel walls.

According to the ACI Code, the working stress $f_c$ in reinforced concrete bearing walls shall be $0.25f'_c$ for walls having a relation of height $h$ to thickness $t$ of 10 or less and shall be reduced proportionally to $0.15f'_c$ for walls where $h/t = 25$. If the wall supports concentrated loads, the length of wall to be considered effective for each load shall not exceed the distance center to center of loads, nor shall it exceed the width of the bearing plus four times the wall thickness. If the wall reinforcement is designed, placed, and anchored in position as for tied columns, such reinforcement may be considered effective in resisting vertical loads in the same manner as for tied columns.

**8.18. Basement Walls.** In determining the thickness of basement walls, the lateral pressure of the earth, if any, must be considered in addition to the other structural features. If part of a bearing wall, the lower portion may be designed either as a slab supported by the basement and first floors, or as a retaining wall, depending upon the type of construction. If columns and wall beams are available for support, each basement wall panel of reinforced concrete may be designed to resist the earth pressure as a simple slab reinforced in either one or two directions.

A minimum thickness of 8 in. is generally specified for reinforced concrete basement walls, with minimum reinforcement as specified in Art. 8.16 for panel or curtain walls. In wet ground a minimum thickness of 12 in. should be used. In any case, the thickness cannot be less than that of the wall above.

Care should be taken to brace a basement wall thoroughly from the inside if the earth is backfilled before the wall has obtained sufficient strength to resist the lateral pressure without such assistance.

**8.19. Parapet Walls.** In the case of buildings with flat roof slabs on which drainage slopes are built, parapet walls are necessary architecturally to give a more finished appearance to the top of the structure, and practically to provide a backing for the drainage slopes. They are usually of brick or concrete or a combination of both. Concrete is, in most

cases, preferable from an economical standpoint. In order to give a better appearance it may have a veneer of brick or terra cotta.

The chief point to be considered in the construction of parapet walls is the necessity of providing adequate reinforcement to prevent cracks caused by excessive temperature changes (on account of exposed position) and expansion and contraction at corners.

The thickness is usually required to be the same as the wall below, but not more than 12 in. in any case. Parapet walls should extend at least 2 ft above the roof surface.

**8.20. Veneer for Exterior Walls.** In order to give an attractive appearance to the building, it sometimes becomes necessary to cover up the entire exterior wall surface with a veneer of brick, stone, terra-cotta tile, marble, or other finishing material. A method of securing a covering of face brick to concrete consists in placing corrugated copper or galvanized steel ties, usually about $^3/_4$ in. wide and 6 in long, at frequent intervals in the wall or column forms so that about 4 in. of the tie strip will project into the concrete when poured and the remaining 2 in. will lie flat against the form and tacked lightly to it. When the form is removed, this latter portion is bent outward and is bonded into the brick veneer by means of the joint mortar. Terra-cotta and stone facings are generally supported by ledges in the concrete or by steel angles, and are provided with steel anchors commensurate with the size of the veneer units.[1]

**8.21. Partition Walls.** Interior walls used for the purpose of subdividing the floor area may be made of concrete, metal lath and plaster, clay tile, plaster block, metal partitions or brick Reasonably adequate fire protection is afforded by a solid concrete wall 3 or 4 in thick The reinforcement should be similar to that in curtain walls but need not be so great in quantity. Suitable anchorage may be obtained by permitting the vertical rods to project into the floor and ceiling. If it is convenient, as is usually the case, to pour the wall after the structural framework of the building is completed, a groove should be left in the floor and one in the ceiling to receive the partition Two objectionable features of the solid concrete partition wall are its weight and cost of installation. In buildings where many lives would be endangered by a rapid spread of a fire once started, these objectionable features become insignificant.

The most common form of metal lath and plaster partition consists of some form of vertical steel studding suitably anchored to the floor and ceiling, with metal lath fastened to both sides. Each side is plastered with a mixture of lime and cement mortar, thus forming a hollow wall

[1] See Ramsey and Sleeper, 'Architectural Graphic Standards," for anchorage details.

from 3 to 6 in. thick, which has, if proper bond is secured between the plaster and lath, a fair amount of fire resistance.

Clay-tile partitions are usually made of blocks from 4 to 6 in. thick, although the blocks may be obtained in thicknesses varying from 2 to 12 in. This type of partition is light in weight and is satisfactory under ordinary conditions. The blocks may be plastered on one or both sides, the thickness of wall being increased by about $\frac{3}{4}$ in. for each plastered side.

Partitions made of plaster blocks usually vary from 4 to 8 in. in thickness. The blocks are made of gypsum or plaster of paris, with an admixture of cinders, asbestos fiber, wood chips or vegetable fiber, and laid in gypsum plaster or cement mortar tempered with lime. They are light and easy to handle and place, but offer decidedly poor resistance to the action of fire and water.

The main use of brick inner walls is for the enclosing of stairs and elevator shafts, and in fire walls, the express purpose of which is to divide the floor area into sections to prevent the spreading of fire from one part of the structure to another. Reinforced concrete partitions are also used for the same purpose. These, as well as all other permanent partitions, should be independently supported at each floor on the fireproof construction of the floor. The use of partitions of pressed metal and glass or of wood and glass should be restricted to the subdivision of rooms or spaces enclosed by fireproof partitions.

## STAIRWAYS

**8.22. Types of Concrete Stairs.** The simplest form of reinforced concrete stairway consists of an inclined slab supported at the ends upon beams, with steps formed upon its upper surface. Such a stair slab is usually designed as a simple slab with a span equal to the horizontal distance between supports. This method of design requires steel to be placed only in the direction of the length of the slab. Transverse steel, usually one bar to each tread, is used only to assist in distribution of the load and to provide temperature reinforcement. It sometimes becomes necessary to include a platform slab at one or both ends of the inclined slab. Many successful designs made as outlined above for the simple inclined slab indicate that the effect of the angle that occurs in a slab of this type can safely be disregarded.

It is advisable to keep the unsupported span of a stair slab reasonably short. If no break occurs in the flight between floors, intermediate beams, supported either by the structural framework of the building or by additional short posts from the floor below, may be employed. If the stair between floors is divided into two or more flights, beams as described above may be used to support the intermediate landing, and these in

turn supported as above for the long straight flight, or the intermediate slab may be suspended from a beam at the upper floor level by means of rod hangers. Where conditions permit, the intermediate slab may be supported directly by the exterior walls of the building.

When it becomes necessary to use a stair slab of comparatively great length with no possibility of intermediate support, inclined side beams may be used. The stair slab is then reinforced with transverse bars to carry the load to the side beams, which are designed to transmit this load to the floor beams. This is normally not an economical form of construction.

**8.23. Beams around Stair Wells.** In buildings with flat-slab floor construction, it will be necessary to insert special beams at each floor level around the stairway opening as shown in Figs. 8.38 and 8.40 properly to support the stair slabs. In buildings with beam-and-girder floors, the regular floor beams, with dimensions modified as necessary, may be used to support the stair slabs, or special beams may be inserted for that purpose. In any case, the arrangement of beams must be such as to ensure proper transmission of all loads to the columns or walls.

**8.24. Building Code Requirements.** The required number of stairways, and many of the details, are governed to a large extent by the provisions of the governing building code. Among other things, these provisions stipulate the maximum distance from the most remote point in the floor area to the stairway, the minimum width of stairway, the maximum height of any straight flight, the maximum height (or rise) of a single step, the minimum distance (the run) between the vertical faces of two consecutive steps, and the required relation between the rise and the run to give safety and convenience in climbing.

In most codes the minimum width of any stair slab and the minimum dimension of any landing are about 44 in. The maximum rise of a stair step is usually specified as about $7\frac{3}{4}$ in., and the minimum run or tread width, exclusive of nosing, is $9\frac{1}{2}$ in. A rise of less than $6\frac{1}{2}$ in. is not considered generally satisfactory. In order to give a satisfactory and comfortable ratio of rise to run, various rules have been adopted. One requires that for steps without nosings, the sum of the rise and run shall be $17\frac{1}{2}$ in., but the rise shall not be less than $6\frac{1}{2}$ in. or more than $7\frac{3}{4}$ in. The New York City Building Code requires that run and rise shall be so proportioned that the product of the run, exclusive of nosing, and the rise in inches, shall be not less than 70 or more than 75, but risers shall not exceed $7\frac{3}{4}$ in. in height, and treads, exclusive of nosing, shall be not less than $9\frac{1}{2}$ in. wide.

The maximum height of a straight flight between landings is generally given as 12 ft, except for stairways serving as exits from places of assembly, where a maximum of 8 ft is normally specified.

The number of stairways is governed by the width of the stair slab, the number of probable occupants on each floor, and the dimensions of the floor area. One typical code specification is that the distance from any point in an open floor area to the nearest stairway or exit shall not exceed 100 ft; that the corresponding distance along corridors in a particular area shall not exceed 125 ft; and that the combined width of all stairways in any story shall be such that the stairs may accommodate at one time the total number of persons occupying the largest floor area served by such stairs above the floor area under consideration on the basis of one person for each 22 in. of stair width and one and one-half treads on the stairs, and one person for each 3½ sq ft of floor area on the landings and halls within the stairway enclosure.

If the exact number of probable occupants is not known, the codes usually assume that there will be two occupants for every 10 sq ft of floor area in dance halls and places of assembly; one person for every 25 sq ft of floor area in stores, lodging houses, and reading rooms; one person for every 50 sq ft of floor area in offices and show rooms; one person for every 100 sq ft in hospitals, hotels, and residence buildings; and one for every 150 sq ft in warehouses and garages.

In all buildings over 40 ft in height, and in all mercantile buildings regardless of height, the required stairways must be completely enclosed by fireproof partitions, and at least one stairway must continue to the roof. An open ornamental stairway can be used from the main entrance floor to the floor next above, provided it is not the only stairway. The live load usually specified for use in the design of stairways is 100 psf of horizontal area.

**8.25. Construction Details.** The usual practice is to construct the stairways after the main structural framework of the building has been completed, in which event recesses should be left in the beams to support the stair slab, and dowels should be provided to furnish the necessary anchorage. Occasionally, however, the stairs are poured at the same time as the floors, in which event negative-moment reinforcement should be furnished over the supports of the stair slab, as in any continuous beam construction. The steps are usually poured monolithically with the slab, but they may be molded after the main slab is in place. In the latter instance, provision must be made for securing the step to the slab. The nosing, where used, may be constructed by offsetting the upper portion of the vertical form of the step. Metal or slate treads embedded in the concrete are often used for a wearing surface. A complete design of a concrete stairway, with details of construction, is given in Art. 8.58.

## DESIGN OF A BEAM-AND-GIRDER FLOOR

**8.26. Data and Specifications.** In order to illustrate the application of the principles of reinforced concrete design to the design of a concrete

floor of the slab–beam-and-girder type, typical portions of the second floor of the building shown in Figs. 8.18 and 8.19 will be designed. The building will be designed to sustain a live load of 200 psf. The approximate methods of frame analysis recommended by the ACI Code (see Art. 3.6) will be used. The outside faces of all the exterior columns will be kept in the same plane so that the pilasters will have smooth faces, and all the wall beams and girders will be placed flush with the outer faces of the pilasters so that the entire outer walls, except for the windows, will be in one plane. A 1-in. bonded wearing surface, which will add to the dead load but will not be considered as part of the effective depth, will be placed on the slab. The allowable unit stresses are to be as specified in the ACI Code for a 3000-psi concrete and intermediate-grade steel.

For purposes of illustration, most of the computations for this design will be presented on computation sheets of the type that are typical in design office practice.

**8.27. Design of the Slabs.** The computations for the design of the floor slabs *S*1 and *S*2 are shown on Computation Sheet 8.1. The clear spans were obtained from the assumptions that the exterior basement columns will be 24 in. deep from the face of the basement wall and that the floor beams will be 8 in. in width. In computing the negative moments at the faces of supports, the average of the two adjacent clear spans was used. A negative-moment coefficient at the interior face of the exterior support of $\frac{1}{24}$ was assumed in accordance with the discussion in Art. 3.6. Reinforcing steel is placed with centers 1 in. from the top or bottom of the slab.

From Diagram 2, Appendix D:

*S*1:

50 per cent positive-moment reinforcement required for

$$0.23 \times 7 \text{ ft} - 12 \times \tfrac{3}{8} \text{ in.} = 1 \text{ ft-3 in.}$$

from support; bent at 45°, reaching top at 1 ft-1 in. from support. Negative-moment reinforcement required at top

$$0.09 \times 7 \text{ ft} + 4\tfrac{1}{2} \text{ in.} = 1 \text{ ft-0 in.}$$

from outer support. At continuous end, 50 per cent positive-moment reinforcement required for 1 ft-3 in. from support and 50 per cent negative-moment reinforcement at

$$0.09 \times 7 \text{ ft} + 4\tfrac{1}{2} \text{ in.} = 1 \text{ ft-0 in.}$$

*S*2:

50 per cent of positive-moment reinforcement required at

$$0.25 \times 6 \text{ ft-4 in.} - 4\tfrac{1}{2} \text{ in.} = 1 \text{ ft-2}\tfrac{1}{2} \text{ in.}$$

**Beam - and - Girder Floor** — **Computation Sheet 1**

Design of floor slab

| | |
|---|---|
| Live load | 200.0 lb / sq ft |
| Dead load (assume 4" slab + 1" finish) | 62.5 |
| Total = | 262.5 lb / sq ft |

Clear span for $S$-1 = 7'-0"
Clear span for $S$-2 = 6'-4"
} Ave. clear span = 6'-8"

## $S$-1

Center:

$M = +\frac{1}{14}(262.5)(7.0)^2(12) = 11{,}000$ in-lb

At continuous end

$M = -\frac{1}{12}(262.5)(6.67)^2(12) = -11{,}680$ in-lb

At wall end:

$M = -\frac{1}{24}(262.5)(7.0)^2(12) = -6{,}400$ in.-lb

From Table 5

$k = 0.403$, $j = 0.866$, $K = 235$

$d = \sqrt{\frac{11{,}700}{12(235)}} = 2.01"$

Use a minimum $d$ of 3"

Slab thickness = 4" as assumed

Center:

$A_s = \frac{11{,}000}{20{,}000(0.866)(3)} = 0.21$ sq in.

At continuous end:

$A_s = \frac{11{,}700}{20{,}000(0.866)(3)} = 0.22$ sq in.

At wall end:

$A_s = \frac{6{,}400}{20{,}000(0.866)(3)} = 0.12$ sq in.

## $S$-2

Center:

$M = +\frac{1}{16}(262.5)(6.33)^2(12) = 7{,}900$ in.-lb

At first interior support:

$M = -\frac{1}{12}(262.5)(6.67)^2(12) = -11{,}700$ in.-lb

At all other interior supports:

$M = -\frac{1}{12}(262.5)(6.33)^2(12) = -10{,}500$ in.-lb

Use 4" slab thickness throughout.

Center:

$A_s = \frac{7{,}900}{20{,}000(0.866)(3)} = 0.15$ sq in.

At first interior support:

$A_s =$ 0.22 sq in.

At other interior supports:

$A_s = \frac{10{,}500}{20{,}000(0.866)(3)} = 0.20$ sq in.

| Outer support | $S$-1 center | First interior support | $S$-2 center | Other supports |
|---|---|---|---|---|
| 0.12 | 0.21 | 0.22 | 0.15 | 0.20 |
| #3 bent @ 12" | #3 bent @ 12"<br>#3 str. @ 12" | #3 bent @ 6" | #3 bent @ 12"<br>#3 str. @ 12" | #3 bent @ 6" |
| $A_s = 0.11$ | $A_s = 0.22$ | $A_s = 0.22$ | $A_s = 0.22$ | $A_s = 0.22$ |

Temperature steel

$A_s = 0.002(12)(3) = 0.072$ sq in. Use #3 at 18"

COMPUTATION SHEET 8.1

from support. 50 per cent of negative-moment reinforcement required at

$$0.09 \times 6 \text{ ft-4 in.} + 4\tfrac{1}{2} \text{ in.} = 0 \text{ ft-}11\tfrac{1}{2} \text{ in.}$$

from support. Number 3 bars at bottom 12 in. center to center are carried into exterior support 6 in. and are continuous over other supports. Alternate No. 3 bars 12 in. center to center are bent bars bent up 1 ft-2 in. from face of all supports. Bent-up bars must be extended into the adjacent span a length of

$$0.21 \times 7 \text{ ft} + 4\tfrac{1}{2} \text{ in.} = 1 \text{ ft-}10 \text{ in.}$$

All details for the slab are shown in Fig. 8.18.

**8.28. Design of the Beams.** The computations for the design of beams *B*1 and *B*2 are shown on Computation Sheets 8.2 and 8.3. The clear spans for the beams were obtained using the same assumptions made in the design of the slab with the further assumption that the beams will be framed into girders 12 in. in width. The beams framing into girders were designed, since these beams have clear spans slightly longer than those framing into columns; however, to simplify the construction, all beams are of the same dimensions. In the beams no bars will be bent. Stirrups are provided to take the diagonal tension stresses. Although the ACI Code allows a unit shearing stress of $0.08f'_c$ for this type of reinforcement, a depth selected on this basis would require heavy horizontal reinforcement, and therefore the larger depth shown on Computation Sheet 8.2 was selected. Two rows of reinforcement are used in most cases both at the top and bottom of the beam. The center of the row nearest the surface is placed $2\frac{1}{2}$ in. from the surface with the other row spaced 2 in. center to center vertically below or above the first row.

From Diagram 2, Appendix D:

*B*1:

Negative-moment reinforcement at outer support required for $0.09 \times (22 \text{ ft-}6 \text{ in.}) + 12 \times 1 \text{ in.} = 3 \text{ ft-}1 \text{ in.}$ from face of support. 50 per cent of positive-moment reinforcement required for

$$0.23 \times (22 \text{ ft-}6 \text{ in.}) - 12 \times \tfrac{7}{8} = 4 \text{ ft-}3 \text{ in.}$$

from face of outer support. Remaining two No. 7 bars carried 6 in. into support. At continuous support, two No. 7 bars of positive-moment reinforcement are cut off 4 ft-3 in. from face of support; 50 per cent of negative moment reinforcement at continuous support is cut off at $0.11 \times (22 \text{ ft-}6 \text{ in.}) + 12 \times 1.128 \text{ in.} = 3 \text{ ft-}7 \text{ in.}$ from face of support and remaining two No. 9 bars

$$0.28 \times (22 \text{ ft-}6 \text{ in.}) + 13.5 \text{ in.} = 7 \text{ ft-}5 \text{ in.}$$

from face of support.

**Beam - and - Girder Floor** — **Computation Sheet 2**

Design of Beams:

Uniform load per foot from floor = 7(262.5) = 1840 lb/ft
Dead load of stem (Assumed) = 170 lb/ft
Total 2010 lb/ft

Clear span for $B$-1 = 22'-6"
Clear span for $B$-2 = 22'-0"
} Ave. clear span = 22' - 3"

## $B$-1

Center:

$M = +\frac{1}{14}(2010)(22.5)^2(12) = 874{,}000$ in.-lb

At continuous end:

$M = -\frac{1}{10}(2010)(22.25)^2(12) = 1{,}194{,}000$ in.-lb

At wall end:

$M = -\frac{1}{24}(2010)(22.5)^2(12) = 510{,}000$ in.-lb

$$V_{max} = \frac{1.15(2010)(22.5)}{2} = 26{,}000 \text{ lb}$$

$$b'd = \frac{26{,}000}{0.06(3{,}000)(7/8)} = 165 \text{ sq in.}$$

Use $b' = 8"$, $d = 21"$

Assuming 2 rows of steel, wt. of stem = 171 lb/ft O.K.

Center:

$$A_s = \frac{874{,}000}{20{,}000(21-2)} = 2.30 \text{ sq in.}$$

Use 4-#7 = 2.41 sq in.

At continuous end:

$M_1 = 235(8)(21)^2 = 829{,}000$ in.-lb

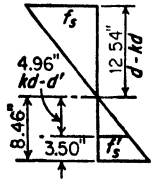
$$A_{s_1} = \frac{829{,}000}{20{,}000(0.866)(21)} = 2.28 \text{ sq in.}$$

$M_2 = 1{,}194{,}000 - 829{,}000 = 365{,}000$ in.-lb

$$A_{s_2} = \frac{365{,}000}{20{,}000(21-3.5)} = 1.04 \text{ sq in.}$$

$A_s = 3.32$ sq in.

Use 4-#9 = 4.00 sq in.




$$f'_s = \frac{2(20{,}000)(4.96)}{12.54} = 15{,}820 \text{ psi}$$

$$A'_s = \frac{365{,}000}{15{,}820(21-3.5)} = 1.32 \text{ sq in.}$$

Supplied by 4-#7 carried through.

## $B$-2

Center:

$M = +\frac{1}{16}(2010)(22)^2(12) = 730{,}000$ in.-lb

At first interior support

$M = -\frac{1}{11}(2010)(22.25)^2(12) = 1{,}085{,}000$ in.-lb

At all other interior supports:

$M = -\frac{1}{11}(2010)(22)^2(12) = 1{,}061{,}000$ in.-lb

$$V_{max} = \frac{2010(22)}{2} = 22{,}100 \text{ lb}$$

Beam dimensions will be kept the same throughout.

Center:

$$A_s = \frac{730{,}000}{20{,}000(21-2)} = 1.92 \text{ sq in.}$$

Use 4-#7 = 2.41 sq in.

First interior support is the same as the continuous end of $B$-1.

At other interior supports:

$M_1 = 829{,}000$ in-lb

$A_{s_1} = 2.28$ sq in.

$M_2 = 1{,}061{,}000 - 829{,}000 = 232{,}000$ in.-lb

$$A_{s_2} = \frac{232{,}000}{20{,}000(21-3.5)} = 0.66 \text{ sq in.}$$

$A_s = 2.94$ sq in.

Use 4-#8 = 3.14 sq in.

$f'_s = 15{,}820$ psi

$$A'_s = \frac{232{,}000}{15{,}820(21-3.5)} = 0.84 \text{ sq in.}$$

Supplied by 4-#7 carried through.

COMPUTATION SHEET 8.2

**Beam-and-Girder Floor** — **Computation Sheet 3**

Design of Beams - Cont:

At wall end, assuming 1 row of steel

$$A_s = \frac{510{,}000}{20{,}000\,(0.866)(22)} = 1.34 \text{ sq in.}$$

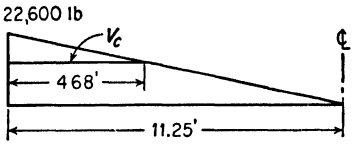
Use 2-#8 = 1.57 sq in.

Shear:

At wall end:

$$V = \frac{2010\,(22.5)}{2} = 22{,}600 \text{ lb}$$

$$V_c = (0\,03)(3{,}000)(\tfrac{7}{8})(8)(21) = 13{,}200 \text{ lb}$$



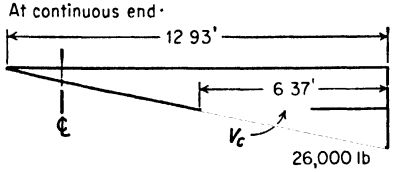

$$u = \frac{22{,}600}{6.3(\tfrac{7}{8})(22)} = 187 \text{ psi} < 210$$

$$s = \frac{(0\,22)(20{,}000)(\tfrac{7}{8})(21)}{9{,}400} = 8.6"$$

Placing the first stirrup 2" from the face of the support, the spacings will be:

1 @ 2", 3 @ 8", 3 @ 10"

At continuous end:




$$u = \frac{26{,}000}{14\,2(\tfrac{7}{8})(21)} = 100 \text{ psi}$$

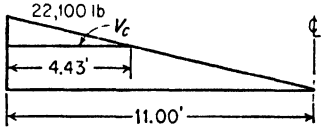
$$s = \frac{(0\,22)(20{,}000)(\tfrac{7}{8})(21)}{12{,}800} = 6.3"$$

Placing the first stirrup 2" from the face of the support, the spacings will be:

1 @ 2", 4 @ 6", 5 @ 8", 1 @ 10"

Shear:

At either end:




Shear in $B$-2 is symmetrical about the ℄

$$u = \frac{22{,}100}{12.6\,(\tfrac{7}{8})(21)} = 96 \text{ psi}$$

$$s = \frac{(0\,22)(20{,}000)(\tfrac{7}{8})(21)}{8{,}900} = 9.1"$$

Placing the first stirrups 3" from the faces of the supports, the spacings at each end will be:

1 @ 3", 3 @ 8", 3 @ 10"

COMPUTATION SHEET 8.3

*B*2:

50 per cent of positive-moment reinforcement required for

$$0.25 \times 22 \text{ ft} - 12 \times \tfrac{7}{8} \text{ in.} = 4 \text{ ft-7 in.}$$

from face of support. Negative-moment reinforcement at first interior support same as *B*-1. At other interior supports 50 per cent of negative-moment reinforcement may be cut off at

$$0.10 \times 22 \text{ ft} + 12 \times \tfrac{7}{8} \text{ in.} = 3 \text{ ft-1 in.}$$

from face of support and remaining two No. 7 bars

$$0.24 \times 22 \text{ ft} + 12 \times {}^{7}8 \text{ in.} = 6 \text{ ft-2 in.}$$

from face of support.

**8.29. Design of the Girders.** A portion of the sample computations for the design of the girders is shown in Computation Sheets 8.4 and 8.5. The clear spans here were obtained by making use of all previous assumptions and further assuming that the exterior columns supporting the second floor have a depth of 14 in., measured from the face of the wall, and that the interior first-floor columns are 16 in. in diameter. The average clear span was used in computing moments caused by the uniformly distributed load and the average distance between beams and adjacent columns in *G*1 for computing moments caused by concentrated loads. These assumptions are justifiable since they produce moments slightly larger than those which actually exist in either span. The inherently approximate nature of calculations of this sort does not warrant closer refinement in computations. Reinforcing steel will be placed with centers $2\frac{1}{2}$ in. from the bottom of the girder, since the girder steel will be at a different level from the beam steel. However, at the columns, the negative reinforcing steel in the beams and in the girders cross; therefore girder steel will be placed with centers $3^{1}2$ in. from the top so that it will cross below the beam steel without interference. In all cases, parallel rows of bars will be placed 2 in. on centers vertically. The different positions of steel relative to the top and bottom of the girder account for the difference of effective depths used at the center and ends, since the maximum shear occurs at the faces of the columns and the required depth is supplied at that point. Note that a slight excess of steel is used at the center of each girder to provide a sufficient number of bars to satisfy the reinforcement requirements at the supports.

The points where the bars may be bent up, in pairs, are determined by assuming that the positive moment is zero at a distance of $\frac{2}{3} \times l/3$ from the concentrated load to the center line of the support, measured toward the support, and that the moment diagram is a straight line between the maximum and zero values. The latter assumption is in error only

**Beam - and - Girder Floor** — **Computation Sheet 4**

Design of Girders

Load from floor beams = 44,200 lb

Live load on girder = 200 lb/ft

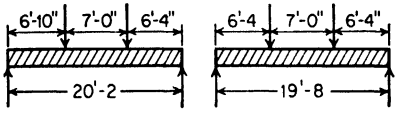
Dead load of floor finish on girder = 13

Dead load of girder (assumed) = 350

Total = 563 lb/ft

Clear span of $G$-1 = 20'-2"
Clear span of $G$-2 = 19'-8" } Ave clear span used = 19'-11"

Simple span maximum moment




$\frac{1}{8}(563)(19.92)^2(12) = 335{,}000$ in.-lb
$+ 44{,}200\,(6.58)(12) = 3{,}490{,}000$
$M_s = 3{,}825{,}000$ in-lb

## $G$-1

Center

$M = +\frac{8}{14} M_s = 2{,}185{,}000$ in-lb

At continuous end

$M = -\frac{8}{10} M_s = -3{,}060{,}000$

At wall end

$M = -\frac{8}{16} M_s = -1{,}912{,}000$

$V_{max} = 1.15\left[44{,}200 + \frac{563(20.2)}{2}\right] = 57{,}500$ lb

$b'd = \frac{57{,}500}{(0.08)(3{,}000)(\frac{7}{8})} = 274$ sq in

Use $b' = 12$ in, $d = 23$ in

Assuming two rows of steel, wt of girder = 344 lb/ft

Center

$A_s = \frac{2{,}185{,}000}{20{,}000\,(24.2)} = 4.96$ sq in

Use 6-#9 = 6.00 sq in

## $G$-2

Center

$M = +\frac{8}{16} M_s = 1{,}912{,}000$ in-lb

At ends

$M = -\frac{8}{11} M_s = 2{,}780{,}000$ in-lb

$V_{max} = 44{,}200 + \frac{563(19.67)}{2} = 49{,}700$ lb

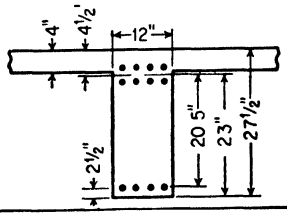
Girder dimensions will be kept the same throughout

Center

$A_s = \frac{1{,}912{,}000}{20{,}000\,(24.2)} = 4.35$ sq in

Use 6-#9 = 6.00 sq in

Cross section of girder at supports, assuming two rows of tension steel and one row of compression steel




COMPUTATION SHEET 8.4

**Beam - and - Girder Floor** **Computation Sheet 5**

Design of Girders - Cont.

At continuous end:

$M_1 = 235(12)(23)^2 = 1{,}495{,}000$ in.-lb

$$A_{s_1} = \frac{1{,}495{,}000}{20{,}000(0.866)(23)} = 3.75 \text{ sq in.}$$

$M_2 = 3{,}060{,}000 - 1{,}495{,}000 = 1{,}565{,}000$ in.-lb

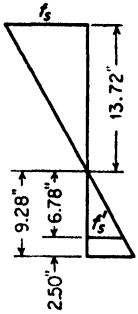
$$A_{s_2} = \frac{1{,}565{,}000}{20{,}000(20.5)} = 3.82 \text{ sq in.}$$

$A_s = 7.57$ sq in.

Use 8-#9 = 8.00 sq in.

Bend up 4 from each beam

$$u = \frac{57{,}500}{28.4(7/8)(23)} = 101 \text{ psi}$$




$$f'_s = \frac{2(20{,}000)(6.78)}{13.72} = 19{,}800 \text{ psi}$$

$$A'_s = \frac{1{,}565{,}000}{19{,}800(20.5)} = 3.85 \text{ sq in.}$$

2-#9 from each beam = 4.0 sq in.

At wall end

$$A_s = \frac{1{,}912{,}000}{20{,}000(0.866)(22)} = 5.01 \text{ sq in.}$$

Bend up 4-#9 from center and add 1-#9

$$u = \frac{49{,}700}{17.7(7/8)(22)} = 146 \text{ psi}$$

Compression steel required 1.03 sq in. furnished by 2-#9 carried into support.

At supports:

$M_1 = 2{,}780{,}000$ in.-lb

$$A_s = \frac{2{,}780{,}000}{20{,}000(0.866)(22)} = 7.29 \text{ sq in.}$$

Use steel same as $G$-1

Compression steel

$M_2 = 2{,}780{,}000 - 1{,}495{,}000 = 1{,}285{,}000$ in-lb

$$A'_s = \frac{1{,}285{,}000}{19{,}800(20.5)} = 3.17 \text{ sq in.}$$

furnished by 2-#9 from each span.

COMPUTATION SHEET 8.5

because of the uniformly distributed weight of the girder, and this is very small in comparison with the large concentrated loads. The first pair of bars may be bent up at a distance of

$$\tfrac{1}{3} \times \tfrac{2}{3} \times (21 \times 12)/3 = 18.7 \text{ in.}$$

to the left of the first concentrated load, or at point $a$ in Fig. 8.17. The next pair may be bent at a distance 37.4 in. from the first load, or at point $b$.

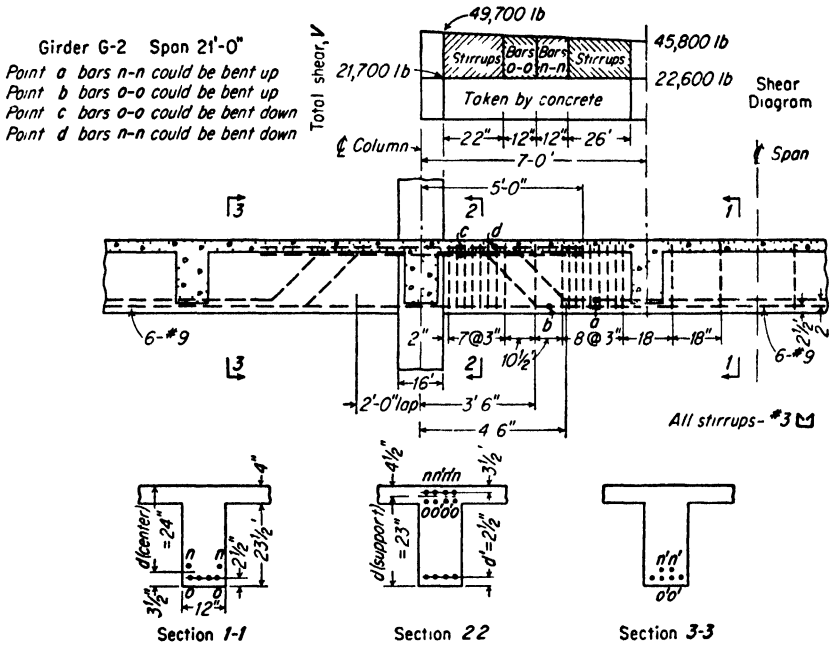
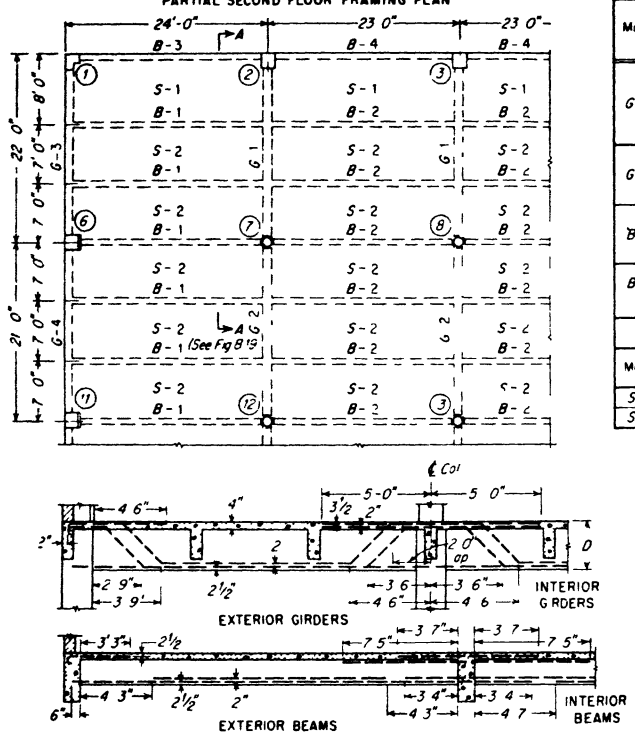



FIG. 8.17. Details of girder.

A similar assumption for the negative-moment variation can be made; the point of zero negative moment being, therefore, $\frac{2}{3} \times l/3 = 56$ in. from the center line of the support. The overlapping of the positive- and negative-moment diagrams which results from these assumptions provides for any variation in the moments that might result from unequal placing of the live load on adjoining spans. Assuming that the eight bars over the support are stressed equally, one pair may be bent down at a distance of $\frac{1}{4} \times 56 = 14$ in. from the support, or at point $c$ in Fig. 8.17, and another pair may be bent down at point $d$. Bend points are located to satisfy these requirements and to enable the inclined bars to take as much of the diagonal tension as possible. In spacing inclined bars or stirrups, attention must be paid to the ACI Code requirement

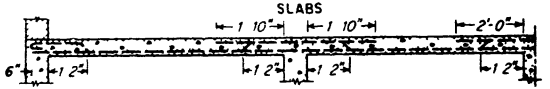



**BEAM AND GIRDER SCHEDULE**

| Mark | Size | | Reinforcing | | | |
|---|---|---|---|---|---|---|
| | | | Longitudinal | | Stirrups | |
| | B' | D | No Size | Remarks | No Size | Spacing From Face of Support |
| G 1 | 12 | 27½ | 2 – #9 | Bot | 40 – #3 | 1 @ 2", 7 @ 3", 2 @ 10½", 8 @ 3", 2 @ 18", Each end |
| | | | 2 – #9 | Bent Lower layer | | |
| | | | 2 – #9 | Bent Upper layer | | |
| | | | 1 – #9 | Top Wa l end Hook n wall | | |
| G-2 | 12 | 27½ | 2 – #9 | Bot | 40 – #3 | 1 @ 2", 7 @ 3", 2 @ 10½", 8 @ 3", 2 @ 18", Each end |
| | | | 2 – #9 | Bent Lower layer | | |
| | | | 2 – #9 | Bent Upper layer | | |
| B 1 | 8 | 24½ | 4 – #7 | Bot 2 Layers | 7 – #3 | 1 @ 2", 3 @ 8", 3 @ 10", Wall end |
| | | | 2 – #8 | Top Wall end | 11 – #3 | 1 @ 2", 4 @ 6", 5 @ 8" 1 @ 10", Cont end |
| | | | 4 – #9 | Top Cont end 2 Layers | | |
| B-2 | 8 | 24½ | 4 – #7 | Bot 2 Layers | 14 – #3 | 1 @ 3", 3 @ 8", 3 @ 10", Each end |
| | | | 4 – #8 | Top 2 Layers At Sup Com to Adj B 2's | | |

**SLAB SCHEDULE**

| Mark | Depth | Reinforcing | | |
|---|---|---|---|---|
| | | Main Steel | Temp Steel | Remarks |
| S 1 | 4 | #3 @ 6 | #3 @ 18 | Alternate straight and bent |
| S 2 | 4 | #3 @ 6 | #3 @ 18 | Alternate straight and bent |

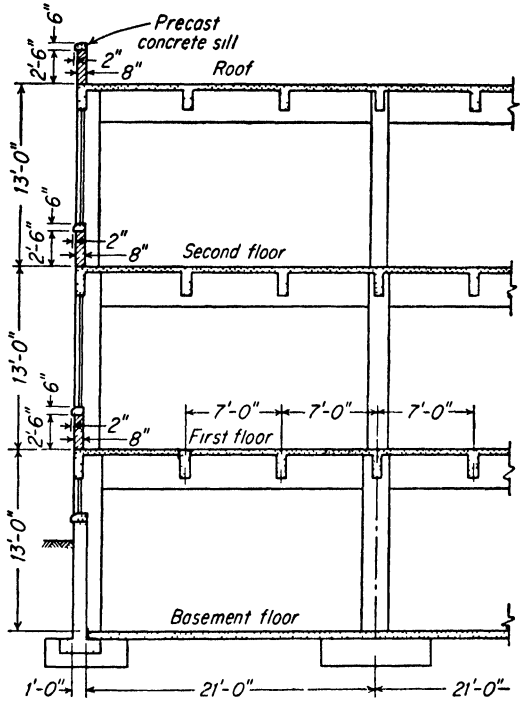



*NOTES*

1 All concrete to have 3000 pounds per square inch compressive strength in 28 days
2 Reinforcing steel shall be intermediate grade deformed in accordance with latest A S T M specifications with deformation conforming to A S T M - A 305
3 Bends to be made as shown on diagrams
4 Place main reinforcing steel so that bottom of steel is 2" above forms in beams and girders and ¾ in slabs
5 Provide bar supports in accordance with A C I Detailing Manual
6 Stirrups to have 2-#3 spacers for length of stirrup spacing

FIG 8 18 Engineering drawing of beam-and-girder floor.

which states that in beams with a shearing stress in excess of $0.06f'_c$ every 45° line extending from the mid-depth of the beam to the longitudinal tension bars shall be crossed by at least two lines of web reinforcement. The maximum permissible stirrup spacing is therefore $\frac{3}{4} = 6$ in. For inclined bars twice this spacing may be used.




Section A-A Partial Cross Section

Fig. 8.19

The amount of external shear that can be resisted by the concrete is $V_c = 90 \times 12 \times \frac{7}{8} \times 23 = 21{,}700$ lb at the support and

$$V_c = 90 \times 12 \times \frac{7}{8} \times 24 = 22{,}600 \text{ lb}$$

away from the support. The shear diagram (Fig. 8.17) shows that web reinforcement is required only from the support to the first concentrated load.

Stirrups are required between the support and the point where the outermost row of inclined bars pass the mid-depth of the beam. They are also required between the concentrated load and the point of bending of the innermost row of bars. With No. 3 U stirrups, the required spacing at the support is

$$s = \frac{2 \times 0.1104 \times 20{,}000 \times \frac{7}{8} \times 23}{49{,}700 - 21{,}700} = 3.18 \text{ in.}$$

Between the concentrated load and the innermost inclined bars

$$s = \frac{2 \times 0.1104 \times 20{,}000 \times \frac{7}{8} \times 24}{(49{,}700 - 4.50 \times 562.5) - 22{,}600} = 3.78 \text{ in.}$$

A 3-in. spacing is used throughout both regions. It is advisable to use No. 3 U stirrups, spaced at about 18 in., over the remaining portions of the girder to assist in securing unity of action of the two parts of the T beam, as shown in Fig. 8.17.

The bent bars are continued 60 in. beyond the center of the column; the straight bars should be continued for a distance beyond the support at least equal to

$$\frac{19{,}800}{4 \times 300} \times 1.128 = 18.5 \text{ in.}$$

Complete engineering details for the portion of the building designed above are shown in Figs. 8.18 and 8.19.

## DESIGN OF A RIBBED FLOOR WITH STEEL-TILE FILLERS

**8.30. Dead Loads.** The weight of a ribbed floor can be computed from the known concrete dimensions. The general form of the cross section of the steel tiles must be known, since the taper of the sides and the chamfering of the upper corners affect the volume of concrete in the ribs or joists. The cross section can be obtained from the manufacturers' catalogues. The average values in Table 8.2 can be used for joists (or ribs) 5 in. wide at the bottom and 25 in. on centers. These values assume that the steel cores will be removed. If permanent cores are used, from 1 to 2 psf should be added.

The values given in this table do not include the weight of an extra floor finish, or the weight of a plastered ceiling below the floor. The former can be computed from the specified thickness of finish or type of floor surface. In computing design loads, an allowance of 10 psf is usually made for a plastered ceiling. An additional allowance of from 10 to 20 psf is made for the weight of partitions, when these partitions are not definitely located on the architect's plans, or where there is a possibility of future rearrangement of partitions. The latter condition is very likely to occur in buildings of the types to which ribbed-floor construction is adapted. When definitely located partitions are parallel with the joists, a thicker joist than the normal 5-in. joist is usually placed under these partitions, or, if the partition is located between two joists, both of these joists are made thicker than the others and the slab thickness between them is also increased. The increased slab thickness is

TABLE 8.2. Weight of Concrete in Ribbed Floors with Steel-tile Fillers

| Depth of joist below slab, in. | Thickness of slab, in. | Average weight of floor, psf |
|---|---|---|
| 6 | 2 | 42 |
| | $2\frac{1}{2}$ | 48 |
| 8 | 2 | 48 |
| | $2\frac{1}{2}$ | 54 |
| 10 | 2 | 55 |
| | $2\frac{1}{2}$ | 61 |
| 12 | 2 | 61 |
| | $2\frac{1}{2}$ | 67 |
| | 3 | 73 |
| 14 | 2 | 69 |
| | $2\frac{1}{2}$ | 75 |
| | 3 | 81 |

obtained by using shallower tiles in the one row, or by lowering the tiles in that row. When definitely located partitions are perpendicular to the joists, the partition weight is considered as a concentrated load in designing the joists. Transverse bridging ribs 4 in. wide and the same depth as the joists are commonly used at the center of spans less than 24 ft and

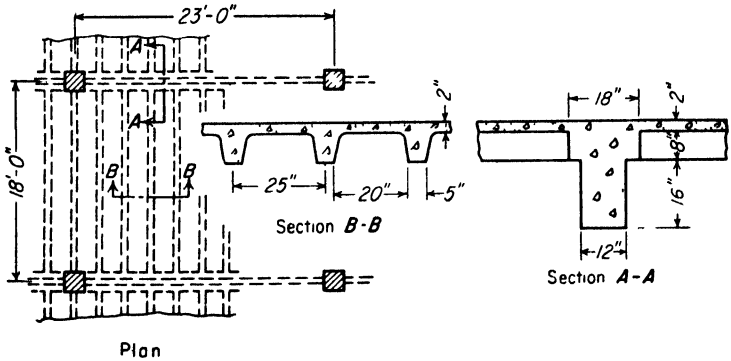



FIG. 8.20. Details of ribbed floor with removable steel tiles.

at the third points of longer spans to stiffen the joists and distribute concentrated floor loads to adjacent joists.

**8.31. Data for Design.** A typical interior panel of a hotel floor is to be built as a ribbed floor, using removable steel-tile cores. The joists are to be supported at the ends on concrete girders, as shown in Fig. 8.20. The span of the girders is 23 ft-0 in., and the distance center to

center of girders is 18 ft-0 in. The live load is 50 psf, and allowances for dead loads other than the weight of the concrete are to be made as follows: partitions, 10 psf; plastered ceiling, 10 psf; wood floor, with sleepers in cinder-concrete fill, 15 psf. A 2500-lb concrete is to be used in the floor, and intermediate-grade deformed bars, for which $f_s = 20{,}000$ psi, are to be used for reinforcement. It is required to design the panel.

**8.32. Design of Joists.** *Computation of Moments and Shears.* The joists are designed as regular continuous T beams, with a flange width equal to the distance center to center of joists. In order to compute the weight of the floor, the depth of the joists below the slab and the thickness of the slab must be assumed. A combination of 8-in. joists and 2-in. slab will be tried. The weight of the concrete per square foot of floor is 48 lb, and the total load is $50 + 10 + 10 + 15 + 48 = 133$ psf. With 5-in. joists, spaced 25 in. on centers, each joist supports $^{25}/_{12} = 2.08$ sq ft of floor per foot of joist, and the total load on each joist is

$$133 \times 2.08 = 277 \text{ lb per lin ft}$$

The panel is fully continuous, and the effective span for moment is equal to the clear distance between the stems of the girders. If the width of the girder stems is assumed as 12 in., the joist span is $18 - {}^{12}/_{12} = 17$ ft. The maximum negative moment at the face of the support is

$$M = {}^{1}/_{11} \times 277 \times 17^2 \times 12 = 87{,}330 \text{ in.-lb}$$

and the maximum positive moment at the center of the span is

$$M = {}^{1}/_{16} \times 277 \times 17^2 \times 12 = 60{,}040 \text{ in.-lb}$$

In computing the maximum shear in the joists, the effective span can be taken as the clear distance between the flanges of the girders which support the joists. If, as in Fig. 8.20, the girder-flange width is assumed as 18 in., the effective span of the joists is $18 - {}^{18}/_{12} = 16.5$ ft, and the maximum shear is

$$V = 277 \times \frac{16.5}{2} = 2285 \text{ lb}$$

*Design for Shear.* It is rather difficult to place stirrups in narrow joists, such as are used in this form of construction; hence the shearing unit stress should be kept below the maximum allowable value of

$$0.03 \times 2500 = 75 \text{ psi}$$

The effective shearing width of a joist should be taken as the width at the bottom of the joist, neglecting the increased section due to side slope of the tiles This practice is reasonably conservative for continuous joists, since the critical shear stresses occur in the upper, or tension, part of the member. If necessary, tapered tiles can be used at the ends of each row

to increase the shearing area, or deeper tiles can be used to serve the same purpose. With 8-in. joists under a 2-in. slab, the effective depth is $10 - 1\frac{1}{8} = 8.87$ in., which will allow for a minimum clear insulation under the bars of about $\frac{3}{4}$ in.

If straight end tiles are used, the approximate maximum unit shearing stress is

$$v = \frac{V}{b'jd} = \frac{2285}{5 \times \frac{7}{8} \times 8.87} = 59 \text{ psi}$$

This is less than the allowable stress, and the joist size is satisfactory for shear resistance; therefore no taper is required. If the problem of deciding whether to use tapered end tiles or deeper tiles throughout arises, it should be remembered that the steel area required decreases with an increase in the depth of the joist, and both the cost of the steel and the cost of the concrete should be considered. The cost of the forms can be omitted in the comparison, because this cost does not vary materially.

*Computation of Steel Area.* Design at the support will be based on the moment value of 87,330 in.-lb at the face of the girder stem, although the actual maximum compressive stress occurs at the edge of the girder flange, where the moment is about 7 per cent less. The added width of concrete on the compression face afforded by the girder flange will be neglected.

$$K = 196 \qquad b = 5 \text{ in.} \qquad d = 8.87 \text{ in.} \qquad d' = 1.12 \text{ in.}$$

$$M_1 = Kbd^2 = 196 \times 5 \times (8.87)^2 = 77{,}100 \text{ in.-lb}$$

$$M_2 = 87{,}300 - 77{,}100 = 10{,}200 \text{ in.-lb}$$

$$A_{s_1} = pbd = 0.0113 \times 5 \times 8.87 = 0.50 \text{ sq in.}$$

$$A_{s_2} = \frac{10{,}230}{20{,}000(8.87 - 1.12)} = 0.07 \text{ sq in.}$$

$$A_s = 0.50 + 0.07 = 0.57 \text{ sq in.}$$

$$A'_s = A_{s_2} = 0.07 \text{ sq in.}$$

The approximate required area of steel reinforcement in each joist for positive moment is

$$A_s = \frac{M}{f_s(d - 1/2t)} = \frac{60{,}040}{20{,}000(8.87 - \frac{2}{2})} = 0.38 \text{ sq in.}$$

*Arrangement of Reinforcement.* As shown in Fig. 8.21, two No. 4 bars ($A_s = 0.39$ sq in.) will be used for positive-moment reinforcement. Bending up one bar in each span and adding one No. 4 straight top bar will provide sufficient steel ($A_s = 0.58$ sq in.) to meet the negative-moment requirements. Bent-up bars and top bars are extended to the point of inflection in each joist. This point may be taken at a distance from the edge of the stem of the girder equal to one-quarter of the clear

span of the joist. The remaining positive reinforcement is continued beyond the far face of the girder support a distance of 9 in. Ample compression reinforcement is thus provided at the support by this steel.

*Investigation of Maximum Stress in Concrete.* It is necessary to review the joist at the center, in order to make sure that the compression stress in the concrete is less than the allowable value of

$$0.45 \times 2500 = 1125 \text{ psi}$$

With $n = 12$, $d = 8.87$ in., $A_s = 0.39$ sq in., $b = 25$ in., and $t = 2$ in., the beam is found to act as a rectangular beam, the values of $k$ and $j$, as

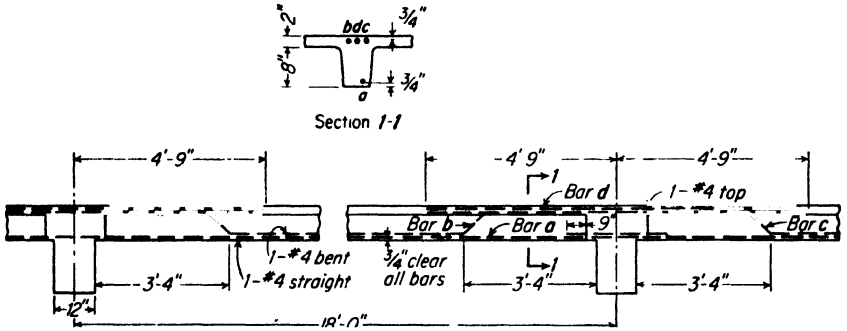



FIG. 8.21. Joist details.

obtained from Table 6, Appendix D, being 0.195 and 0.938, respectively. The maximum compression stress in the concrete is

$$f_c = \frac{60{,}040 \times 2}{0.938 \times 0.195 \times 25 \times (8.87)^2} = 336 \text{ psi}$$

This stress is satisfactory.

*Investigation of Bond Resistance.* The critical section for bond in the negative reinforcement is at the edge of the flange of the supporting girder. The unit bond stress at this section is

$$u = \frac{2285}{4.7 \times \frac{7}{8} \times 8.87} = 63 \text{ psi}$$

The allowable unit bond stress is $0.07 \times 2500 = 175$ psi.

**8.33. Design of Girder.** *Computation of Loads.* The loads which are brought to the girder from the joists are theoretically concentrated at the points where the joists frame into the girder. They are spaced so closely, however, that they may be considered as uniformly distributed throughout the length of the girder, without affecting materially the maximum moment in the girder. Each pair of adjoining joists, one on either side of the girder, transmits a load to the girder equal to twice the end

shear of one joist, or $2 \times 2285 = 4570$ lb. Since the joists are 25 in., or 2.08 ft, on centers, the uniform load per foot of girder is $\frac{4570}{2.08} = 2200$ lb. To this must be added the weight of the girder and the floor and ceiling loads directly over and under the flange of the girder. With an assumed cross section as shown in Fig. 8.20, the weight of the flange is

$$[(18 \times 10)/144] \times 150 = 188 \text{ lb per ft}$$

and the weight of the stem is

$$[(12 \times 16)/144] \times 150 = 200 \text{ lb per ft}$$

The live load, the ceiling load, and the weight of the flooring and partitions on the 18-in. flange width are $(50 + 10 + 15 + 10)\frac{18}{12} = 128$ lb per lin ft. The total load on the girder is, therefore,

$$2200 + 188 + 200 + 128 = 2716 \text{ lb per ft}$$

*Computation of Moment and Shear.* The girder is continuous at both ends, so that with 18-in. columns, the effective span is

$$23 - \frac{18}{12} = 21.5 \text{ ft}$$

The maximum negative moment at the face of the support is

$$M = \frac{1}{11} \times 2716 \times (21.5)^2 \times 12 = 1{,}370{,}000 \text{ in.-lb}$$

and the maximum positive moment at the center of the span is

$$M = \frac{1}{16} \times 2716 \times (21.5)^2 \times 12 = 942{,}000 \text{ in.-lb}$$

The maximum shear is

$$V = 2716 \times \frac{21.5}{2} = 29{,}300 \text{ lb}$$

*Design at Center.* The girder is a T-shaped beam, but, because of the comparatively thick flange, the neutral axis will probably be in the flange and the girder must be designed for moment as a rectangular beam with a width equal to the width of the flange. For shear, the effective width $b'$ is the width of the stem, or 12 in. Web reinforcement will be provided as necessary. The design will be based on a maximum unit shearing stress of $0.06 \times 2500 = 150$ psi. The effective depth required for shear is

$$d = \frac{V}{vjb'} = \frac{29{,}300}{150 \times \frac{7}{8} \times 12} = 18.6 \text{ in., or } 19 \text{ in.}$$

Allowing for two rows of bars, the over-all depth must be $19 + 3 = 22$ in. and the depth below the flange is $22 - 10 = 12$ in. The assumed depth of 16 in. will be maintained to reduce the amount of reinforcement

required. With an effective depth of 23 in., the flange width required for moment is

$$b = \frac{M}{Kd^2} = \frac{942{,}000}{196 \times (23)^2} = 9.1 \text{ in.}$$

The assumed width of 18 in. will be maintained to provide sufficient room for the steel-tile form supports.

Assuming $j = \frac{7}{8}$, the approximate required steel area is

$$A_s = \frac{M}{f_s jd} = \frac{942{,}000}{20{,}000 \times 0.875 \times 23} = 2.34 \text{ sq in.}$$

Six No. 6 bars with an area of 2.65 sq in. will be used.

*Design at Support.* At the support, it will be necessary to provide compression reinforcement in the bottom of the beam. The resisting moment of the concrete section is

$$M_1 = 196 \times 12 \times (23)^2 = 1{,}242{,}000 \text{ in.-lb}$$

The steel required to develop the strength of the concrete is, assuming $j = \frac{7}{8}$,

$$A_{s_1} = \frac{1{,}242{,}000}{20{,}000 \times \frac{7}{8} \times 23} = 3.09 \text{ sq in.}$$

$$M_2 = 1{,}370{,}000 - 1{,}242{,}000 = 128{,}000 \text{ in.-lb}$$

$$A_{s_2} = \frac{128{,}000}{20{,}000(23 - 3)} = 0.32 \text{ sq in.}$$

From Eq. (3.30), the amount of compression steel required is

$$A'_s = \frac{A_{s_2}}{2} \times \frac{1 - k}{k - (d'/d)} = \frac{0.32}{2} \times \frac{1 - 0.403}{0.403 - \frac{3}{23}} = 0.35 \text{ sq in.}$$

The total tension steel required is

$$A_s = 3.09 + 0.32 = 3.41 \text{ sq in.}$$

Three bars will be bent up and continued across the support to the point of inflection in the adjoining girder, which will be assumed at the quarter point of the span of that girder. Two additional straight No. 6 bars will be added at the top over the support, giving a total amount of steel (eight No. 6 bars, $A_s = 3.53$ sq in.) sufficient to resist the tension caused by the negative moment. The top bars will also be extended to the inflection point in each span. The other three bars in each span will be run straight through the support for a distance of 1 ft beyond the center of the column, which will be enough to form a proper splice with the straight bars from the adjoining span. The effective compression reinforcement (three No. 6 bars, $A'_s = 1.32$) is ample.

*Investigation of Bond Resistance.* The critical section for bond in the negative-moment steel is at the face of the column, where the shear is 29,300 lb, and the sum of the perimeters of the bars is

$$8 \times 2.356 = 18.85 \text{ in.}$$

The unit bond stress is

$$u = \frac{29{,}300}{18.85 \times \frac{7}{8} \times 23} = 77 \text{ psi}$$

This is well below the allowable value of $0.07 \times 2500 = 175$ psi.

*Design of Web Reinforcement.* Resistance to diagonal tension stresses will be provided by bending up the three bars and by placing vertical stirrups in the proper positions. The computations are the same as for any uniformly loaded, continuous beam; a typical example is given in Art. 8.28.

**8.34. Detail Drawings.** Each floor plan must show the number of ribs, or joists, in each panel, the number of rows of tiles, and the arrangement of the joists. All identical joists are given the same mark, which consists of a numeral indicating the floor, the letter *R* or *J* indicating ribs or joists, and a following numeral which is the mark of identity. Thus, 1*R*22 would designate first-floor joist number 22. The outlines of the rows of tile are shown by dotted lines. Where there are several adjacent identical rows, forming identical joists, only the end rows and joists in the group are shown, and the numbers of rows and joists are placed on dimension lines with arrows pointing to the boundaries of the group. The depth of the stems of the joists and the thickness of the slab are also given on the floor plan. Temperature steel, usually about No. 2 bars at 10 in. on centers, is placed in the slab at right angles to the joists; this is detailed on the plan in the usual manner. Quite often welded wire mesh is used to reinforce the slab.

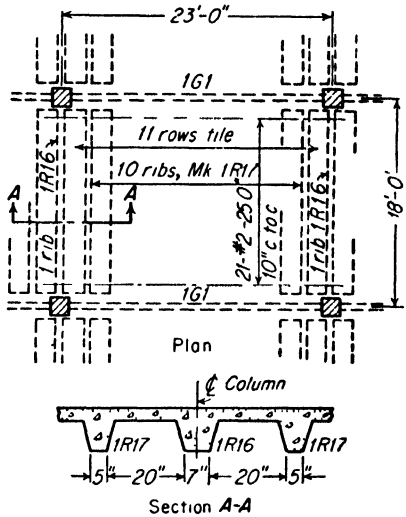



Fig. 8.22 Method of detailing ribbed floors with concrete girders.

A detail floor plan for the typical interior panel which was designed in the preceding articles is shown in Fig. 8.22. The girders are marked on the plan in the usual manner. At least one typical cross section through the joists and through the girders should be given on the drawing. The joists 1*R*16 between the columns of Fig. 8.22 are 7 in. wide at the bottom

instead of 5 in., in order to fill up the panel. It is usually desirable to have a joist such as 1*R*16 between the columns, in order to stiffen the columns in this direction. In tall structures subject to wind pressure, a deeper beam would be used in place of the joist 1*R*16, in order to add to the rigidity of the entire structure.

## DESIGN OF A RIBBED FLOOR WITH CLAY-TILE FILLERS

**8.35. Dead Loads.** The dead weight of the floor includes the weight of the concrete in the joists and in the slab, and the weight of the tiles. Table 8.3 gives average weights of clay-tile-and-concrete floors in pounds per square foot, for 4-in. ribs spaced 16 in. on centers.

**TABLE 8.3. Weights of Clay-tile-and-concrete Ribbed Floors**

| Depth of joist below slab, in | Thickness of slab, in. | Average weight of floor, psf |
|---|---|---|
| 4 | 2 | 49 |
| | $2\frac{1}{2}$ | 55 |
| 6 | 2 | 60 |
| | $2\frac{1}{2}$ | 66 |
| 8 | 2 | 73 |
| | $2\frac{1}{2}$ | 79 |
| 10 | 2 | 82 |
| | $2\frac{1}{2}$ | 88 |
| | 3 | 95 |
| 12 | 2 | 92 |
| | $2\frac{1}{2}$ | 98 |
| | 3 | 104 |

The values given above do not include the weight of an extra floor finish or plastered ceiling. Allowances for these items and for partitions should be made in computing design loads, as explained in Art. 8.30.

**8.36. Design of Joists.** The general method of designing the joists is the same as that explained in Art. 8.32 for joists which are formed by the use of steel-tile cores. If the joists are 4 in. wide and the tiles 12 in. wide, the distance center to center of joists is 16 in., and each joist supports $\frac{16}{12} = 1.33$ sq ft of floor per linear foot of joist. Both dead and live loads must be included in the computations for moment and shear.

Because of the comparatively great strength of the walls of structural-clay tiles and the thorough bonding to the concrete which is obtained by the projections on the tile blocks, part of the side walls can be assumed

as an effective part of the joists in resisting bending and shearing stresses. As a usual rule, the effective width of the joist is assumed as 1 in. or sometimes 1½ in. greater than the actual width of concrete between the surfaces of adjacent rows of tile. Thus, if 12-in. tile blocks are laid in rows which are 16 in. on centers, the normal width of the joist is 4 in., but the width that would be used in the design of the joist is 5 or 5½ in. The former value is preferred by conservative designers, but the latter has proved thoroughly safe.[1]

If the maximum unit shearing stress in a joist of given dimensions is greater than that allowable for concrete without web reinforcement, this stress may be reduced by placing one or more pieces of tile 8 in. in width at each end of each row, thus increasing the width of the joists at the point of maximum shear by 4 in. This is somewhat analogous to the use of tapered steel tiles in ribbed floors with steel-tile fillers.

**8.37. Design of Girders.** The method of designing the girders which support the joists is exactly the same as that used in the design of similar members in ribbed floors with steel-tile fillers (see Art. 8.33). The methods of detailing the joists and girders are also the same in the two types of construction.

## FLAT-SLAB FLOORS

**8.38. Description of General Type.** A flat-slab floor, as its name implies, is one consisting of a reinforced concrete floor slab built monolithically with the columns and supported directly by the columns without the aid of beams and girders. The slab may be of uniform thickness throughout the entire floor area, or a part of it, symmetrical about the column, may be made somewhat thicker than the rest of the slab, the thickened portion of the slab thus formed constituting what is known as a dropped panel, or drop (see Fig. 8.25).

Dropped panels are used to reduce the shearing stresses in the slab within the area of the drop. The increase in the effective slab thickness which is provided by the drop also decreases the compression stresses in the concrete and reduces the amount of steel which is required over the column heads. In general, the use of dropped panels is not economical for live loads less than 150 psf. Dropped panels are usually square, the width being approximately equal to one-third of the panel length. The American Concrete Institute Building Code requires that the thickness of the drop panel below the slab shall not be more than one-fourth the distance from the edge of the column capital to the edge of the drop panel. This usually results in a drop panel with a thickness of one-

[1] The ACI Code states that, if the fillers are so placed that the joints in alternate rows are staggered, the shells of the fillers in contact with the joists may be included in the calculations involving shear or negative bending moment. No other portions of the fillers may be included in the design calculations.

quarter to one-half the thickness of the slab outside the drop. When drops are used over interior columns, they are normally used over the wall columns also. Such drops should have a width parallel to the wall equal to the corresponding width of the interior-panel drops; at right angles to the wall they should project beyond the center of the column a distance equal to one-half of the corresponding total width of the interior drops.

The columns in practically all cases flare out toward the top, forming a capital of a shape somewhat similar to an inverted truncated cone. This capital gives a wider support for the floor slab, which results in a decrease in the bending moment which the slab is called upon to resist, and a decrease in the shearing stresses around the perimeter of the column, and tends toward a more rigid structure. The effective diameter of the capital should be taken as the diameter of the circle at the point where a 45° line from the base of the capital intersects the bottom of the slab or dropped panel. The 45° line must fall within the concrete of the cap at all points. Some codes require a minimum effective diameter $c$ of the cap equal to $0.20l$, where $l$ is the average span of the panel. A value of $c$ equal to $0.225l$ is often used in practice.

At the wall columns, a bracket is often used in place of a regular half-capital. The width of the bracket is equal to the width of the column parallel to the wall; the sloping side of the bracket forms an angle of 45° with the horizontal; the projection of the bracket beyond the face of the column is the same as, or somewhat less than, the projection of the interior-column capital.

**8.39. Advantages of Flat-slab Floors.** Structurally, a flat-slab floor has many advantages over the ordinary beam-and-girder floor. The most important of these may be enumerated as follows:

1. For ordinary spans with heavy loads, under average conditions, the flat-slab floor is more economical than the beam-and-girder floor.

2. In a multistoried building, the same number of stories of a given clear height may be obtained with a smaller total building height, because of the smaller floor thickness.

3. The slab formwork is much simplified.

4. The flat-slab floor, owing to the lack of many sharp corners, is better able to resist continued exposure to fire than the beam-and-girder floor. It has been found by actual experience that the worst damage caused to reinforced concrete by severe fires has occurred at places where there may be spalling, that is, at exposed edges and sharp corners.

5. Automatic sprinkler protection may be made more complete under a flat-slab floor since the nozzles may be placed well up near the under side of the slab without obstruction to the path of the spray.

6. More light may be admitted into the building if desired, by placing

the wall beams above the floor level, and thus allowing the windows to be extended to the under side of the slab. The absence of deep beams and girders also removes the obstruction to the passage of light within the building.

7. Owing to the large number of smaller bars extending in several directions over the entire area of the floor, the danger of sudden failure or collapse is less than in the beam-and-girder type of floor. The relatively large breadth of structure also makes the effect of local variations in the concrete less than would be the case for narrow members like beams.

8. The opportunity for inspecting the position of the reinforcement is excellent, and the conditions attending deposition and placing of the concrete are favorable to securing uniformity and soundness in the concrete.

**8.40. Bending Moments in Flat-slab Floors.** Figure 8.23 represents a portion of a flat-slab floor including four column supports, the load on the floor being uniformly distributed. The full circles represent the column heads underneath the slab. It is evident that, considering any radial line from the column center, the curvature of the slab along this line will be convex upward for a certain distance, then concave upward, then convex upward again. This implies that at some point along each radial line there is a point of inflection where the radial bending moment changes from positive to negative. The locus of all these points may be represented by the dotted approximate circles centered about the column capitals.

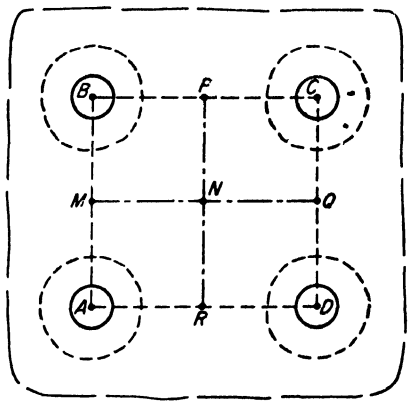


FIG. 8.23

As the slab is loaded, deflection occurs. The point $N$ at the midpoint of the panel, being the farthest away from the support, will deflect more than a point $M$, $P$, $Q$, or $R$ midway between any two adjacent columns. The points $M$, $P$, $Q$, $R$ will therefore be higher than point $N$ but lower than the supports. This results in a negative moment along the line $MQ$ at $M$, and a positive moment at $N$. The condition is similar along line $PR$.

The analysis of a flat-slab floor is a statically indeterminate problem and the elastic properties of the slab and the relative stiffness of its various parts must be considered in determining accurately the moments

and shears in the slab. An arbitrary but satisfactory method of obtaining the theoretical bending moments is to divide each panel into rectangular strips, a middle strip $A$ (Fig. 8.24) with a width equal to $l_1/2$ and two equal column strips $B$ which occupy the outer portions of the panel. Similar strips $A'$ and $B'$ are assumed at right angles to those shown in Fig. 8.24. A system of imaginary beams is thus established and the load on the panel is transferred to the columns by these beams.

Consider strip $A$ and the two strips $B$ to form one wide but shallow continuous beam, rigidly supported at the four corners by the columns, and partially supported between the columns along the lines $AB$ and $CD$ by the perpendicular strips $B'$. In the resulting beam, negative moments exist along the lines $AB$ and $CD$, and positive moments along the line $EF$. Strips $B$ are obviously stiffer than strip $A$, and hence both the positive and negative moments in strips $B$ (combined) will be greater than those in strip $A$.

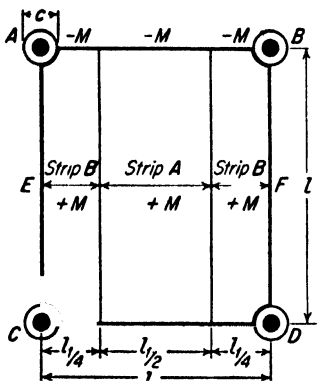



FIG. 8.24

It is a well-known fact that, in any continuous beam, the sum of the maximum positive moment in any span of the beam and the average of the negative moments at the adjacent supports are equal to the maximum moment in a corresponding simply supported beam. In a paper entitled "Statical Limitations upon the Steel Requirement in Reinforced-concrete Flat-slab Floors," John R. Nichols[1] recommends that, in flat-slab analysis, the span of the corresponding simply supported beam be taken as $l - \frac{2}{3}c$, where $l$ is the panel length center to center of columns and $c$ is the diameter of the capital. Thus, in Fig. 8.24,

$$M_A + M_{EF} = \frac{1}{8} w l_1 \left(l - \frac{2}{3} c\right)^2 = \frac{1}{8} W l \left(1 - \frac{2c}{3l}\right)^2$$

where $w$ is the dead and live load on the slab, per unit of area, $W$ is the total load on one panel, $l$ is the distance center to center of columns parallel to the strips under consideration, $l_1$ is the distance center to center of columns perpendicular to the strips under consideration, and $c$ is the diameter of the column capital.

The distribution of this total moment to the positive- and negative-moment sections $EF$ and $AB$, respectively, must be determined and then the total positive moment in the section $EF$ and the negative moment in the section $AB$ must be apportioned to the strips $A$ and $B$. H. M.

[1] *Trans. ASCE*, vol. 77.

Westergaard and W. A. Slater presented a solution of this problem of distribution in a paper entitled "Moments and Stresses in Slabs" which was published in the *Proceedings of the American Concrete Institute*, volume 17. For a square interior panel with column capitals but without drop panels, they recommend the following average percentages of the total moment ($M_{AB} + M_{AF}$) for each of the various moment sections:

| | |
|---|---|
| Column strips, negative moment | 48 per cent |
| Column strips, positive moment | 21 per cent |
| Middle strip, negative moment | 17 per cent |
| Middle strip, positive moment | 14 per cent |
| Total | 100 per cent |

The effect of a dropped panel is to stiffen the negative-moment portion of the column strip. This causes an increase in the negative moment in strip $B$ and a corresponding decrease in the positive moment at the middle of the strip.

The preceding equation can be applied to the determination of moments in square panels and in rectangular panels in which the longer side of the panel is not more than 1.33 times the shorter side. In rectangular panels, separate moments must obviously be computed for the various sections of the two rectangular directions, using for $l$ the length of the side of the panel in the direction parallel to the strip under consideration.

**8.41. Moments in Interior Panels.** Any theoretical analysis, such as the one outlined above, gives only approximate results, because of the many assumptions which are necessary. Tests of full-sized panels show that the actual stresses in the steel are less than would be obtained from the usual rectangular-beam equations with moments as indicated in Art. 8.40. By comparing the stresses determined by a sound theoretical analysis with those obtained by actual tests, moment coefficients can be obtained which will give rational and safe results. This process has resulted in the development of flat-slab regulations which are a part of practically all municipal building codes and technical society standards.

In addition to specifying moment coefficients, these codes usually give minimum requirements for slab thickness, cap and drop dimensions, and column sizes. They specify the arrangement of the reinforcement, and outline the method of computing stresses due to shear and bending. The formulas and methods of one code may differ from those of another, but there is sufficient similarity between them so that the designer who is familiar with the application of the provisions of one code has little difficulty in following those of any other code.

The code which is to govern a particular design depends largely on the location of the proposed structure. Practically every city of any size has its own flat-slab regulations which form a part of the complete building code. The provisions of these regulations must be adhered to (or

exceeded) if the design is to be approved by the building department. In the absence of a governing municipal code, any recognized code may be followed.

The recommendations of the American Concrete Institute are given in Appendix C. These recommendations provide for two methods of design of flat-slab structures. One method applies to flat-slab floors in which (*a*) there are three or more panels in each direction, (*b*) the successive span lengths differ by not more than 20 per cent of the shorter span, and (*c*) the ratio of length to width of panel does not exceed 1:33. These conditions are usually met in practice. When they do not obtain, the second method is used. This method consists of recommendations governing the analysis of the structure as a series of rigid frames in the two directions of reinforcement. Only the former method will be considered in this chapter.

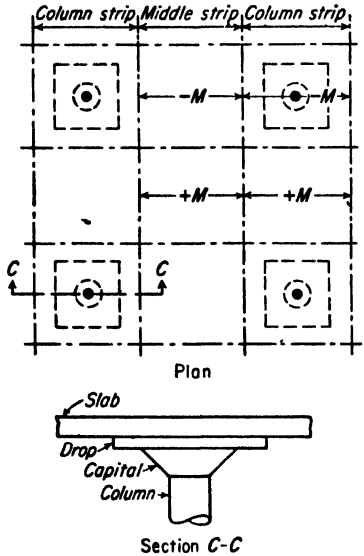



Fig. 8.25

In the ACI Code the total moment $M_{AB} + M_{EF}$ (Fig. 8.24) is specified as somewhat less than the value given in the equation on page 290, and the distribution varies slightly from the percentages tabulated in Art. 8.40. For convenience in specifying moment coefficients and to simplify the subsequent detailing of the reinforcement, it is desirable to consider two adjoining column strips in adjacent panels as one band. This band is centered about the column line and has a width equal to one-half of the panel width. In all the following discussions, any reference to the "column strip" will imply the combined strips or band described above, as indicated in Fig. 8.25.

The moment distribution for interior panels, as given in the ACI Code, is as follows:

Let $M_0 = M_{AB} + M_{EF}$ (Fig. 8.24), in ft-lb

$W$ = total load on the panel, including the weight of the drop, lb

$l$ = span of the strip under consideration, center to center of columns, ft

$c$ = diameter of the column capital, ft

$$M_0 = 0.09Wl\left(1 - \frac{2c}{3l}\right)^2$$

Moments in Interior Panels—ACI Code

| Strip | Moments in slabs without drops | | Moments in slabs with drops | |
|---|---|---|---|---|
| | Negative | Positive | Negative | Positive |
| Column strip ..... | $-0.46M_0$ | $+0.22M_0$ | $-0.50M_0$ | $+0.20M_0$ |
| Middle strip..... | $-0.16M_0$ | $+0.16M_0$ | $-0.15M_0$ | $+0.15M_0$ |

**8.42. Moments in Exterior Panels.** In wall panels, the strips which are perpendicular to the wall are analogous to the end spans of a beam which is continuous over a number of supports. In such a beam, if the spans are all equal, the load uniform, and if the ends of the beam rest freely on the end supports, the maximum positive moment in the end spans is greater than in the intermediate spans; the negative moment at the second support is greater than the negative moment at any other interior support; the moment at the first support is equal to zero. If the ends of the beam are restrained instead of being freely supported, moments are induced at the end supports, and the maximum positive moments in the end spans as well as the negative moment at the second support are also affected. The latter two values will normally be greater than the corresponding moments in a typical interior span.

In flat-slab construction, the restraint which is offered at the end supports may vary considerably. The wall panels may but seldom do rest freely on brick walls, they may be built monolithically with the wall columns and with marginal beams (wall beams) along the outer edge of the panels which are capable of resisting torsional stresses, or they may be built monolithically with the columns but without the rigid marginal beams. Obviously, the moment coefficients for sections parallel to the wall (strips perpendicular to the wall) would differ materially for these various conditions. Many flat-slab regulations do not take the condition of end restraint into consideration, but specify merely the coefficients that shall be used for the moment sections in all exterior panels. The ACI Code provides separate moment coefficients for different support conditions.

The value of $c$ which shall be used in computing $M_0$ for the strips perpendicular to the wall in exterior panels is defined in Appendix C. Since brackets are normally used at exterior columns, and since the sloping face of these brackets usually projects beyond the columns a sufficient amount to make the distance from the center of the column to the extremity of the bracket equal to one-half of the diameter of the interior-column capital, the values of $c$, and hence $M_0$, for the exterior-panel strips

perpendicular to the wall are generally the same as for an interior panel. The moments at the various sections of these strips can therefore usually be obtained by multiplying the moments at the corresponding sections in the interior panel by the percentages given in the preceding paragraph.

The moments to be used in the design of the middle strip parallel to the wall in an exterior panel are the same as for the corresponding sections in an interior panel The moments to be used in the design of the half column strip adjacent and parallel to the wall depend upon the relative size of the marginal or wall beam. If the depth of this beam is $1\frac{1}{2}$ times the thickness of the slab, or less, the moments in the half column strip, both positive and negative, are taken as one-half of the corresponding moments specified for a typical interior-panel column strip; but if the depth of the marginal beam is greater than $1\frac{1}{2}$ times the depth of the slab, the moments in the half column strip are one-quarter of the interior-panel moments. In the latter case the comparative stiffness of the marginal beam relieves the half column strip of some of its load and thus reduces the moments in this strip According to the ACI Code, the marginal beam must then be designed to carry one-quarter of the total live load and dead load on the panel, in addition to the load directly superimposed on it. Where the depth of the marginal beam is $1\frac{1}{2}$ times the slab thickness, or less, the beam need be designed only for the superimposed load (*i.e.*, the wall load).

**8.43. Thickness of Slab.** In order to prevent undue deflection, certain limitations are placed on the minimum slab thickness that can be used in a given floor panel. Inasmuch as the actual deflection of a flat slab cannot be computed with any appreciable degree of accuracy, these limitations were developed from a study of the observed deflections in actual structures. The ACI Code specifies that the slab thickness, exclusive of the drop, shall not be less than one-fortieth of the longer dimension for slabs with drops, and not less than one-thirty-sixth of the same dimension for slabs without drops. Of course the slab must also be thick enough so that allowable unit compressive and shear stresses will not be exceeded.

**8.44. Methods of Reinforcing Flat-slab Floors.** There are, in the main, four different methods or systems of reinforcing the slab in this type of floor: (1) two-way system, (2) four-way system, (3) three-way system, and (4) circumferential system.

Virtually all flat-slab floors built in this country at the present time have a two-way system of reinforcement. Originally all flat-slab systems were covered by patents. The basic patents have expired, and at present the two-way system is the one in general use. It has the advantage of being simple to design and construct and has proved thoroughly satisfactory. The other systems are chiefly of interest historically and to engineers working on renovations or additions to old buildings.

In the two-way system, bars are placed parallel to the lines of columns over the entire area of the floor at small intervals. The maximum allowable spacing varies in the different codes and specifications. The ACI Code permits a maximum bar spacing of three times the slab thickness. Most other codes are considerably more conservative in this respect.

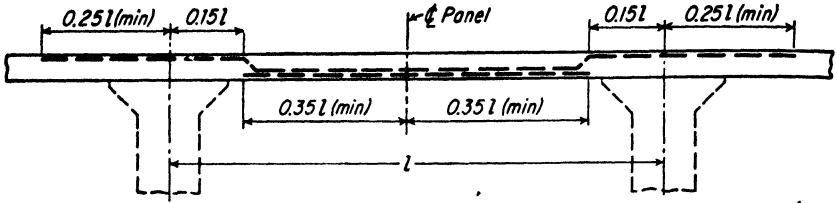
The four-way system consists of two main bands of steel running parallel to the lines of columns, each band centered about the column lines, and two diagonal bands of sufficient width to fill up the floor area left uncovered by the direct bands. In some cases short bars are placed near the top of the slab at right angles to the direct bands over the middle portion of the band to resist the negative moment over that portion; these latter bars constitute what are known as the across-direct bands.

The three-way system involves a special arrangement of columns, such that the lines connecting their center lines form a series of equilateral triangles. The reinforcement then follows the sides of these triangles, each band being centered about one of the panel sides.

In the circumferential system, radial steel emanating from the column head and circumferential steel in the form of concentric rings symmetrical about the column head are used. Concentric rings are also placed about the mid-point of the slab, and about the mid-points of the four edges of each panel.

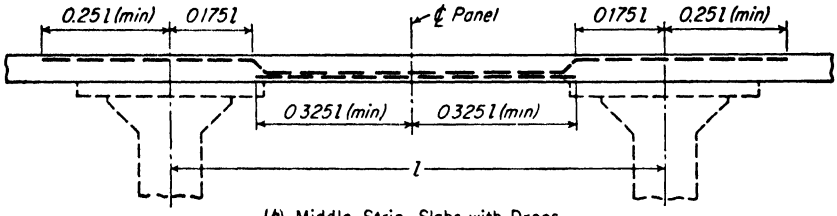
The area of steel required at each section is obtained from the equation $M_s = A_s f_s jd$, where $M_s$ is the moment specified for the particular section under consideration and $d$ is the distance from the compression face to the center of the tension steel (effective depth) at the section. Part of the positive-moment reinforcement is usually bent up to furnish part or all of the negative-moment steel in accordance with the provisions of the governing code. The current ACI Code treats reinforcement arrangement in flat slabs (Art. 8, Appendix C of this book) in general terms only. More specific recommendations were contained in the 1936 edition of the ACI Code. Since the 1936 provisions furnish sensible guides for the determination of reinforcement cutoff points, and since they are in general agreement with most current codes, they are reproduced graphically in Fig. 8.26. These requirements are for interior panels of flat-slab floors which are regular in column arrangement and loading. Good engineering judgment must be used in laying out reinforcement in the irregular panels which frequently occur in practice. The straight bottom bars shown in the figure usually comprise about one-third or more of the total steel needed for positive reinforcement at the center line of the panel. Some codes require that these bars be carried almost to the column line to provide resistance to injurious spalling of the concrete in case of fire.

**8.45. Factors to Be Considered in the Design of Flat-slab Buildings.** Flat-slab floors are ordinarily designed to carry only a uniform load over

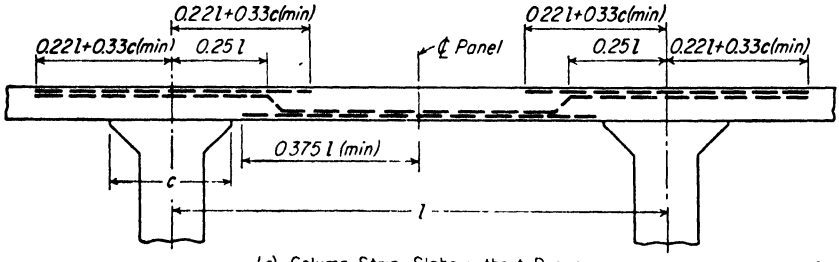



(a) Middle Strip, Slabs without Drops

Note: Not less than ½ of all bars are to be bent as shown in (a) and (b)

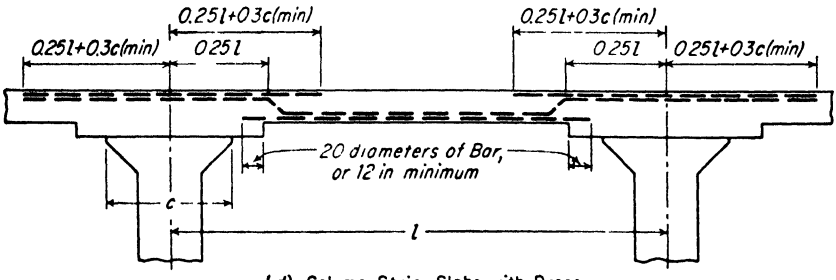



(b) Middle Strip, Slabs with Drops




(c) Column Strip, Slabs without Drops

Note: Not less than 4/10 of all bars to be bent as shown in (c) and (d)




(d) Column Strip, Slabs with Drops

FIG. 8.26. American Concrete Institute 1936 specifications for placing reinforcement in two-way flat slabs.

the entire surface, the assumption being that no breaks in the continuity occur. Where heavy concentrated loads are to be sustained in addition to the uniform load, beams should be introduced in such positions as will enable them to carry the weight of the concentrations. Where openings in the slab occur, they should be framed by beams which will have the effect of restoring continuity to the slab. These beams should be designed to carry a portion of the floor load in addition to any concentrated loads that may rest upon them.

The columns should be designed to provide for bending stresses such as might be caused by unequally loaded panels. This is especially important in the exterior columns where both the dead and live loads cause continual bending, and where the direct loads are relatively small The interior roof columns are not likely to be subjected to such eccentric loading, and the ratio of the possible bending stress to the direct load stress decreases as the number of floors to be supported increases. Hence, bending in the interior columns is not so important as in the exterior columns. It should be investigated, however, especially in the upper stories. The amount of bending moment to be assumed is usually stated in the various regulations governing flat-slab design. One method of analysis is outlined in Art. 8.52 for interior columns and in Art. 8.53 for exterior columns. The spacing of columns is governed by practically the same factors as in the case of the beam-and-girder type.

To provide proper drainage, a slight pitch may be given to the roof slab without any change in the theoretical computations. Sudden changes in slope, or steps, on the other hand, require special attention.

It should be remembered at all times that careful compliance with the building code pertaining to the place of construction is not only necessary to the acceptance of the design, but is also conducive to safety. As in all other types of construction, failures of flat-slab buildings have occurred. The causes of such failures may in most cases be traced to one or more of the following:

1. Strong commercial competition leading to the tendency to use thinner sections than good design dictates, especially in the absence of good building codes.

2. Faulty construction, such as poor mixing of concrete, inaccurate placing of the steel, too early removal of forms, or placing of concrete in freezing weather without adequate precaution.

3. Overloading of the floors beyond the load allowed in the design.

4. Faulty design, due to a lack of knowledge on the part of the designer.

The first essential of a safe and economical design is the removal of all agencies, such as are stated above, that might lead to failure or, on the other hand, to needless waste.

## DESIGN OF A FLAT-SLAB BUILDING

**8.46. Data and Specifications.** The method of design of flat-slab floors and other details involved in a reinforced concrete building are illustrated in the following articles which contain a complete design of a building 66 by 105 ft in plan, consisting of two upper stories and a basement. The height of the upper stories, floor to floor, is 12 ft-0 in. and that of the basement is 10 ft-0 in. The floor plan is shown in Fig. 8.38. The live load to be supported by the floors is 200 psf, and by the roof 40 psf. An additional load of 40 psf is to be considered in the dead weight of the roof, to provide for a cinder concrete fill and for the roof covering. Adequate drainage will be provided by inclining the exterior slabs in the short direction of the building and by varying the thickness of the surfacing over the middle panel. A 3000-lb concrete will be used for all of the construction. The ACI Code will be used in the design of the floor and roof slabs, beams, columns, and footings. Intermediate-grade steel will be used throughout.

**8.47. Design of Interior Floor Panel.** *Slab.* Article 6(*d*), Appendix C. Two-way reinforcement with dropped panel. Assume the thickness of the slab exclusive of the drop panel to be

$$t_s \geq \frac{22 \times 12}{40} \geq 6.6 \text{ in.}$$

Try a thickness $t_3$ of 8 in.

*Capital.* Metal cap forms will be used, standard diameters being multiples of 6 in. A diameter $c$ of cap of approximately 0.20 times the average span of the panel will be used.

$$0.20 \times 21.5 \times 12 = 51.6 \text{ in.}$$

Use 4 ft-6 in.

*Drop.* The length of each side of the drop panel will be made approximately equal to 0.33 times the longer span.

$$0.33 \times 22 \times 12 = 88 \text{ in.}$$

Use 7 ft-6 in.

The thickness $t_2$ of the drop panel is controlled by Art. 6(*e*), Appendix C.

$$(t_2 - t_3) \leq \frac{1}{4}\left(\frac{7.50}{2} - 4.50\right) \times 12 = 4\tfrac{1}{2} \text{ in.}$$

Try a thickness $t_2$ of 12 in.

*Shearing Stresses.* Article 6(*b*), Appendix C. The unit shearing stress ($v = V/\frac{7}{8}bd$) on a vertical section $t_2 - 1\frac{1}{2}$ in. from the edge of the capital, and concentric with it, shall not exceed $0.025f'_c = 75$ psi when less than 25 per cent of the total negative reinforcement in the column strip passes directly over the column capital.

$$t_2 - 1\tfrac{1}{2} = 10\tfrac{1}{2} \text{ in.}$$
$$4 \text{ ft-6 in.} + 2 \times 10\tfrac{1}{2} \text{ in.} = 6 \text{ ft-3 in.} = 6.25 \text{ ft}$$

The total load on the panel $= 300 \times 21 \times 22 + \frac{4}{12} \times .150(7.50)^2$
$= 141{,}300$ lb.

At the section described above,

$$V = 141{,}300 - 350 \times \frac{\pi \times (6.25)^2}{4} = 130{,}600 \text{ lb}$$

$$v = \frac{130{,}600}{\frac{7}{8} \times (\pi \times 6.25 \times 12) \times 10.5} = 61 \text{ psi}$$

The unit shearing stress on a vertical section $t_3 - 1\frac{1}{2}$ in. from the edge of the dropped panel, and parallel with it, shall not exceed $0.03f'_c = 90$ psi.

$$t_3 - 1\tfrac{1}{2} = 6\tfrac{1}{2} \text{ in.}$$
$$7 \text{ ft-6 in.} + 2 \times 6\tfrac{1}{2} \text{ in.} = 8 \text{ ft-7 in.} = 8.58 \text{ ft}$$
$$V = [(21 \times 22) - (8.58)^2] \times 300 = 116{,}400 \text{ lb}$$
$$v = \frac{116{,}400}{\frac{7}{8} \times (4 \times 8.58 \times 12) \times 6.5} = 50 \text{ psi}$$

The assumed dimensions of the capital and drop need no revision.

*Bending Moments.* Table 4(*a*), Appendix C. Since the panel is nearly square, moments and steel areas will be computed for the long direction and the same steel will be placed in the short direction. $W = 141{,}300$ lb; $l = 22.0$ ft; $c = 4.5$ ft.

$$M_0 = 0.09Wl\left(1 - \frac{2c}{3l}\right)^2 = 0.09 \times 141{,}300 \times 22.0\left(1 - \frac{2 \times 4.5}{3 \times 22.0}\right)^2$$
$$= 209{,}000 \text{ ft-lb} = 2{,}510{,}000 \text{ in.-lb}$$

Column strip, positive moment: $M = +0.20 \times 2{,}510{,}000$
$= +502{,}000$ in.-lb

Column strip, negative moment: $M = -0.50 \times 2{,}510{,}000$
$= -1{,}256{,}000$ in.-lb

Middle strip, positive moment: $M = +0.15 \times 2{,}510{,}000$
$= +376{,}000$ in.-lb

Middle strip, negative moment: $M = -0.15 \times 2{,}510{,}000$
$= -376{,}000$ in.-lb

*Steel Areas.* In computing steel areas required for those moment sections in which bars are placed in two directions (*i.e.*, the positive-moment section of the middle strip and the negative-moment section of the column strip) the governing row will be assumed as that row farthest from the tension face of the slab. This removes all restrictions as to the order of placing the steel in the two directions. Values of $j$ are obtained from Table 5, Appendix D. $f_s = 20{,}000$; $f_c = 0.45 \times 3000 = 1350$ psi;

$n = 10$. Equation (3.5) is used to compute the steel areas required at each of the moment sections. Bars which are parallel to a section (*i.e.*, perpendicular to the strip under consideration) obviously cannot be considered effective at that section.

COLUMN-STRIP, POSITIVE-MOMENT SECTION. Assuming No. 5 bars and $\frac{3}{4}$-in. clear insulation, $d = 8 - 1\frac{1}{16} = 6\frac{15}{16}$ in.

$$A_s = \frac{502{,}000}{20{,}000 \times 0.866 \times 6.93} = 4.19 \text{ sq in.}$$

Fourteen No. 5 bars furnish 4.30 sq in.

COLUMN-STRIP, NEGATIVE-MOMENT SECTION. Assuming No. 5 bars and $\frac{3}{4}$-in. clear insulation, $d = 12 - 1\frac{11}{16} = 10\frac{5}{16}$ in.

$$A_s = \frac{1{,}256{,}000}{20{,}000 \times 0.866 \times 10.31} = 7.02 \text{ sq in.}$$

Twenty-three No. 5 bars furnish 7.06 sq in.

Nine bars, or slightly less than two-thirds of the total required for positive moment, will be bent up to provide resistance to negative moment.

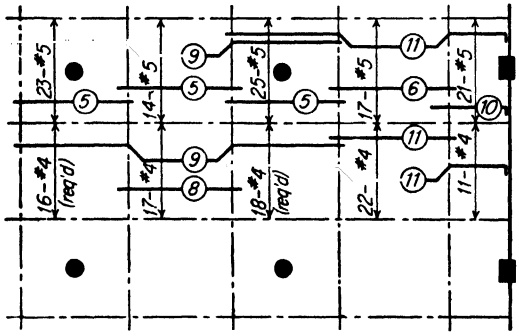



FIG. 8.27. Arrangement of reinforcement in floor slab.

Hence at each negative-moment section a total of 18 bars is furnished from the two adjacent positive areas. Five straight bars in the top of the slab over the column head are sufficient to complete the steel area required at the negative moment section (see Fig. 8.27).

MIDDLE-STRIP, POSITIVE-MOMENT SECTION. Assuming No. 4 bars and $\frac{3}{4}$-in. clear insulation, $d = 8 - 1\frac{1}{2} = 6\frac{1}{2}$ in.

$$A_s = \frac{376{,}000}{20{,}000 \times 0.866 \times 6.5} = 3.34 \text{ sq in.}$$

Seventeen No. 4 bars furnish 3.33 sq in.

MIDDLE-STRIP, NEGATIVE-MOMENT SECTION. Assuming No. 4 bars and $\frac{3}{4}$-in. clear insulation, $d = 8 - 1 = 7$ in.

$$A_s = \frac{376{,}000}{20{,}000 \times 0.866 \times 7} = 3.10 \text{ sq in.}$$

According to Fig. 8.26 not less than five-tenths of the steel required in the positive-moment section of the middle strip should be bent up at both ends so as to reinforce the two adjacent negative-moment sections. Bending 9 of the 17 bars required in the positive-moment section provides a total of 18 bars in each negative-moment section, which is more than ample in the present case.

*Fiber Stress in Concrete.* The critical section is in the negative-moment portion of the column strip. According to Art. 3(*c*), Appendix C, the effective width of the column-head section, for compression, shall be taken as three-fourths of the width of the dropped panel, 67.5 in.

$$p = \frac{7.06}{67.5 \times 10.31} = 0.0102$$

From Table 6, Appendix D, $k = 0.360$ and $j = 0.880$.

$$f_c = \frac{2 \times 1{,}256{,}000}{0.360 \times 0.880 \times 67.5 \times (10.31)^2} = 1100 \text{ psi}$$

The allowable stress $= 0.45 \times 3000 = 1350$ psi.

**8.48. Design of Exterior Floor Panel.** A comparison of Tables 4(*a*) and 4(*b*), Appendix C, shows that the column-strip negative moment at the wall is 90 per cent of that specified for the interior panel; the column-strip positive moment in the exterior panel and the middle-strip positive moment in that panel are each 125 per cent of the corresponding interior-panel moments; the middle-strip negative moment at the wall is $66\frac{2}{3}$ per cent of that specified for the interior panel; and both the column- and middle-strip negative moments at the first interior support are 110 per cent of the corresponding moments in interior panels. Hence, the steel areas required at the various sections in the exterior panel (for bands perpendicular to the wall) may be obtained by multiplying the corresponding interior-panel required areas by the above percentages, as follows:

Middle-strip, Positive-moment Section:

$A_s = 1.25 \times 3.34 = 4.17$ sq in.
Twenty-two No. 4 bars furnish 4.32 sq in.

Middle-strip, Negative-moment Section at Wall:

$A_s = 0.667 \times 3.10 = 2.07$ sq in.
Eleven No. 4 bars furnish 2.16 sq in.

Middle-strip, Negative-moment Section at First Interior Support:

$A_s = 1.10 \times 3.10 = 3.41$ sq in.
Eighteen No. 4 bars furnish 3.54 sq in.

Column-strip, Positive-moment Section:

$$A_s = 1.25 \times 4.19 = 5.24 \text{ sq in.}$$

Seventeen No. 5 bars furnish 5.22 sq in.

Column-strip, Negative-moment Section at Wall:

$$A_s = 0.9 \times 7.02 = 6.32 \text{ sq in.}$$

Twenty-one No. 5 bars furnish 6.45 sq in.

Column-strip, Negative-moment Section at First Interior Support:

$$A_s = 1.10 \times 7.02 = 7.72 \text{ sq in.}$$

Twenty-five No. 5 bars furnish 7.67 sq in.

**8.49. Arrangement of Reinforcement.** The proposed method of placing and bending the steel so as to furnish the necessary areas at the various sections in both the interior and exterior panels is shown diagrammatically in Fig. 8.27. The points at which the bars are bent are obtained from Fig. 8.26. The minimum spacing is governed by the necessity of allowing space for the concrete to be deposited conveniently and effectively. A minimum spacing of about 3 in. center to center of bars should be maintained if possible. Bars less than $\frac{1}{2}$ in. in diameter are difficult to handle because of their lack of stiffness, and bars greater than $\frac{3}{4}$ in. in diameter cannot be bent in place.

A complete engineering drawing for the floor slab is shown in Fig. 8.38. This drawing follows the recommendations in the American Concrete Institute of Standard Practice for Detailing Reinforced Concrete Structures (ACI 315-51).

**8.50. Design of Roof Slab.** The design of the interior and exterior panels of the roof slab is carried out in a manner similar to that used in the design of the floor slab. The total load on the roof includes the live load (40 psf), the weight of the roofing material (40 psf), and the dead weight of the slab itself. The thickness of slab beyond the drop is $6\frac{1}{2}$ in. and the total thickness through the dropped panel is 10 in. The diameter of the capital is 4 ft-6 in., and the drop is 7 ft-6 in. square in plan. The unit shearing stress on a vertical section at a distance $t_2 - 1\frac{1}{2}$ in. from the capital is 43 psi, and the corresponding stress at a distance $t_3 - 1\frac{1}{2}$ in. from the edge of the dropped panel is 36 psi. At the column-head section the fiber stress in the concrete is 950 psi.

The number of $\frac{1}{2}$-in. round bars required in the various sections of the slab and the proposed method of placing and bending the bars so as to furnish the required steel areas at all sections are shown diagrammatically in Fig. 8.28. It has been assumed, for simplicity, that the stairway and elevator shaft openings do not extend through the roof.

**8.51. Design of Interior Columns.** The interior columns are to be made of 3000 lb concrete, with reinforcement of intermediate-grade steel.

They are to be round, with spirals, and designed in accordance with the ACI specifications. The fundamental principles involved in the design are explained in Art. 4.2. In flat-slab construction the columns are an important factor in adding to the rigidity of the slab, and many codes specify a minimum column size greater than the usual minimum for beam-and-girder floors. For columns supporting heavily loaded floors, a minimum over-all dimension of one-fifteenth of the average span of the panel is considered satisfactory. Roof columns may be somewhat lighter; one-eighteenth of the average span is specified by some codes as a minimum diameter. In the present case $\frac{1}{15} \times 21.5 \times 12 = 17.2$, or 18 in., will be taken as the minimum size for floor columns and 16 in. for columns supporting the roof.

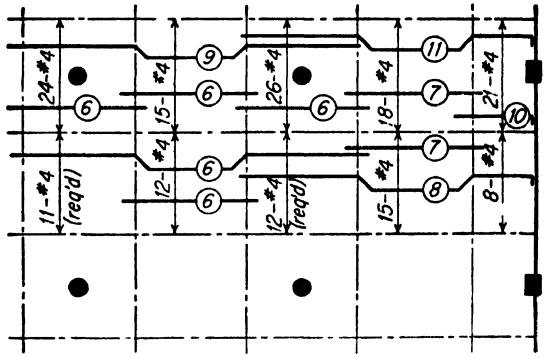



FIG. 8.28. Arrangement of reinforcement in roof slab.

According to the ACI Code, values of $p_g$ may vary from 0.01 to 0.08 for columns with spirals, but values greater than 0.02 to 0.025 are normally inadvisable because of the crowding of the steel which occurs just above the floor line where the bars are spliced by lapping. In general, greater over-all economy is obtained by the use of the smaller values of $p_g$. Steel ratios approaching the maximum may be necessary, however, in heavily loaded columns, in order to keep the size of the columns within reasonable limits. In some cases architectural limitations may require a smaller column with the resulting greater steel ratio. Metal column forms, which are generally used for round columns, are available in multiples of 2 in., and column sizes should be selected accordingly.

Table 8.4 gives a summary of the loads (to the nearest thousand pounds) which are carried in each tier of columns. The weights of the columns were taken from Table 9, Appendix D. The sizes of the columns and the reinforcement required to support these loads were obtained from Table 7,and spiral details from Table 10. The excess strength of the top column cannot be avoided, since the minimum size columns and the minimum percentage of reinforcement have been used. The excess

strength of the intermediate column is required to provide for bending stresses, as shown by the investigation in Art. 8.52.

The bars in the basement and first-floor columns are extended 1 ft-6 in. above the first- and second-floor levels, respectively, in order to lap with the bars in the columns next above. A lap of 20 diameters is required by the ACI Code when $f'_c$ is 3000 psi. In order to prevent too sharp a bend in the bars at the top of any column in getting them within the area of the column next above, wherever possible the diameters of two successive tiers of columns should not vary by more than about 6 in. The bend in each bar is usually made in a height of about 18 in., and the maximum slope of the bend is then 3 in. in 18 in., which is the maximum permitted by the Code.

**TABLE 8.4. Interior Columns, Design for Direct Load Only**

| Column | Load from | Amount of load, lb | Diameter and area of column | Vertical bars | Spirals (Table 10) | Load carried by concrete, lb | Load carried by steel, lb | Total load, lb |
|---|---|---|---|---|---|---|---|---|
| Top | Roof | 76,000 | 16 in | 5 #6 | | | | |
| | Column | 2,000 | 201 sq in | 2.21 sq in | No. 3-1¾ in. | 136,000 | 35,000 | 171,000 |
| | Capital | 2,000 | | $p_g$ 0.0110 | | | | |
| | Total | 80,000 | | | | | | |
| Intermediate | Floor | 141,000 | 20 in | 9 #6 | | | | |
| | Column | 3,000 | 314 sq in | 3.98 sq in | No. 3-1¾ in. | 212,000 | 64,000 | 276,000 |
| | Capital | 2,000 | | $p_g$ 0.0127 | | | | |
| | Total | 226,000 | | | | | | |
| Basement | Floor | 141,000 | 24 in | 10-#7 | | | | |
| | Column | 4,000 | 452 sq in | 6.01 sq in | No. 3 2 in | 305,000 | 96,000 | 401,000 |
| | Capital | 2,000 | | $p_g$ = 0.0133 | | | | |
| | Total | 373,000 | | | | | | |

**8.52. Investigation for Bending Stresses.** Bending stresses due to unequally loaded panels are not apt to affect the design of the interior columns, except possibly in the upper tiers. This is because the bending is usually caused by the live load only, and thus the maximum unit stress due to bending is in most cases less than the allowable *increase* in fiber stress as compared with the stress permitted when no bending is considered.

The method of investigation for bending is explained in Sec. 5; the average stress due to direct load ($N/A_t$) is added to the extreme fiber stress due to bending ($Mc/I$) in order to obtain the maximum combined stress $f_c$. The allowable unit combined stress as specified in the ACI Code for both spiral and tied columns is given in the equation

$$f_c = f_a\left(\frac{t + De}{t + CDe}\right)$$

where $f_a$ = average permissible stress on an equivalent axially loaded plain concrete column (*i.e.*, based on the area of the transformed section of the actual reinforced concrete column), or

$f_a = \dfrac{0.225f'_c + f_s p_g}{1 + (n - 1)p_g}$ for spiral columns and 0.8 of this value for tied columns

$e$ = eccentricity of the resultant load $N$ on the column ($e = M/N$)

$D = t^2/2R^2$ = a factor, usually varying from 3 to 9

$t$ = over-all depth of column section

$R$ = least radius of gyration of the column section

$C$ = ratio of $f_a$ to the permissible fiber stress for members in flexure ($C = f_a/0.45f'_c$)

Values of $f_c$ may be taken directly from Diagrams 9 to 12, Appendix D, for spiral columns and from Diagrams 13 and 14 for tied columns, these diagrams being based on the preceding equation.

With unequally loaded panels, the amount of bending moment transferred to the columns depends upon the relative stiffness of the slab on both sides of the columns and of that of the columns themselves. It is recommended that a moment[1] of $\frac{1}{10}W_1 l$ shall be divided between the columns immediately above and below any floor, in direct proportion to their stiffness factors (values of $I/h$), where $W_1$ = the total live load on one panel, $l$ = the average span of the panel, $I$ = the moment of inertia of the column, and $h$ = its unsupported height.

In flat-slab construction, the unsupported height of a column is equal to the distance from the floor to the bottom of the capital. The moment

[1] The recommended moment of $\frac{1}{10}W_1 l$ is based on the following analysis: Due to live load only, the negative moment in a column strip, for two-way slabs with drops and with $c = 0.225l$ (a representative value), is $0.325W_1 l$. This unbalanced moment (assuming live load to be placed only on one side of the columns under consideration) is resisted by the slab on the unloaded side and the columns above and below the floor, more or less in direct proportion to the stiffness factors ($I/h$) of these members. Assuming that the stiffness factor of the slab is one-third of the sum of the stiffness factors of the two columns, the moment to be resisted by the two columns is

$$\tfrac{3}{4} \times 0.0325W_1 l = 0.0244W_1 l$$

or approximately $\frac{1}{40}W_1 l$.

In the investigation of the exterior columns (see Art. 8.53), since there is no slab beyond the wall the entire negative moment in the column strip at the wall is resisted by the two columns. For $c = 0.225l$ this moment is $0.029Wl$, or approximately, $\frac{1}{35}Wl$. Here the full load (dead plus live) on the panel is used, because both the dead and live loads are unbalanced loads. The total moment of $\frac{1}{35}Wl$ is distributed to the columns in direct proportion to the stiffness factors.

*of inertia of the column is equal to the sum of the moment of inertia of* the over-all cross section of the column and $(n - 1)$ times that of the longitudinal steel. The longitudinal steel is placed directly inside the spirals, making the diameter of the circle on which the steel is placed equal to the diameter out to out of spirals, less two diameters of spiral and one diameter of the longitudinal bars. With $1\frac{1}{2}$ in. insulation to the spirals, as required by the ACI Code, this steel-circle diameter will vary from $4\frac{1}{4}$ to $5\frac{1}{4}$ in. less than the over-all diameter of the column. The moment of inertia of the steel may be taken from Table 12, which is based on a steel-circle diameter 5 in. less than the over-all column diameter.

The maximum combined stress occurs only when the panels at the floor line under consideration are unequally loaded, with a full live load on alternate panels and no load on the remaining panels. A full live load is assumed on all floors above the one considered. In the investigation for the combined stress at the top of any column, the total direct load is equal to the load in the preceding table (the full load) minus the weight of the column and minus one-half of a live panel load.

**TABLE 8.5. Interior Columns. Computation of Moments of Inertia and Stiffness Factors**

| Column | $I_g$ (Table 9), in.$^4$ | $(n-1)I_s$ (Table 12), in.$^4$ | $I = I_g + (n-1)I_s$, in.$^4$ | $h$, in. | $\frac{I}{h}$ |
|---|---|---|---|---|---|
| Top | 3,217 | 302 | 3,519 | 113 | 31 1 |
| Intermediate | 7,854 | 950 | 8,804 | 113 | 77 9 |
| Basement | 16,286 | 2,440 | 18,726 | 91 | 206 0 |

Tables 8.5 and 8.6 contain a summary of computations which are necessary to determine the unit stresses due to bending and direct stress in the interior columns. Diagrams 15 to 17 would give essentially the same results.

For the roof,

$$M = \tfrac{1}{40} \times 40 \times 21 \times 22 \times 21.5 \times 12 = 119{,}000 \text{ in.-lb}$$

For the other floors,

$$M = \tfrac{1}{40} \times 200 \times 21 \times 22 \times 21.5 \times 12 = 596{,}000 \text{ in.-lb}$$

In the present case, the investigation for bending and direct stress shows that no revision of the original columns, as designed for direct load only, is necessary. It should be noted, however, that in the original selec-

tion, some excess strength was intentionally provided in the intermediate column. The amount there provided is shown to be more than is necessary for bending; but the minimum percentage of steel, approximately, has been used, and subsequent investigation indicated that an 18-in. column (the next smaller standard size) would require more than 3 per cent of steel in order to keep the combined stress within the allowable limits. The design as shown in Tables 8.5 and 8.6 is therefore satisfactory.

**TABLE 8.6. Interior Columns. Computation of Stresses Due to Bending and Direct Load**

| Column | Point | $A_g$, sq in. | $N$ (effective), lb | $f_c$ (direct), psi | $M$, in.-lb | $f_c$ (bending), psi | $f_c$ (total), psi | $e$ / $t$ | $f_c$ (allowable), psi |
|---|---|---|---|---|---|---|---|---|---|
| Top | Top | 221 | 69,000 | 312 | 119,000 | 270 | 582 | 0.108 | 960 |
| | Bottom | | 80,000 | 362 | 170,000 | 387 | 749 | 0.133 | 990 |
| Intermediate | Top | 350 | 177,000 | 506 | 426,000 | 185 | 991 | 0.120 | 990 |
| | Bottom | | 226,000 | 646 | 165,000 | 187 | 831 | 0.036 | 850 |
| Basement | Top | 504 | 323,000 | 640 | 430,000 | 276 | 916 | 0.055 | 910 |
| | Bottom | | 373,000 | 740 | 0 | 0 | 740 | 0 | 770 |

**8.53. Design of Exterior Columns.** The exterior columns are to be made of 3000-lb concrete, with longitudinal bars of intermediate-grade steel. They are to be rectangular in section, and the longitudinal bars are to be tied together by means of ties spaced 12 in. on centers, the diameter of tie being $\frac{3}{8}$ in. for the upper two tiers and $\frac{1}{2}$ in. for the lower tier, in accordance with the rule suggested in Art. 4.3.

In addition to the load from the floors, the exterior columns must support the weight of the walls enclosing the story next above. In estimating the weight of the enclosure walls, the wall beams and brick spandrels underneath the windows are assumed 12 in. thick, the spandrel 2 ft-6 in. deep, and the wall beam 2 ft-0 in. deep. The brick parapet wall at the roof is assumed 12 in. thick and 3 ft-6 in. deep. The weight of windows, including sash, is taken as 8 psf. The weight of the brick masonry is assumed as 120 pcf. The general arrangement of a typical wall panel is shown in Fig. 8.29.

Bending stresses should always be considered in the design of the exterior columns, especially in the upper tiers. The direct loads on these columns are comparatively small and the bending moments due to unsym-

metrical loading are large since both live load and dead load act together in causing these moments. According to footnote 1, page 305, the wall columns in flat-slab construction shall be designed to resist a bending moment from the slab of $\frac{1}{35}Wl$, where $W$ is the total load (dead and live) on one panel and $l$ is the average span of the panel. In the present design, at the first- and second-floor levels, $W$ = 141,300 lb, $l$ = 21.5 ft, and

$$M = \tfrac{1}{35}(141{,}300 \times 21.5 \times 12) = 1{,}042{,}000 \text{ in.-lb}$$

At the roof level, $W$ = 76,300 lb, and

$$M = \tfrac{1}{35}(76{,}300 \times 21.5 \times 12) = 565{,}000 \text{ in.-lb}$$

Countermoments due to the weight of the structure that projects beyond the column center lines and countermoments due to the eccentricity of one column with respect to the column beneath may be deducted from the value of $\frac{1}{35}Wl$, and the resulting reduced moment is then divided between the two columns immediately above and below a given floor line in proportion to the stiffness factors ($I/h$) of these columns.

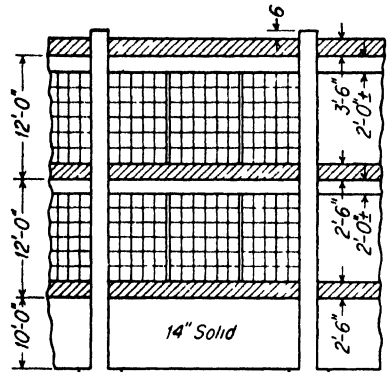



Fig 8.29

Inasmuch as bending stresses constitute a large proportion of the total stresses in exterior columns, particularly in the upper tiers, it is of no use to design the columns first for direct load only. The general procedure is to assume column sizes and reinforcement, compute the combined extreme fiber stresses, including bending, and then revise the assumed sizes if necessary. Where the concrete is to be left exposed on the exterior face, the dimension along that face should be the same for all tiers of columns and of sufficient amount to give a satisfactory architectural appearance to the face. If this dimension is too small, the building will appear to be unstable or insecure. If it is too large, the building will appear squatty and the design will be needlessly uneconomical. An elevation drawn to scale, as in Fig. 8.35, will show whether or not an assumed face dimension is satisfactory. Better architectural appearance is obtained by allowing the columns to project 2 in. or more beyond the face of the wall. In the present case, a column width of 22 in. is selected. The minimum required dimension of 18 in. for floor columns and 16 in. for roof columns (see Art. 8.51) will be maintained in the direction perpendicular to the wall. For axially loaded tied columns, steel

ratios may vary from 0.01 to 0.04. For tied columns designed to withstand combined bending and axial stress, the steel ratio may vary from 0.01 to 0.08, provided that the amount of steel spliced by lapping shall not exceed a steel ratio of 0.04 in any 3 ft. length of column. As stated before, the smaller ratios of reinforcement usually result in more economical designs. Where it is desirable to keep the column sizes small, larger percentages may be desirable. The column dimensions and steel areas ultimately selected for trial are shown in Table 8.7.

TABLE 8.7. Exterior Columns. Computation of Loads

| Column | Load from | Amount of load, lb | Size of column, in | Vertical steel | $A_g$, sq in | $g$ |
|---|---|---|---|---|---|---|
| Top | Roof | 38,000 | 22 × 16 | 6-#7 | 385 | 0.688 |
| | Parapet* | 14,000 | $A_g$ = 352 sq in | $A_s$ = 3.61 sq in | | |
| | Column† | 6,000 | | $p_g$ = 0.01025 | | |
| | Total | 58,000 | | | | |
| Intermediate | Floor | 71,000 | | | | |
| | Wall | 12,000 | 22 × 18 | 6-#8 | 438 | 0.721 |
| | Column | 5,000 | $A_g$ = 396 sq in | $A_s$ = 4.71 sq in | | |
| | Total | 146,000 | | $p_g$ = 0.0119 | | |
| Basement | Floor | 71,000 | | | | |
| | Wall | 12,000 | 22 × 20 | 8-#8 | 497 | 0.750 |
| | Column | 1,000 | $A_g$ = 440 sq in. | $A_s$ = 6.28 sq in | | |
| | Total | 233,000 | | $p_g$ = 0.0143 | | |

* This value includes the weight of the stem of the wall beam

† This value includes the weight of the column above the roof (see Fig. 8.29). The weight of the bracket in each tier has been neglected, since it varies only from about 300 lb for the basement column to 500 lb for the top column.

The countermoment at the top of the intermediate column is equal to the product of the wall load and the distance from the center of the wall beam to the center of the column plus the moment due to column eccentricity, which is equal to the load on the top column multiplied by the distance between column centers. In the design under consideration, the wall beams are set back 2 in. from the faces of the columns, as shown in Fig. 8.30. At the top of the basement columns the countermoment due to column eccentricity is equal to the load on the intermediate column multiplied by the distance between the column centers, which is $\frac{1}{2}(20 - 18) = 1$ in. In this design the countermoment at the second-floor level is 70,000 in.-lb and that at the first-floor level is 170,000 in.-lb.

Tables 8.7, 8.8, and 8.9 contain a summary of the design. For all

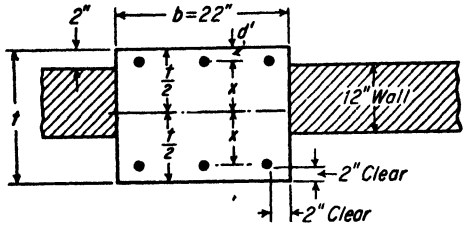



FIG 8.30

**TABLE 8.8. Exterior Columns. Computation of Moment of Inertia and Stiffness Factors**

| Column | $I_c = \frac{bt^3}{12}$, in.$^4$ | $(n-1)I_s = 9(A_s r^2)$, in.$^4$ | $I = I_c + (n-1)I_s$, in.$^4$ | $h$, in. | $\frac{I}{h}$ |
|---|---|---|---|---|---|
| Top | 7,500 | 980 | 8,480 | 113 | 75 1 |
| Intermediate | 10,700 | 1,790 | 12,490 | 112 | 110 3 |
| Basement | 14,670 | 2,390* | 17,060 | 91 | 187 5 |

* Here $A_s$ is the area of six bars, since the $I$ of the other two bars is zero.

**TABLE 8.9. Exterior Columns. Computation of Stresses Due to Bending and Direct Load**

| Column | Point | $A_t$, sq in | $N$, lb | $f_c$ (direct), psi | $M$ column, in.-lb | $f_c$ (bending), psi | $f_c$ (total), psi | $\frac{e}{t}$ | $f_c$ (allowable), psi |
|---|---|---|---|---|---|---|---|---|---|
| Top | Top | 385 | 51,000 | 140 | 565,000 | 533 | 673 | 0 655 | 1,090 |
| | Bottom | | 58,000 | 151 | 394,000* | 372 | 523 | 0 424 | 1,010 |
| Intermediate | Top | 438 | 141,000 | 322 | 578,000 | 416 | 738 | 0 228 | 900 |
| | Bottom | | 146,000 | 334 | 323,000 | 233 | 567 | 0 123 | 810 |
| Basement | Top | 497 | 229,000 | 460 | 549,000 | 322 | 782 | 0 120 | 810 |
| | Bottom | | 233,000 | 469 | 0 | 0 | 469 | 0 | 650 |

* Moment from slab − countermoment = 1,042,000 − 70,000 = 972,000 in.-lb.

$$394{,}000 = \frac{75.1}{75.1 + 110.3} \times 972{,}000$$

critical sections, the ratio $c/t$ is less than 1. Therefore, according to the ACI Code, the section may be considered uncracked (see Art. 5.3) and the stresses computed in the same manner as those in the interior columns.

The allowable stress, where bending is included, is computed from the same formula as for the interior columns (Art. 8.52), except that the value of $f_a$ is 0.8 of the value used for the interior columns, in accordance with the specifications of the ACI Code as explained in Art. 4.3. Actually, the allowable stresses were taken from Diagram 13. The apparent excess strength of the top and intermediate columns cannot be avoided,

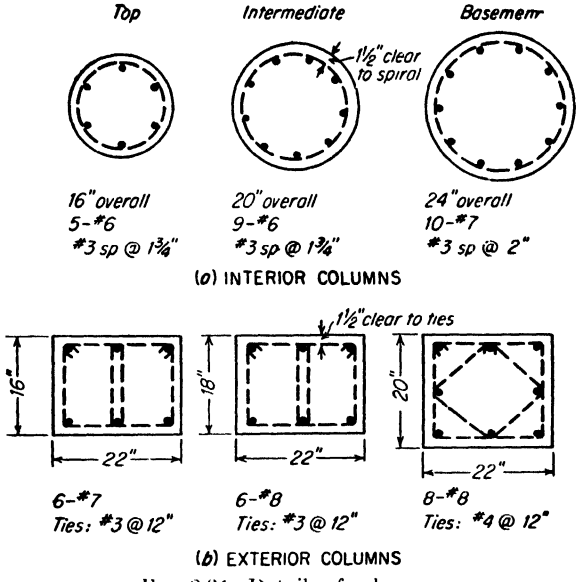



Fig. 8.31. Details of columns.

inasmuch as the minimum column size, and practically the minimum steel ratio, have been used.

The complete column details are shown in Fig. 8.31.

**8.54. Design of Interior-column Footings.** The interior-column footings are to be square, and reinforced in two directions. They will be made of 3000-lb concrete, with intermediate-grade steel. The allowable soil pressure is 2 tons per sq ft.

The interior basement column is a 24-in. round column, with a total load of 373,000 lb. According to Art. 7.10, in the design of the footing the round column must be replaced with a square column of the same area. The equivalent square has a side dimension equal to

$$\sqrt{452} = 21.3 \text{ in.} = 1.78 \text{ ft}$$

The footing design from this point on is similar to the one in Art. 7.13. Only the essential computations are shown here.

The bearing area required is

$$\frac{373{,}000 + 28{,}000}{4000} = 100.2 \text{ sq ft}$$

A base 10 ft-0 in. square is selected. The net upward pressure is

$$\frac{373{,}000}{100} = 3730 \text{ psf}$$

$$M = 3730 \times 10.00 \times 49.3/12 \times 24.6 = 3{,}780{,}000 \text{ in.-lb}$$

$$d = \sqrt{\frac{3{,}780{,}000}{10.00 \times 12 \times 235}} = 11.6 \text{ in.}$$

Assume that, for shear, an effective depth of 18 in. will be adequate. The width of the critical section for shear is then

$$21.3 + 2 \times 18 = 57.3 \text{ in.}$$

and

$$V = \frac{4.78 + 10.00}{2} \times 2.61 \times 3730 = 71{,}900 \text{ lb}$$

The allowable unit shear is 75 psi; hence,

$$75 = \frac{71{,}900}{57.3 \times 0.9 \times d}$$

Therefore,

$$d = 18.6 \text{ in.}$$

The assumed value of 18 in. may be considered satisfactory, in view of the many assumptions that are involved in the computations. The total thickness of the footing is then 22 in., and the weight 27,500 lb, which agrees with the assumed weight

$$A_s = \frac{0.85 \times 3{,}780{,}000}{20{,}000 \times 0.9 \times 18} = 9.9 \text{ sq in.}$$

The allowable unit bond stress is $0.08 \times 3000 = 240$ psi, and the maximum shear to be used in the equation for unit bond stress is

$$0.85 \times 3730 \times 10.00 \times 4.11 = 130{,}000 \text{ lb}$$

$$\Sigma_0 \text{ (required)} = \frac{130{,}000}{240 \times 0.9 \times 18} = 33.5 \text{ in.}$$

Thirteen No. 8 bars are selected; $A_s = 10.21$ sq in., and $\Sigma_0 = 40.9$ in. The spacing of the bars is about $9\frac{1}{4}$ in., which is satisfactory.

Ten No. 7 dowels will be placed in the footing. The dowels should extend into the columns 20 bar diameters or, for No. 7 bars, approxi-

mately 18 in. The embedment in the footing should be sufficient to develop the full working value of the dowel at the allowable stress in bond. Therefore

$$l \text{ (required)} \times \pi \times \frac{7}{8} \times 300 = \frac{\pi}{4} \times \left(\frac{7}{8}\right)^2 \times 16{,}000$$

$$l \text{ (required)} = 11.7 \text{ in.}$$

This length can be accommodated within the footing; therefore no pedestal is required. A total length of 30 in. will be used. The bearing

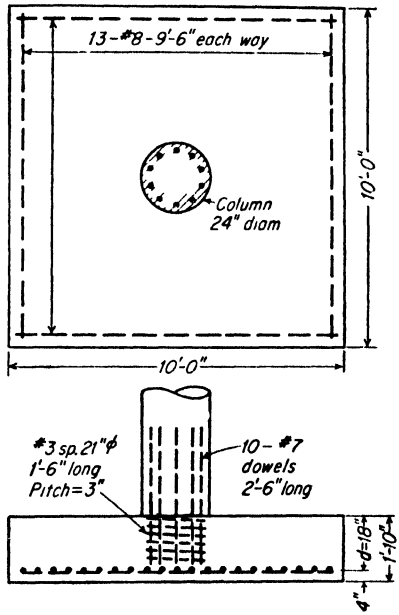



FIG. 8.32. Details of interior-column footing.

stress 373,000/452 = 825 psi is less than the allowable value of 1125 psi. The complete details of the footing are shown in Fig. 8.32.

**8.55. Design of Exterior-column Footings.** Each exterior-column footing will consist of a solid block, rectangular in plan and reinforced in two directions. A 3000-lb concrete will be used, with intermediate-grade reinforcing steel. The basement column is 22 by 20 in., and the total load on this column is 233,000 lb.

The bearing area required is

$$\frac{233{,}000 + 15{,}000}{4000} = 62 \text{ sq ft}$$

A base 7 ft-6 in. by 8 ft-6 in. (area = 63.8 sq ft) is selected. The net upward pressure is 233,000/63.8 = 3650 psf.

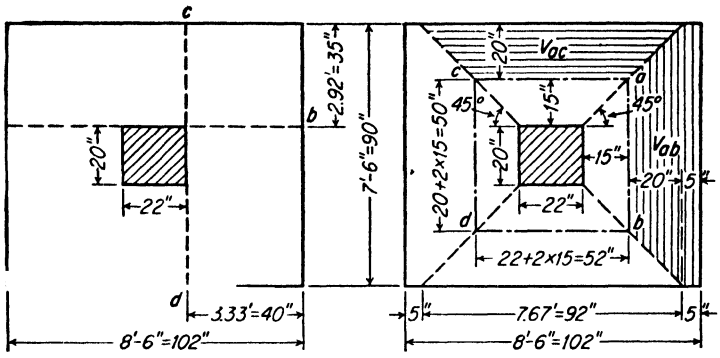



FIG. 8.33

In the short direction (see Fig. 8.33),

$$M_{ab} = 3650 \times 8.5 \times {}^{35}\!/_{12} \times 17.5 = 1{,}590{,}000 \text{ in.-lb}$$

In the long direction,

$$M_{cd} = 3650 \times 7.5 \times {}^{40}\!/_{12} \times 20 = 1{,}830{,}000 \text{ in.-lb}$$

$$d = \sqrt{\frac{1{,}830{,}000}{90 \times 235}} = 9.3 \text{ in.}$$

Assume that, for shear, an effective depth of 15 in. will be adequate. The width of the critical section $ab$ (Fig. 8.33) is then

$$20 + 2 \times 15 = 50 \text{ in.} = 4.17 \text{ ft}$$

and $ac$ is $22 + 2 \times 15 = 52$ in. $= 4.33$ ft.

$$V_{ab} = 3650\left(\frac{5}{12} \times 7.5 + \frac{7.5 + 4.17}{2} \times \frac{20}{12}\right) = 17{,}300 \text{ lb}$$

$$V_{ac} = 3650\left(\frac{7.67 + 1.33}{2} \times \frac{20}{12}\right) = 36{,}500 \text{ lb}$$

The depth required for shear will therefore obviously be governed by the critical section $ab$; hence, since the allowable unit shear is 75 psi,

$$75 = \frac{17{,}300}{50 \times 0.9 \times d}$$

Therefore,

$$d = 14.1 \text{ in.}$$

The assumed effective depth of 15 in. is satisfactory. The total thickness of the footing is then 19 in., and the weight 15,100 lb, as assumed.

In the long direction,

$$A_s = \frac{0.85 \times 1{,}830{,}000}{20{,}000 \times 0.9 \times 15} = 5.75 \text{ sq in.}$$

In the short direction,

$$A_s = \frac{0.85 \times 1{,}590{,}000}{20{,}000 \times 0.9 \times 15} = 4.99 \text{ sq in.}$$

Subsequent investigations for bond stresses (see Art. 7.10) show that thirteen No. 6 bars are required in each direction, furnishing an area of 5.74 sq in. and a total perimeter of 30.6 in. Since the two sides of the footing are so nearly equal, the bars in each band can be spaced uniformly over the entire width of the footing, with about 3 in. insulation

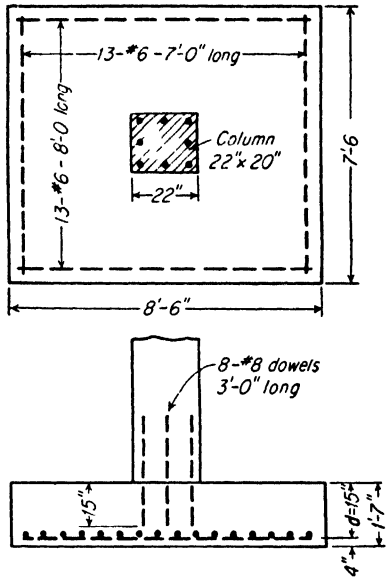



Fig. 8.34 Details of exterior-column footing.

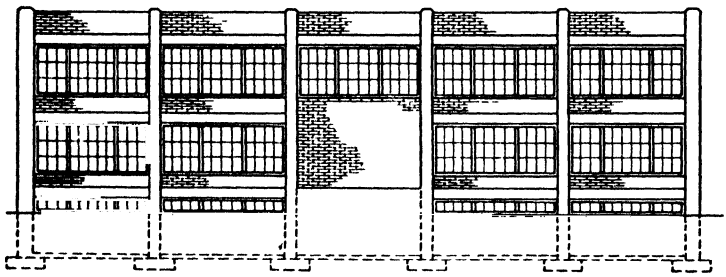
at the sides of the band. The requirements for placing the short bars, as given in Art. 7.9, would not materially alter this arrangement.

Eight No. 8 dowels 3 ft-0 in. long are required in each footing. They will project into the footing 15 in., which is sufficient to develop the full working value of each bar. The unit bearing stress at the base of the column is 233,000/22 × 20 = 530 psi, which is much less than the allowable of 1125 psi. Complete details of the footing are shown in Fig. 8.34.

**8.56. Design of Wall Beams.** With the proposed framing as shown in Figs. 8.35 and 8.38, the unsupported spans of the end wall beams are 18 ft-9 in. and 17 ft-9 in., while those of the intermediate wall beams are 20 ft-2 in. and 19 ft-2 in. for the short and long sides of the building, respectively. Since it is desirable to keep the depth of the wall beams

constant throughout the building on account of architectural appearance, it is first necessary to determine the cross section required for the maximum moment and shear. The width of the beams is taken as 12 in. in all cases. As stated in Art. 8.46, $f'_c = 3000$ psi, and $f_s = 20{,}000$ psi.

The maximum shear occurs in the first- and second-floor intermediate beams along the short side of the building. According to Table 4(*c*), Appendix C, a marginal beam which has a depth greater than 1½ times the minimum slab thickness shall be designed to carry, in addition to the load superimposed directly upon it, a uniformly distributed load



REAR ELEVATION
FIG. 8.35

equal to at least one-quarter of the total live and dead load for which the adjacent panel is designed. The total load on the wall beam per linear foot is therefore

| | | |
|---|---|---|
| Brick sill: $2.5 \times 1 \times 120$ | = | 300 lb |
| Windows: $7.5 \times 8$ | = | 60 lb |
| Stem of beam, assumed | = | 165 lb |
| Floor load: $\frac{1}{4} \times 300 \times 21$ | = | 1575 lb |
| | | 2100 lb |

The maximum shear occurs at the interior supports of the end spans on the short side of the building (see also Art. 3.6):

$$V = 1.15 \times 2100 \times \frac{18.75}{2} = 22{,}700 \text{ lb}$$

Assuming that web reinforcement will be used,

$$d \text{ (for shear)} = \frac{22{,}700}{360 \times \frac{7}{8} \times 12} = 6.0 \text{ in.}$$

The maximum moment occurs in the end spans on the short side of the building. These beams are continuous over one support only and are

designed for a negative moment of $\frac{1}{10}wl'^2$ where $l'$ is the average of the two clear spans adjacent to the interior support (see Art. 3.6).

$$M = \frac{1}{10} \times 2100 \times (19.46)^2 \times 12 = 955{,}000 \text{ in.-lb}$$

Since the beams are, in effect, T beams with the tee on one side only, the effective width of flange, according to the ACI Code (see Art. 3.39), is

$$b = \frac{1}{12} \times 18.75 \times 12 + 12 = 30.75 \text{ in.}$$

On account of the relatively thick flange it is apparent that the neutral axis will undoubtedly be in the flange. With this assumption, the beam must be designed as a rectangular beam 30.75 in. wide, and the required depth is obtained from Eq. (3.6), Art. 3.4.

$$d \text{ (for moment)} = \sqrt{\frac{955{,}000}{235 \times 30.75}} = 11.5 \text{ in.}$$

In order to give the desired architectural appearance, a total height of 21 in. is adopted for all the wall beams. The effective depth at the center of the beam, allowing for one row of steel with $2\frac{1}{2}$ in. insulation, is $18\frac{1}{2}$ in., and with the proposed arrangement of the steel (Fig. 8.38) the effective depth at the support is 19 in. The weight of the stem per foot is then 165 lb, as assumed.

**8.57. Design of *L*1 and *L*2.** Since it has been determined above that a cross section of 12 by 21 in. is satisfactory for all wall beams, it is now merely necessary to determine the area of steel required for each beam and to provide for shearing stresses.

The design positive moment in the end beam *L*1 is (see Art. 3.6):

$$M = \frac{1}{14} \times 2100 \times (18.75)^2 \times 12 = 634{,}000 \text{ in.-lb}$$

and the positive moment in the interior beam *L*2 is

$$M = \frac{1}{16} \times 2100 \times (20.17)^2 \times 12 = 640{,}000 \text{ in.-lb}$$

Assuming $j = 0.9$, the required steel areas are as follows:
Beam *L*1, positive-moment section:

$$A_s = \frac{634{,}000}{20{,}000 \times 0.9 \times 18.5} = 1.91 \text{ sq in.}$$

Beam *L*1, interior support:

$$A_s = \frac{955{,}000}{20{,}000 \times 0.9 \times 19} = 2.79 \text{ sq in.}$$

Beam $L2$, positive-moment section:

$$A_s = \frac{640{,}000}{20{,}000 \times 0.9 \times 18.5} = 1.92 \text{ sq in.}$$

Two No. 6 and two No. 8 bars ($A_s = 2.45$ sq in.) will be placed in the bottom of beam $L1$; two No. 6 and two No. 7 ($A_s = 2.08$ sq in.) in the bottom of beam $L2$. By bending up the No. 7 and No. 8 bars and carrying them over the interior supports a sufficient amount of steel can be provided there ($A_s = 2.77$). The corner columns are relatively stiff. Therefore the No. 8 bars in beam $L1$ will be bent up and an additional No. 7 top bar added at the exterior end to provide for a negative moment of $\frac{1}{12}wl'^2$ (see also Art. 3 6).

The maximum unit bond stress is

$$u = \frac{22{,}700}{(5.5 + 6.3) \times \frac{7}{8} \times 19} = 116 \text{ psi}$$

The points at which the two No. 8 bars in $L1$ may be bent up are obtained from Diagram 2. The bars may be bent $0.20 \times 18.75 = 3.75$ ft from the edge of the column. In $L2$, the two No. 7 bars may be bent $0.23 \times 20.17 = 4.64$ ft from the edge of the column.

In beam $L1$ at the interior support the bent bars may be bent down $0.13 \times 18.75 = 2.44$ ft from the edge of the column. At the exterior support, assuming a design moment of $wl'^2/12$, the bent-up bars should reach the top not closer than $0.14 \times 18.75 = 2.62$ ft from the edge of the column. The arrangement shown in Fig. 8.38 satisfies these conditions. The bent bars are continued along the top into the adjacent span to the approximate location of the point of inflection. The straight bars are carried 6 in. into the columns.

Since only two bars are bent up in each beam, and these at one place, their strength is disregarded in providing for diagonal tension; stirrups are placed at suitable intervals to furnish all of the web strength necessary. In $L2$, the distance $x_1$ from the edge of the column is

$$x_1 = \frac{20.17}{2} - \frac{90 \times 12 \times \frac{7}{8} \times 18.5}{2100} = 1.8 \text{ ft}$$

At the support,

$$V' = V - V_c = 21{,}200 - 90 \times 12 \times \tfrac{7}{8} \times 19 = 3200 \text{ lb}$$

With $\frac{3}{8}$-in. round U stirrups, and $f_s = 20{,}000$ psi,

$$s = \frac{2 \times 0.1104 \times 20{,}000 \times \frac{7}{8} \times 19}{3200} = 22.8 \text{ in.}$$

The maximum permissible spacing is $\frac{1}{2} \times 19 = 9\frac{1}{2}$ in. The first stir-

rup will be placed 2 in. from the edge of the column; the remainder spaced at 9 in.

In beam $L1$ zero shear will occur 22,700/2100 = 10.8 ft from the edge of the interior column. Therefore

$$x_1 = 10.8 - \frac{90 \times 12 \times \frac{7}{8} \times 18.5}{2100} = 2.5 \text{ ft}$$

At the support $V' = 22{,}700 - 90 \times 12 \times \frac{7}{8} \times 19 = 4700$ lb and

$$s = \frac{2 \times 0.1104 \times 20{,}000 \times \frac{7}{8} \times 19}{4700} = 15.6 \text{ in.}$$

This is again larger than the maximum allowable spacing. The first stirrup will be placed 2 in. from the edge of the column and the remainder spaced at 9 in.

The areas of steel required in the other wall beams, floor, and roof and the arrangement of the web reinforcement are determined in a manner similar to the procedure outlined in the above paragraphs. The roof wall beams are made the same depth as those at the second-floor level for architectural reasons. In investigating the beams on the long side of the building it should be borne in mind that the first interior supports of the line are designed for a moment of $\frac{1}{10}wl'^2$, while at the other interior supports a $\frac{1}{11}$ coefficient is used. The end supports at the corners of the building are arbitrarily reinforced for negative moments of $\frac{1}{12}wl'^2$. The necessary steel details are shown in Fig. 8.38.

**8.58. Design of Stairway Slab.** The stairway, which extends from the basement to the second-floor level, is located as shown in Fig. 8.38. The opening made by the stair well and the future elevator shaft is framed by a series of beams as shown. In order to keep the thickness of the stair slab down to a reasonable minimum, beam $B9$ is placed at the edge of the floor-level landing slab. The stair slab is designed as a simple slab with a span equal to the horizontal distance from the middle of beam $B9$ to the middle of the wall support of the intermediate landing slab. As in Art. 8.16, a 3000-lb concrete is used and $f_s = 20{,}000$ psi.

The dead load on the stair slab is made up of the weight of the intermediate landing slab, the weight of the inclined slab, and the weight of the treads. For purposes of computation it is sufficiently accurate to assume that this total dead load is uniformly distributed over the horizontal span. The live load is assumed as 100 psf of horizontal surface (see Art. 8.24).

The widths of the stair slabs and landing slabs as shown in Fig. 8.40 will prove satisfactory in the ordinary building of this size. The width of each tread, exclusive of the nosing, is made 10¼ in. and the rise about

$7\frac{3}{16}$ in. These values satisfy the rules for convenient climbing (Art. 8.24) and also give a uniform rise of all treads between landing slabs.

Allowing 6-in. bearing on the brick exterior wall, the horizontal span of the stair slab is 12 ft-1 in. The total length of slab between supports is $\sqrt{6^2 + (7.66)^2} + 3.83 = 13.5$ ft. Assuming the weight of the slab as 69 psf, the total load on a 1-ft strip of slab is

$$\text{Treads: } 9 \times \frac{7.19 \times 10.25}{2 \times 144} \times 150 = 350 \text{ lb}$$

$$\text{Slab: } 13.5 \times 69 = 940$$

$$\text{Live load: } 12 \times 100 = 1200$$

$$\text{Total} = 2490 \text{ lb}$$

$$M = \tfrac{1}{8} \times 2490 \times 12.1 \times 12 = 45{,}300 \text{ in.-lb}$$

$$d = \sqrt{\frac{45{,}300}{235 \times 12}} = 4.0 \text{ in.}$$

An effective depth of $4\frac{1}{2}$ in. is selected; with 1-in. insulation the total slab thickness is $5\frac{1}{2}$ in. and the weight is 69 psf, as assumed.

$$A_s = \frac{45{,}300}{20{,}000 \times 0.866 \times 4.50} = 0.58 \text{ sq in. per ft of width}$$

This area is furnished by No. 5 bars, 6 in. center to center. One No. 5 bar, 3 ft-8 in. long, is placed under each tread at right angles to the main reinforcement, as shown in Fig. 8.40.[1]

The floor-level landing slab is made 8 in. thick, and every other bar of the stair-slab reinforcement is continued across this slab. Two No. 5 bars are placed in the intermediate landing slab, at right angles to the main reinforcement, in order to assist in distributing the load on that slab and to provide for temperature stresses. Negative-moment stresses in the stair slabs across beams *B*9 are provided for by means of short bent bars placed in the top of the slabs at these points as shown in Fig. 8.40.

**8.59. Design of Beams Framing Stair Well.** *Beam B*9. See Fig. 8.36. This beam supports a uniform load along its entire length, consisting of one-half of the stairway-slab load and one-half of the floor-level landing-slab load in addition to its own weight. The span of the beam is 9 ft-5 in., the distance center to center of beams *B*6 and *B*7. The beam is designed as a T beam with the tee on one side only; this is

[1] The detailing shown in Fig. 8.40 assumes that the stairway slab will be poured with the structural frame. A more widely used method of construction places the stair slab after the main structure has been completed. In such cases, recesses must be left in the beams *B*-9 and in the brick wall in order to furnish support for the future stair slabs, and dowels must be placed in the beams *B*9 so as to project into the stair slabs when they are poured.

necessary because of the break between the up and down stair slabs at the landing.

*Beam B6.* This beam is a simply supported rectangular beam, the load on which consists of the weight of a 4-in. hollow tile partition and the concentrated load from *B9* in addition to its own weight.

*Beam B8.* This beam carries the concentrated load from beam *B6*, one-half of the load from the landing slab, and the weight of a 4-in. hollow tile partition in addition to its own weight. Beam *B8* has been placed at the edge of the column strip. This fact, together with the improbability of having the full live load on the portion of the floor near *B8*, justifies the assumption that no floor load is supported by *B8*.[1] Because of the open shaft on one side of the beam at the point of maximum moment, it must be designed as a T beam with the tee on one side only. In order to illustrate the method of design where such irregular loads are involved, the design of *B8* is included below.

The loads on the beam (see Figs. 8.36 and 8.38) are as follows:
Uniform load over entire length of beam:

$$\begin{aligned} \text{Partition: } 20 \times 11.5 &= 230 \text{ lb per lin ft} \\ \text{Weight of beam} &= 180 \text{ lb per lin ft} \\ \text{Total} &= 410 \text{ lb per lin ft} \end{aligned}$$

Additional uniform load from stairway landing slab

$$\tfrac{1}{2} \times 3.5 \times 200 = 350 \text{ lb per lin ft}$$

Concentrated load from *B6* is 8570 lb:

$$R_L = \frac{8570 \times 9.42 + 350 \times \frac{(9.42)^2}{2} + 410 \times \frac{21^2}{2}}{21} = 8900 \text{ lb}$$

$$R_R = 11{,}580 \text{ lb}$$

The point of zero shear, which locates the point of maximum moment, occurs under the concentrated load. The maximum moment is

$$M = \left(8900 \times 11.5 - 410 \times \frac{(11.58)^2}{2}\right) \times 12 = 900{,}000 \text{ in.-lb}$$

According to the ACI Code, the effective width of flange is equal to $10 + \frac{1}{12} \times 21 \times 12 = 31$ in. Because of the relatively large thickness of the flange it is reasonable to assume that the neutral axis is in the flange, in which case the equations for rectangular beams apply:

[1] Since beam *B8* and the slab adjacent to it are poured at the same time, they cannot act independently of each other. A more conservative design would assume that a part of the adjacent slab load is supported by *B8*.

$$d = \sqrt{\frac{900{,}000}{31 \times 235}} = 11.1 \text{ in.}$$

An effective depth of 14 in. will be used, in order to avoid interference with the bars in $B6$:

$$b' = \frac{11{,}580}{360 \times \frac{7}{8} \times 14} = 2.7 \text{ in.}$$

A width of stem of 10 in. will be used, in order to give a reasonable amount of resistance to torsional stresses. With 3 in. of insulation, to provide

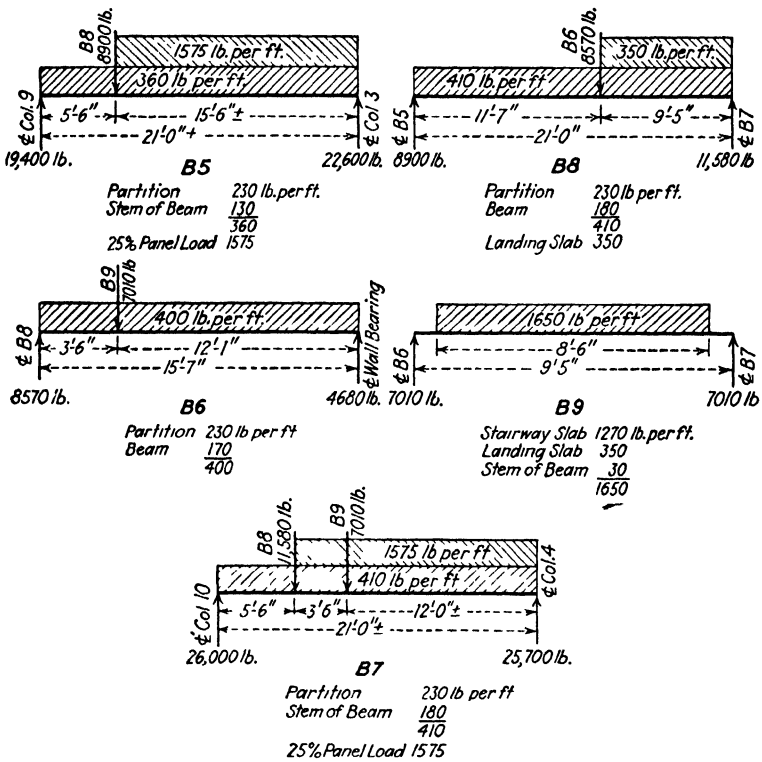



FIG. 8.36. Loading diagrams for beams framing stair well.

for two rows of bars, the total height of the beam is 17 in. and the weight per foot is 180 lb, as assumed.

$$A_s = \frac{900{,}000}{20{,}000 \times 0.875 \times 14} = 3.66 \text{ sq in.}$$

Six No. 7 bars are selected (3.61 sq in.). With

$$np = 10 \times 3.61/(31 \times 14) = 0.085$$

and $\frac{t}{d} = \frac{14}{8} = 0.57$, Diagram 3, Appendix D, shows that the neutral axis is in the flange, so that, from Table 6 with $p = 3.61/(31 \times 14) = 0.0085$, $j = 0.887$, and the revised steel area required is 3.62 sq in. The six No. 7 bars are satisfactory.

Since this beam is poured monolithically with the beams supporting it, there will exist some negative moment at the supports. The amount of this bending moment is dependent upon too many factors to permit of an accurate determination. In order to provide for the stresses of negative moment, one-half of the steel is bent up and hooked over the support. Adequate bond resistance is furnished by the three deformed bars remaining. The points at which this steel may be bent up may best be found by determining the points where the bending moment is one-half of the maximum, by means of an equation involving the distances $x$ from the supports to these points. For the left end of the beam, as shown in Fig. 8.36, this equation is as follows:

$$8900x - 410\frac{x^2}{2} = 37{,}500$$

$$x = 4.7 \text{ ft}$$

The three bars on this side may be bent up 4 ft-6 in. from the center of $B5$. A similar equation for the right end of the beam locates the point where the bending moment is 37,500 ft-lb, at a distance of 3.7 ft from the center of $B7$. The three bars may be bent up 3 ft-6 in. from that point.

Since only three bars are bent up, and since these are bent at a single point, they will not be depended upon for diagonal tension resistance, and vertical stirrups will be used wherever web reinforcement is required. In computing the spacing of stirrups, it is assumed that the concrete can take care of the shear up to a unit value of 90 psi. The total shear that the concrete can resist is, therefore,

$$V_c = 90 \times 10 \times {}^7\!/_8 \times 14 = 11{,}000 \text{ lb}$$

The stirrups will be designed to resist all the shear in excess of this amount.

With $^1\!/_4$-in. round U stirrups, the maximum spacing which can be used at the right support (see Fig. 8.36) is obtained from Eq. (3.15), Art. 3.33.

$$s = \frac{2 \times 0.0491 \times 20{,}000 \times {}^7\!/_8 \times 14}{11{,}580 - 11{,}000} = 41.4 \text{ in.}$$

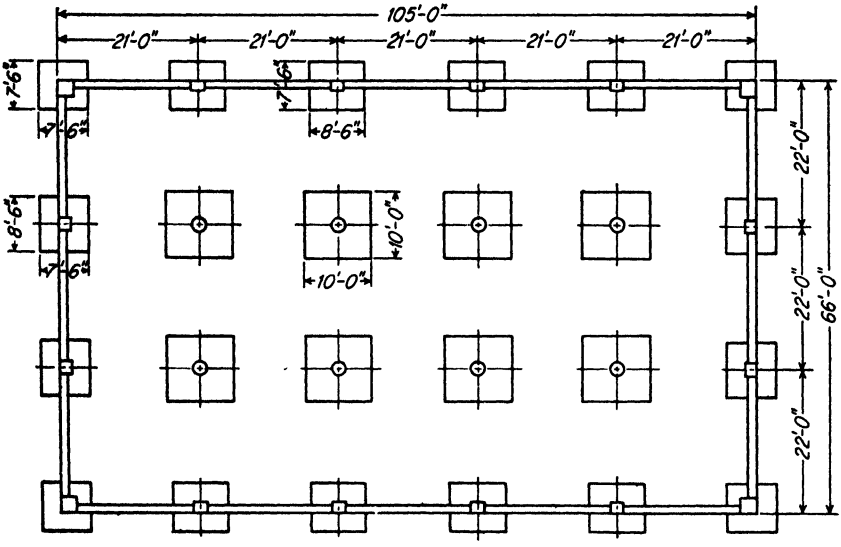
In this case the maximum allowable spacing of $0.5 \times 14 = 7$ in. governs. The distance from the right support where the unit shear is

90 psi is

$$\frac{11{,}580 - 11{,}000}{350 + 410} = 0.76 \text{ ft}$$

At the left end no stirrups are required since $R_L$ is less than $V_c$.

*Beam B5.* In addition to supporting the concentration from beam *B8* and its own weight, beam *B5* must be designed to support the partition as shown and the required proportion of the floor load. The beam is a T beam, with the tee on one side only.

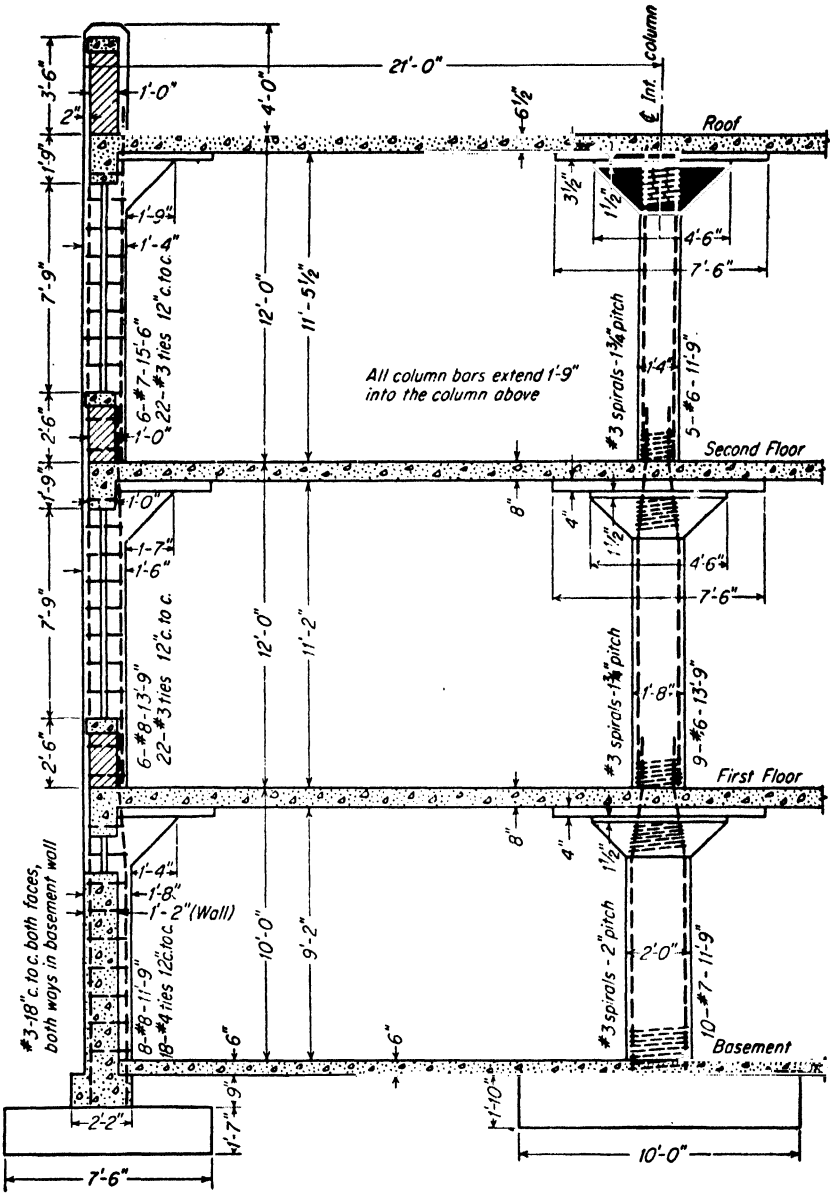



FOUNDATION PLAN

FIG. 8.37

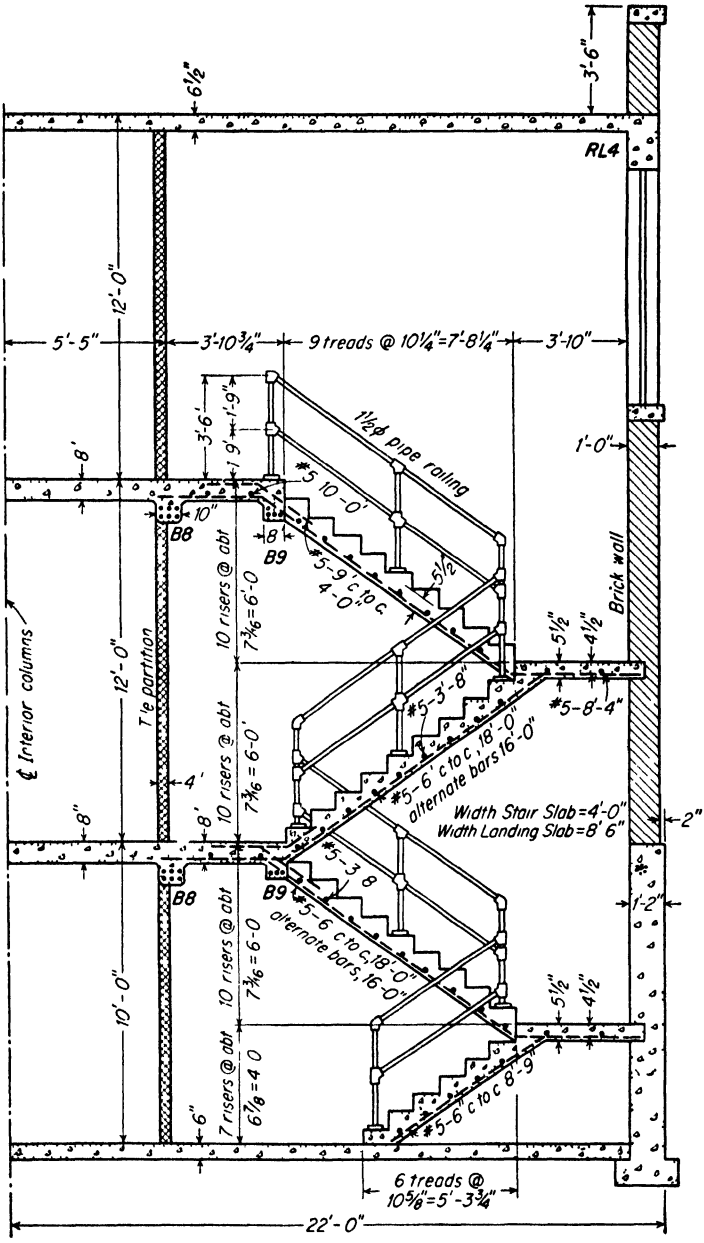
*Beam B7.* This beam supports loads as stated for beam *B5*, and, in addition, the concentrated load from beam *B9*. It is a T beam, with the tee on one side only. The complete framing details of the stairway beams are shown in Fig. 8.38.

**8.60. Engineering Drawings.** Figure 8.38 is an engineering drawing of the second floor, giving the necessary steel information on the concrete and steel for the slab and beams at that floor. This is the type of drawing which would be prepared in the design office. The reinforcing-bar fabricator would use this drawing as a basis for preparing a complete detail drawing showing bar lengths and bends and placing instructions. The Manual of Standard Practice for Detailing Reinforced Concrete Structures (ACI 318 51) of the American Concrete Institute is usually followed in the preparation of both engineering drawings and fabricator's detail drawings.




PARTIAL CROSS SECTION
SECTION X-X
Showing Column Steel

FIG. 8.39




DETAIL OF STAIRWAY
SECTION Y-Y

FIG. 8.40

# SECTION 9

# RETAINING WALLS

**9.1. Function and Types of Retaining Walls.** Retaining walls are used to hold back masses of earth or other loose material where conditions make it impossible to let those masses assume their natural slopes. Such conditions occur when the width of an excavation, cut, or embankment is restricted by conditions of ownership, use of the structure, or economy. Thus, in railway or highway construction the width of the right of way is fixed and the cut or embankment must be contained within that width. Similarly, the basement walls of buildings must be located within the property and must retain the soil surrounding the basement.

Free-standing retaining walls, as distinct from those which form part of structures like basement walls, are of various types, the most common of which are shown in Fig. 9.1. The gravity wall (Fig. 9.1*a*), retains the earth entirely by its own weight. The reinforced concrete cantilever wall (Fig. 9.1*b*) consists of the vertical arm which retains the earth and is held in position by the footing or base slab. In this case the weight of the fill on top of the heel, in addition to the weight of the wall, contributes to the stability of the structure. Since the arm represents a vertical cantilever, its required thickness increases rapidly with increasing height. To reduce the bending moments in vertical walls of great height, counterforts are used spaced at distances from each other equal to or slightly larger than one-third of the height (Fig. 9.1*c*). Property rights or other restrictions sometimes make it necessary to place the wall at the forward edge of the base slab, *i.e.*, to omit the toe. Wherever it is possible, toe extensions of one-third to one-fourth of the width of the base provide a more economical solution.

Which of the three types of walls is appropriate in a given case depends on a variety of conditions, such as local availability and price of construction materials, property rights, and the like. In general, gravity walls are economical only for relatively low walls, possibly up to about 15 ft. Cantilever walls are economical for heights from 10 to 20 ft, while counterforts are indicated for greater heights.

**9.2. Earth Pressure.** In their physical behavior soils and other granular masses occupy a position intermediate between liquids and solids. If sand is poured from a dump truck, it flows but, unlike a frictionless liquid, it will not assume a horizontal surface. It maintains itself in a stable heap with sides subtending with the horizontal an *angle of repose* whose tangent is roughly equal to the coefficient of intergranular friction. If a pit is dug in clay soil, its sides can usually be made vertical over considerable depths without support; that is, the clay will behave like a

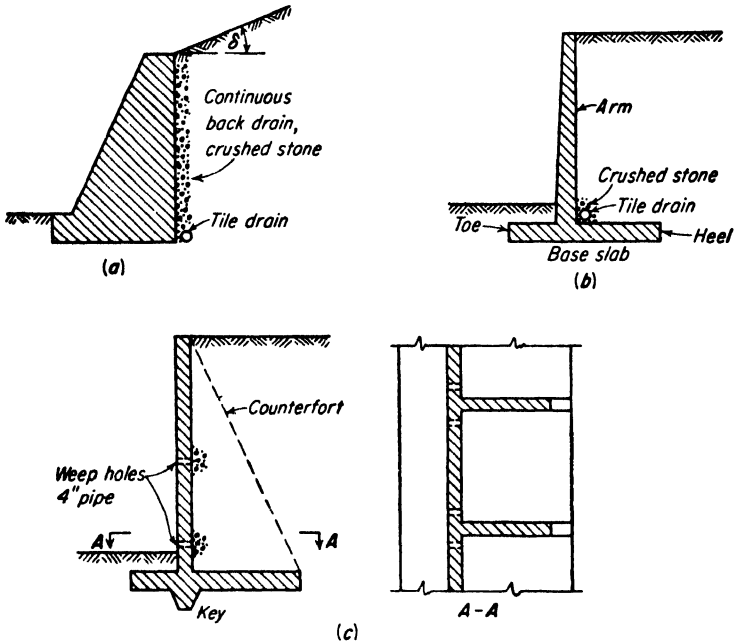



FIG. 9.1. Types of retaining walls and backdrains: (*a*) gravity wall; (*b*) cantilever wall; (*c*) counterfort wall.

solid and will retain the shape it is given. If, however, the pit is flooded, the sides will give way and, in many cases, the saturated clay will be converted nearly into a true liquid. The clay was capable of maintaining its shape by means of its internal cohesion, but flooding reduced that cohesion greatly, often to zero.

If a wall is built in contact with a solid, such as a rock face, no pressure is exerted on it. If, on the other hand, a wall retains a liquid, as in a reservoir, it is subject at any level to the hydrostatic pressure $w_w h$, where $w_w$ is the unit weight of water and $h$ the distance from the surface. If a vertical wall retains soil, the earth pressure similarly increases proportional to the depth but its magnitude is

$$p = Cwh \tag{9.1}$$

where $w$ is the unit weight of the soil and $C$ a constant which depends on the physical characteristics of the soil.

The pressure which soil exerts on an immovable wall, known as the *rest pressure*, has not been investigated thoroughly. The few measurements that have been made seem to indicate that in this case $C = 0.4$ to 0.5 for uncompacted, noncohesive soils, such as sands and gravels, while it may be as high as 0.8 for the same soils in a highly compacted state. For cohesive clay soils, $C$ appears to be of the order of 0.7 to 1.0.

Usually, however, walls move slightly under the action of the earth pressure. Since walls are constructed of elastic material, they deflect under the action of the pressure, and since they generally rest on compressible soils, they tilt and shift away from the fill. For this reason, often the wall is constructed with a slight batter toward the fill on the exposed face, so that, if and when such tilting takes place, it does not

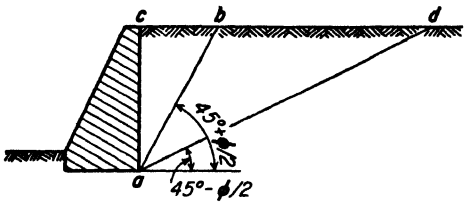



FIG. 9.2

appear evident to the observer. Even if this movement at the top of the wall is only of the order of a fraction of a per cent of the wall height ($\frac{1}{2}$ to $\frac{1}{10}$ per cent according to tests by C. Terzaghi), the rest pressure is materially decreased by it. The magnitude of this pressure against slightly yielding walls is much better explored and known than the rest pressure.

If the wall moves away from the fill, a sliding plane $ab$ (Fig. 9.2) forms in the soil mass, and the wedge $abc$, sliding along that plane, exerts pressure against the wall. Here the angle $\phi$ is known as *angle of internal friction*, *i.e.*, its tangent is equal to the coefficient of intergranular friction which can be determined by appropriate laboratory tests. The corresponding pressure is known as the *active earth pressure*. If, on the other hand, the wall is pushed against the fill, a sliding plane $ad$ is formed and the wedge $acd$ is pushed upward by the wall along that plane. The pressure which this larger wedge exerts against the wall is known as the *passive earth pressure*. (This latter case will occur, for example, at the left face of the gravity wall of Fig. 9.1 when this wall yields slightly to the left under the pressure of the fill.)

The magnitude of these pressures has been analyzed by Rankine, Coulomb, and others. If the soil surface subtends an angle $\delta$ with the

horizontal (Fig. 9.1*a*), then, according to Rankine, the coefficient for active earth pressure is

$$C_a = \cos \delta \frac{\cos \delta - \sqrt{\cos^2 \delta - \cos^2 \phi}}{\cos \delta + \sqrt{\cos^2 \delta - \cos^2 \phi}} \tag{9.2}$$

and that for passive pressure

$$C_p = \cos \delta \frac{\cos \delta + \sqrt{\cos^2 \delta - \cos^2 \phi}}{\cos \delta - \sqrt{\cos^2 \delta - \cos^2 \phi}} \tag{9.3}$$

For the frequent case of horizontal surface, that is, $\delta = 0$ (Fig. 9.2), for active pressure,

$$C_{ah} = \frac{1 - \sin \phi}{1 + \sin \phi} \tag{9.4}$$

and for passive pressure,

$$C_{ph} = \frac{1 + \sin \phi}{1 - \sin \phi} \tag{9.5}$$

Rankine's theory is valid only for noncohesive soils like sand and gravel but, with corresponding adjustments, can be used successfully also for cohesive clay soils.

From Eqs. (9.1) to (9.5) it is seen that the earth pressure at a given depth $h$ depends on the inclination of the surface $\delta$, the unit weight $w$, and the angle of friction $\phi$. The first two of these are easily determined, while little agreement has yet been reached as to the proper values of $\phi$. For the ideal case of a dry, noncohesive fill, $\phi$ could be determined by laboratory test and then used in the formulas. This is impossible for clays only part of whose resistance is furnished by intergranular friction while the rest is due to internal cohesion. For this reason their actual $\phi$ values are often increased by an arbitrary amount to account implicitly for the added cohesion. However, this is often unsafe since, as was shown on the example of the flooded pit, cohesion may vanish almost completely due to saturation and inundation.

In addition, fills behind retaining walls are rarely uniform and, what is more important, they are rarely dry. Proper drainage of the fill is vitally important to reduce pressures (see Art. 9.5), but even in a well-drained fill the pressure will temporarily increase during heavy storms or sudden thaws. This is due to the fact that even though the drainage may successfully remove the water as fast as it appears, its movement through the fill toward the drains causes additional pressure (see page pressure). In addition, frost action and other influences may temporarily increase its value over that of the theoretical, active pressure. Many walls which were designed without regard to these factors have failed, been displaced, or cracked.

It is good practice, therefore, to select conservative values for $\phi$, considerably smaller than the actual test values, in all cases except where extraordinary and usually expensive precautions are taken to keep the fill dry under all conditions. An example of recommended earth-pressure values, which are quite conservative, though based on extensive research and practical experience, can be found in "Soil Mechanics in Engineering Practice."[1] Less conservative values are often used in practical designs, but these should be employed with caution in view of the fact that occasional trouble has been encountered with walls so designed.

Table 9.1 gives average values for $w$ and two columns of values for $\phi$. The first of these has been adapted by the authors from the information by Terzaghi and Peck, the second represents magnitudes which are frequently used in engineering practice and, at least for noncohesive soils, are closer to the actual laboratory test values. In contrast to the first, however, these take no account of probable additional pressures due to porewater, seepage, frost, etc. The table also contains values for the coefficient of friction $f$ between concrete and various soils.

**TABLE 9.1. Unit Weights, Effective Angles of Internal Friction $\phi$, and Coefficients of Friction with Concrete, $f$**

| Soil* | Unit weight, pcf | $\phi$* | $\phi$† | $f$ |
|---|---|---|---|---|
| 1. Sand or gravel without fine particles, highly permeable | 110–120 | 35° | 33–40 | 0.5–0.6 |
| 2. Sand or gravel with silt mixture, low permeability | 120–130 | 32° | 25–35° | 0.4–0.5 |
| 3. Silty sand, sand and gravel with high clay content | 110–120 | 26° | 23–30° | 0.3–0.4 |
| 4. Medium or stiff clay | 100–120 | 0‡ | 25–35° | 0.25–0.4 |
| 5. Soft clay, silt | 90–110 | 0° | 20–25° | 0.2–0.3 |

* Adapted from Terzaghi and Peck.

† Values often used.

‡ For this case the distance $(h - 4$ ft) should be used instead of $h$ in Eq. (9.1).

It is seen that the values $\phi$ according to Terzaghi and Peck, particularly for soils 4 and 5, are considerably more conservative than the values of $\phi$ often used in engineering practice, and reflect the fact that clays and silts, under suitable conditions of saturation, may become entirely liquid (that is, $\phi = 0$).

The low values of $\phi$ for soils 3 to 5 and the resulting high soil pressures indicate that soils of types 1 or 2 should be used for backfills of retaining walls, wherever possible.

[1] Terzaghi and Peck, "Soil Mechanics in Engineering Practice," pp. 315–318, John Wiley & Sons, Inc., New York, 1948.

**9.3. Earth Pressure for Common Conditions of Loading.** In computing earth pressures on walls, four common conditions of loading are most often met: (1) horizontal surface of fill at top of wall; (2) inclined surface of fill sloping up and back from the top of wall; (3) horizontal surface of fill, carrying a uniformly distributed additional load (surcharge) such as from goods in a storage yard, traffic on a road, etc.; (4) horizontal surface of fill carrying a surcharge distributed over a strip of limited width, parallel to the wall, such as the width of a railroad track running along the wall.

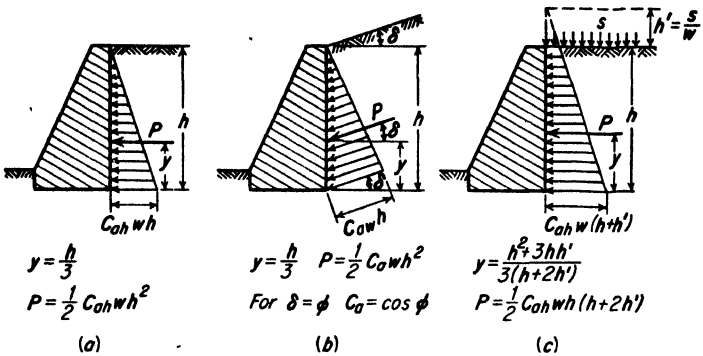



FIG. 9.3. Earth pressures for (*a*) horizontal surface, (*b*) sloping surface; (*c*) horizontal surface with surcharge *s*.

The increase in pressure caused by uniform surcharge $s$ (Case 3) is computed by converting its load into an equivalent, imaginary height of earth $h'$ above the top of the wall such that

$$h' = \frac{s}{w} \tag{9.6}$$

and measuring the depth to a given point on the wall from this imaginary surface. This amounts to replacing $h$ by $(h + h')$ in Eq. (9.1).

The distributions of pressure for Cases 1 to 3 are shown in Fig. 9.3. The total earth thrust $P$ per linear foot of wall is evidently equal to the area under the pressure distribution figure, and its line of action passes through the centroid of the pressure. Figure 9 3 gives information computed in this manner, on magnitude, point of action, and direction of $P$ for these three cases.

Various methods are in use to account for the excess pressure caused by partial surcharge (Case 4). Figure 9.4, as an example, gives the recommendations of the American Railway Engineering Association (Manual, 1946, pp. 8-86) for computing the equivalent height $h'$ from tracks running parallel to the wall, or from buildings founded in the backfill. The following rules apply: (*a*) In calculating the surcharge from a

track, the weight $T$ of track and train, per foot, shall be distributed over a width of 14 ft, and the corresponding fictitious height $h'$ computed as indicated in Fig. 9.4. (*b*) The full effect of the surcharge shall be considered (that is, $h' = T/wb$) where the front edge of the distributed load or structure is located anywhere between the face of the wall and a point vertically above the back edge of the footing. The effect of the surcharge may be neglected if its front edge is located beyond a line drawn

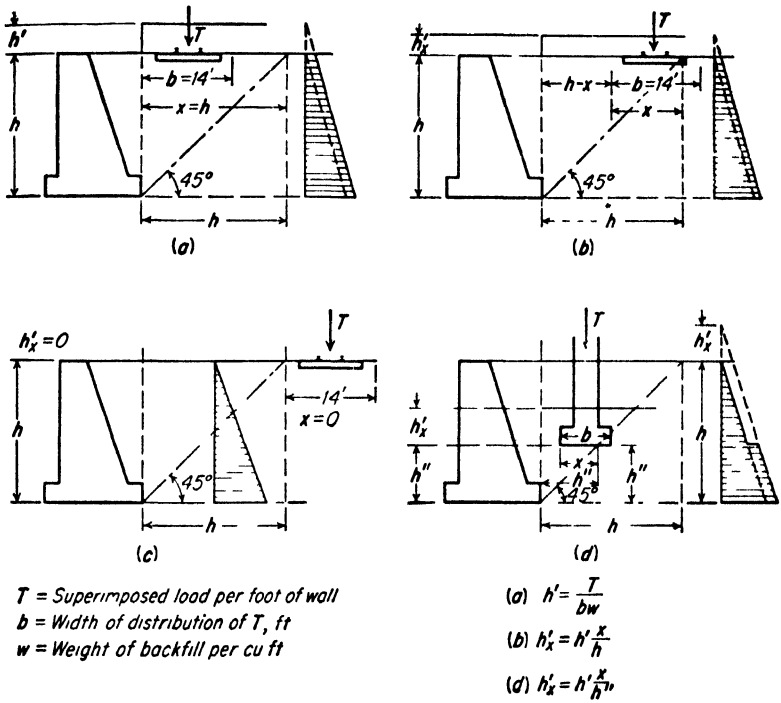



FIG. 9.4. Earth pressures for horizontal surface loaded by a strip surcharge (*a*, *b*, and *c*) or by a wall footing (*d*). (*According to AREA Manual.*)

under 45° from the back corner of the wall. Where the edge of the distributed load is located from the back edge of the footing at a horizontal distance $h - x$ (or $h'' - x$ in Fig. 9.4*d*), the fictitious height of surcharge is $h'_x = h'\ (x/h)$, or $h'_x = h'\ (x/h'')$, respectively.

Occasionally retaining walls must be built for conditions where the ground-water level is above the base of the wall, either permanently or seasonally. In that case the pressure of the soil *above* ground water is determined as usual. The part of the wall *below* ground water is subject to the sum of the water pressure and the earth pressure. The former is equal to the full hydrostatic pressure $p_w = w_w h_w$, where $w_w$ and $h_w$ are, respectively, the unit weight of water and the distance from

ground-water level to the point on the wall. The additional pressure of the soil below ground-water level is computed from Eq. (9.1), where, however, for the portion of the soil below water, $w$ is replaced by $w - w_w$, while $h$, as usual, is measured from the soil surface. That is, for submerged soil, buoyancy reduces the effective weight in the indicated manner. Pressures of this magnitude, which are considerably larger than those of drained soil, will also occur temporarily after heavy rainstorms or thaws in walls without provision for drainage, or if drains have become clogged.

The seeming simplicity of the determination of earth pressure as here indicated should not lull the designer into a false sense of security and certainty. No theory is more accurate than the assumptions on which it is based. It has repeatedly been emphasized that actual soil pressures are affected by irregularities of soil properties, porewater, and drainage conditions and by climatic and other factors which cannot be expressed in formulas. This situation, on the one hand, indicates that involved refinements of theoretical earth-pressure determinations, as sometimes attempted, are of little practical value. On the other hand, the design of a retaining wall is seldom a routine procedure since the local conditions that affect pressures and safety vary from one locality to the other. The advice in the quoted Manual of the American Railroad Engineering Association should be headed: "A careful study should be made of the conditions of design of each wall and it should be remembered that no theoretical formula can be more than an aid to the judgment of the experienced designer."

**9.4. External Stability.** A wall may fail in two different ways: (1) its individual parts may not be strong enough to resist the acting forces, such as when the vertical cantilever wall is cracked by the earth pressure acting on it, and (2) the wall as a whole may be bodily displaced by the earth pressure, without breaking up internally. To design against the first possibility requires the determination of the necessary dimensions, thicknesses, reinforcements, etc., to resist the moments and shears; this procedure, then, is in no way different from that of determining required dimensions and reinforcements of other types of concrete structures. To safeguard the wall against bodily displacements on the other hand, *i.e.*, to ensure its external stability, requires special consideration.

A wall, such as that of Fig. 9.5, together with the soil mass *ijkl* which rests on the base slab, may be bodily displaced by the earth thrust $P$ which acts on plane *ak* by *sliding* along the plane *ab*. Such sliding is resisted by the friction between soil and footing along the same plane. To forestall motion, the forces that resist sliding must exceed those that tend to produce sliding; a factor of safety of 1.5 is generally assumed satisfactory in this connection.

Referring to Fig. 9.5, the force that tends to produce sliding is the horizontal component $P_h$ of the total earth thrust $P$. The resisting friction force is $fR_v$, where $f$ is the coefficient of friction between concrete and soil (see Table 9.1) and $R_v$ is the vertical component of the total resultant $R$, that is, $R_v = W + P_v$ ($W$ = weight of wall + soil resting of footing, $P_v$ = vertical component of $P$). Hence, to provide sufficient safety,

$$f(W + P_v) \geqq 1.5P_h \tag{9.7}$$

Actually, for the wall to slide to the left, it must push with it the earth

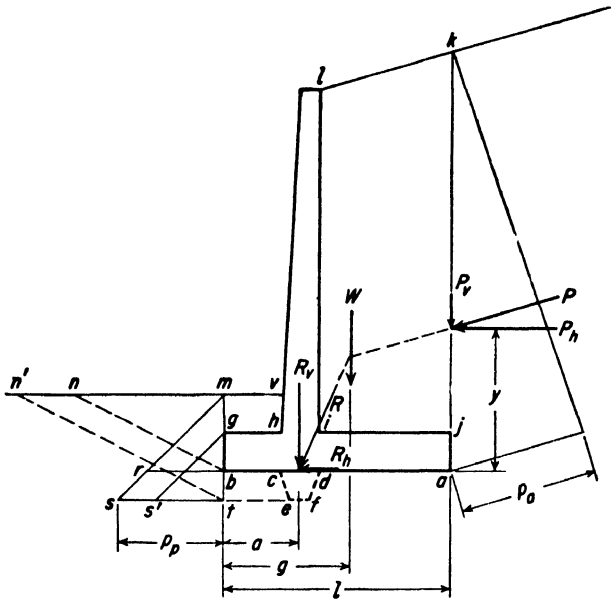



Fig. 9.5

*nmb*, which gives rise to the passive earth pressure indicated by the triangle *rmb*. This passive pressure represents a further resisting force which could be added to the left side of Eq (9 7) This should be done only, however, if the proper functioning of this added resistance is ensured. For that purpose the fill *ghmv* must be placed before the backfill *ijkl* is put in place, and must be secure against later removal by scour or other means throughout the lifetime of the wall. If these conditions are not met, it is better not to count on the additional resistance of the passive pressure.

If the required sliding resistance cannot be developed by these means, a key wall *cdef* can be used to increase horizontal resistance. In this case sliding, if it occurs, takes place along the planes *ad* and *tf*. While, along *ad* and *ef* the friction coefficient $f$ applies, sliding along *te* occurs within

the soil mass. The coefficient of friction that applies in this portion is consequently tan $\phi$, where the value of $\phi$ may be taken from the next to the last column in Table 9.1. In this situation sliding of the front soil occurs upward along *tn'* so that, if the front fill is secure, the corresponding resistance from passive soil pressure is represented by the pressure triangle *stm*. If doubt exists as to the reliability of the fill above the toe, the free surface should more conservatively be assumed at the top level of the footing, in which case the passive pressure is represented by the triangle *s'tg*.

Next, it is necessary to ensure that the pressure under the footing does not exceed the *permissible bearing pressure* for the particular soil. Let $a$ (Fig. 9.5) be the distance from the front edge $b$ to the intersection of the resultant with the base plane, and let $R_v$ be the vertical component of $R$. (This intersection need not be located beneath the vertical arm, as shown, even though an economical wall generally results if it is so located.) Then the base plane $ab$, 1 ft wide longitudinally, is subject to a normal force $R_v$ and to a moment about the centroid $(l/2 - a)R_v$. If these values are substituted in the usual formula for bending plus axial force

$$p_{1,2} = \frac{N}{A} \pm \frac{Mc}{I} \tag{9.8}$$

it will be found that if the resultant is located within the middle third ($a > l/3$), compression will act throughout the section and the maximum and minimum pressures can be computed from the equations in Fig. 9.6*a*. If the resultant is located just at the edge of the middle third ($a = l/3$), the pressure distribution is as shown in Fig. 9.6*b*, and Eq. (9.8) results in the formula there given.

If the resultant were located outside the middle third ($a < l/3$), Eq. (9.8) would indicate tension at and near point $a$. Obviously, tension cannot be developed between soil and a concrete footing which merely rests on it. Hence, in this case the pressure distribution of Fig. 9.6*c* will develop, which implies a slight lifting off the soil of the rear part of the footing. Equilibrium requires that $R_v$ pass through the centroid of the pressure-distribution triangle, from which the formula for $p$ for this case can easily be derived.

It is good practice, in general, to have the resultant located within the middle third. This will not only reduce the magnitude of the maximum bearing pressure but will also prevent too large a nonuniformity of pressure. If the wall is founded on highly compressible soil such as certain clays, a pressure distribution as in Fig. 9.6*b* would result in a much larger settlement of the toe than of the heel, with a corresponding tilting of the wall. In case of foundations on such soils the resultant, therefore, should

strike at or very near the center of the footing. If the foundation is on very incompressible soil, such as well-compacted gravel or rock, the resultant can be allowed to fall outside the middle third (Fig. 9 6c).

A third mode of failure is the possibility of the wall *overturning* bodily around the front edge $b$. For this to occur, the overturning moment $yP_h$ about point $b$ would have to be larger than the restoring moment $(Wg + P_v l)$, Fig. 9 5, which is the same as saying that the resultant would have to strike outside the edge $b$. If, as is mostly the case, the

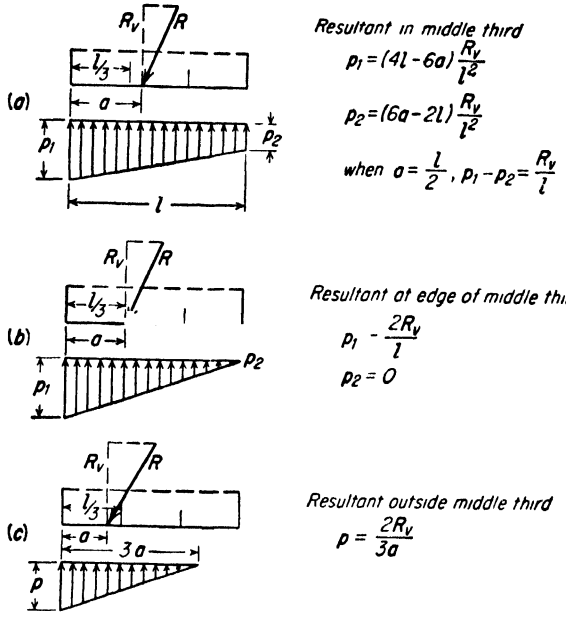



FIG. 9.6. Bearing pressures for different locations of resultant.

resultant strikes within the middle third, adequate safety against overturning exists and no special check need be made If the resultant is located outside the middle third, a factor of safety of at least 1.5 should be maintained against overturning, i e , the restoring moment should be at least 1 5 times the overturning moment

**9.5. Drainage and Other Details.** Such failures or damage to retaining walls as have occasionally occurred were due, in most cases, to one of two causes: overstressing of the soil under the wall with consequent forward tipping or insufficient drainage of the backfill. In the latter case hydrostatic pressure from porewater accumulated during or after rainstorms increases greatly the thrust on the wall; in addition, in subfreezing weather ice pressure of considerable magnitude can develop in such

poorly drained soils. The two causes are often interconnected, since large thrusts correspondingly increase the bearing pressure under the footing.

*Allowable bearing pressures* should be selected with great care. It is necessary, for this purpose, to investigate not only the type of soil immediately underlying the footing but also the deeper layers. Unless reliable information is available at the site, subsurface borings should be made to a depth at least equal to the height of the wall. The foundation must be laid below *frost depth*, which amounts to 4 to 5 ft and more in the northern states, in order to ensure against heaving by the freezing of soils containing moisture.

*Drainage* can be provided in various ways. *Weep holes* consisting of 4-in. pipe embedded in the wall, as shown in Fig. 9.1*c*, are usually spaced horizontally at 5 to 10 ft. In addition to the bottom row, additional rows should be provided in walls of substantial height. To facilitate drainage and prevent clogging, 1 cu ft or more of crushed stone is placed at the rear end of each weeper. Care must be taken that the outflow from the weep holes is carried off safely so as not to seep into and soften the soil underneath the wall. To prevent this, instead of weepers, *longitudinal drains* embedded in crushed stone or gravel can be provided along the rear face of the wall (Fig. 9 1*b*) at one or more levels; the drains discharge at the ends of the wall or at a few intermediate points. The most efficient drainage is provided by a *continuous back drain* consisting of a layer of gravel or crushed stone covering the entire rear face of the wall (Fig. 9.1*a*) with discharge at the ends. Such drainage is expensive, however, unless appropriate material is cheaply available at the site. Wherever possible the surface of the fill should be covered with a layer of low permeability and, in case of a horizontal surface, should be laid with a slight slope away from the wall toward a gutter or other drainage.

In long walls provision must be made against damage caused by *expansion or contraction* from temperature changes and shrinkage. The American Association of State Highway Officials recommends in its Specifications that for gravity as well as reinforced concrete walls expansion joints be made at intervals of 90 ft or less, and contraction joints at not more than 30 ft. The same specifications provide that in reinforced concrete walls horizontal temperature reinforcement of not less than $\frac{1}{8}$ sq in. per foot of depth be provided adjacent to the exposed surface.

**9.6. Design of a Gravity Retaining Wall.** A gravity wall is to retain a bank 11 ft-6 in. high whose horizontal surface is subject to a live-load surcharge of 400 psf. The soil is a sand and gravel mixture with a rather moderate amount of fine, silty particles. It can, therefore, be classified as Class 2 of Table 9.1, with the following characteristics: unit weight $w_s = 120$ pcf; $\phi = 30°$ (with adequate drainage to be provided);

base friction coefficient $f = 0.5$. With $\sin 30° = 0.5$, from Eqs. (9.4) and (9.5), the soil-pressure coefficients are $C_{ah} = 0.333$ and $C_{ph} = 3.0$. The allowable bearing pressure (see Table 7.1) is assumed to be 4 tons per sq ft. This coarse-grained soil has little compressibility so that the resultant can be allowed to strike near the outer-third point (see Art. 9.4). The weight of the concrete is $w_c = 150$ pcf.

The optimum design of any retaining wall is a matter of successive approximation. Reasonable dimensions are assumed by experience and the various conditions of stability are checked for these dimensions. Usually it is found that the assumed dimensions either provide excessive safety, *i.e.*, the design is uneconomical, or that the required dimensions

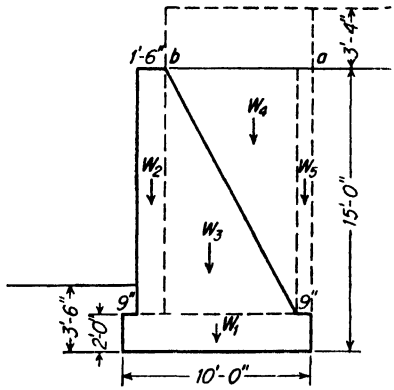



Fig. 9.7

were underestimated so that the wall is unsafe. On the basis of this first trial, dimensions are readjusted correspondingly, and one or two additional trials usually result in a favorable design. In the following, only the final design is analyzed in detail. Preliminary trials have been omitted since the computations required for them are of the same nature as those for the final design. These final dimensions are shown in Fig. 9.7.

The equivalent height of surcharge $h' = {}^{400}/_{120} = 3.33$ ft. From Fig. 9.3c the total earth thrust

$$P = {}^{1}/_{2} \times 0.333 \times 120 \times 15 \times 21.67 = 6500 \text{ lb}$$

and its distance from the base $y = (225 + 150)/(3 \times 21.67) = 5.77$ ft. Hence, the overturning moment $M_o = 6500 \times 5.77 = 37{,}500$ ft-lb. To compute the weight $W$ and its restoring moment $M_r$ about the edge of the toe, individual weights are taken, as shown in Fig. 9.7. With $x$ representing the distance of the line of action of each subweight from the front edge, the following computation obtains:

| | $W$, lb | $x$, ft | $M_r = xW$, ft-lb |
|---|---|---|---|
| $W_1$: 10 × 2 × 150 | 3000 | 5 0 | 15,000 |
| $W_2$: 1.5 × 13 × 150 | 2930 | 1 5 | 4,400 |
| $W_3$: 7½ × 13 × 150 | 6830 | 4 58 | 31,300 |
| $W_4$: 7½ × 13 × 120 | 5460 | 6 82 | 37,200 |
| $W_5$: 0.75 × 13 × 120 | 1170 | 9 62 | 11,250 |
| Total | 19,390 | | 99,150 |

The distance of the resultant from the front edge

$$a = (99{,}150 - 37{,}500)/19{,}390 = 3.18 \text{ ft}$$

which is just outside the middle third. The safety factor against overturning, 99,150/37,500 = 2.64, is ample. From Fig. 9.6c the maximum soil pressure $p = (2 \times 19{,}390)/(3 \times 3.18) = 4070$ psf

The above computations were made for the case that the surcharge extends only to the rear edge of the wall, point $a$ of Fig. 9.7. If the surcharge extends forward to point $b$, the following modification obtains:

$$W = 19{,}390 + 400 \times 7.75 = 22{,}490 \text{ lb}$$
$$M_r = 99{,}150 + 400 \times 7.75 \times 6.13 = 118{,}150 \text{ ft-lb}$$
$$a = \frac{118{,}150 - 37{,}500}{22{,}490} = 3.58 \text{ ft}$$

which is inside the middle third, and, from Fig. 9.6a, the maximum bearing pressure

$$p_1 = (40.0 - 21.5)\frac{22{,}490}{100} = 4170 \text{ lb per sq ft}$$

The situation most conducive to sliding obtains when the surcharge extends only to point $a$, since additional surcharge between $a$ and $b$ would increase the total weight and the corresponding resisting friction. The friction force $F = 0.5 \times 19{,}390 = 9690$ lb. Additionally, sliding is resisted by the passive earth pressure on the front of the wall. Although the base plane is 3.5 ft below grade, the top layer of soil cannot be relied upon to furnish passive pressure since it is frequently loosened by roots and the like or could be scoured out by cloudbursts. For this reason the top 1.5 ft will be discounted in computing the passive pressure, which then becomes $P_p = \frac{1}{2} w_s h^2 C_{ph} = \frac{1}{2} \times 120 \times 4 \times 3.0 = 720$ lb. The safety factor against sliding, (9690 + 720)/6500 = 1.6, is but slightly larger than the required value 1.5, indicating a favorable design.

**9.7. Design of a Cantilever Retaining Wall.** A cantilever wall is to be designed for the same situation as the gravity wall in Art. 9.6. A concrete with $f'_c = 3000$ psi and intermediate-grade steel will be used.

To facilitate computation of weights for checking the stability of the wall, it is advantageous first to ascertain the thickness of the arm and the footing. For this purpose the thickness of the footing is roughly estimated and then the required thickness of the arm is determined at its bottom section. With the bottom of the footing at 3.5 ft below grade

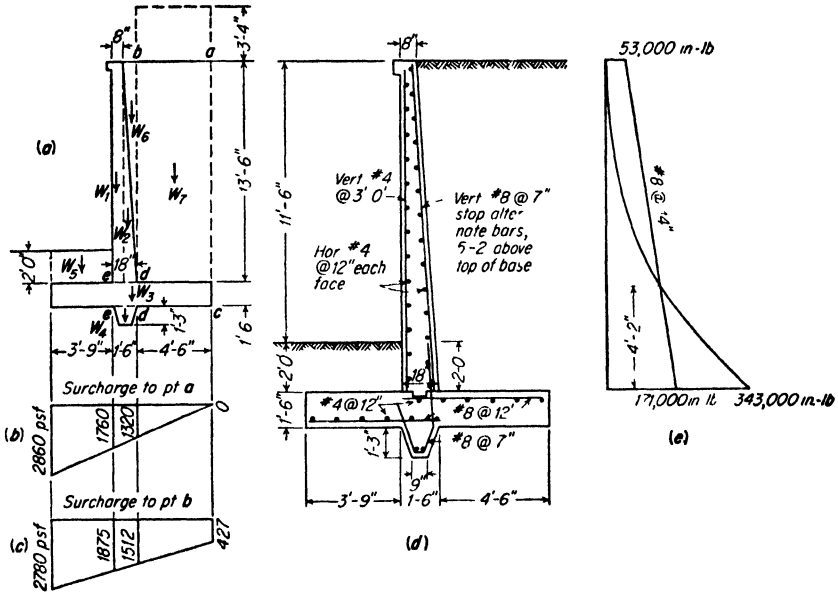



FIG. 9.8

and an estimated footing thickness of 1.5 ft, the free height of the arm is 13.5 ft. Hence, with respect to the bottom of the arm (see Fig. 9.3),

$$P = \tfrac{1}{2} \times 120 \times 13.5 \times 20.16 \times 0.333 = 5440 \text{ lb}$$

$$y = \frac{13.5^2 + 3 \times 13.5 \times 3.33}{3(13.5 + 6.66)} = 5.25 \text{ ft}$$

$$M = 5440 \times 5.25 \times 12 = 343{,}000 \text{ in.-lb}$$

For the given grades of concrete and steel, $K = 235$, and hence the required thickness for bending is $d = \sqrt{343{,}000/(12 \times 235)} = 11.1$ in., and for shear $d = 5440/(12 \times 0.875 \times 90) = 5.2$ in. A protective covering of 3 in. is required for reinforcement in concrete deposited against ground. Hence, the minimum required thickness of the arm at its base is 14 in. This will be increased to 18 in., since the cost of the excess concrete in such structures is usually more than balanced by the simultaneous saving in steel. The thickness of the base is usually the same or slightly larger than that at the bottom of the arm. Hence, the estimated 1.5 ft need not be revised.

Since the moment in the arm decreases with increasing distance from the base and is zero at the top, the arm thickness at the top will be made 8 in. It is now necessary to assume lengths of heel and toe slabs and to check the stability for these assumed dimensions. Intermediate trials are omitted here and the final dimensions are shown in Fig. 9.8*a*. Trial computations have shown that safety against sliding can be achieved only by an excessively long heel or by a key. The latter, requiring the smaller concrete volume, has been adopted.

Weights and moments about the front edge obtain as follows:

| | $W$, lb | $x$, ft | $M_r$, ft-lb |
|---|---|---|---|
| $W_1$: 13.5 × 0.67 × 150 .... | 1,355 | 4.09 | 5,550 |
| $W_2$: 13.5/2 × 0.83 × 150 | 840 | 4.70 | 3,950 |
| $W_3$: 9.75 × 1.5 × 150 | 2,195 | 1.87 | 10,700 |
| $W_4$: (1.25 + 0.67)/2 × 1.5 × 150. | 216 | 4.50 | 970 |
| $W_5$: 3.75 × 2 × 120 | 900 | 1.87 | 1,680 |
| $W_6$: 13.5/2 × 0.83 × 120 | 672 | 4.98 | 3,350 |
| $W_7$: 13.5 × 4.5 × 120 | 7,280 | 7.50 | 54,550 |
| Total | 13,458 | | 79,750 |

The total soil pressure on plane $ac$ is the same as for the gravity wall of Art. 9.6, that is, $P = 6500$ lb, and the overturning moment

$$M_o = 37{,}500 \text{ ft-lb}$$

The distance of the resultant to the front edge is

$$a = (79{,}750 - 37{,}500)\ 13{,}458 = 3.14 \text{ ft}$$

which locates the resultant barely outside the middle third. The corresponding maximum soil pressure at the toe, from Fig. 9.6, is

$$p = 2 \times 13.458\ 3 \times 3.14 = 2860 \text{ psf}$$

The factor of safety against overturning, 79,750 37,500 = 2.12, is ample.

To check the safety against sliding, it will be remembered (Art. 9.4) that if sliding occurs it proceeds between concrete and soil along the heel and key (*i.e.*, length $a - e$ in Fig. 9.5) but takes place within the soil in front of the key (*i.e.*, along $t - e$ in Fig. 9.5). Consequently, the coefficient of friction that applies for the former length is $f = 0.5$, while for the latter it is equal to the internal soil friction, *i.e.*, tan 30° = 0.577.

The bearing-pressure distribution is shown in Fig. 9.8*b*. Actually, since the resultant is at a distance $a = 3.14$ ft from the front, *i.e.*, outside the middle third, formal computation would indicate that the last 4 in. of the heel have zero pressure (see Fig. 9.6). This length is so small

that only a negligible error will be made if, for simplification, it is assumed that the bearing pressure becomes zero exactly at the edge of the heel, as shown in Fig. 9.8*b*.

The resisting force is then computed as the sum of the friction forces of the rear and front portion, plus the passive soil pressure in front of the wall. For the latter, as in Art. 9.6, the top 1.5 ft layer of soil will be discounted as unreliable. Hence

| | |
|---|---|
| Friction, toe: $(2860 + 1760)/2 \times 3.75 \times 0.577$ | = 4930 lb |
| Friction, heel and key: $1760/2 \times 6.0 \times 0.5$ | = 2640 lb |
| Passive earth pressure: $3.25^2/2 \times 120 \times 3.0$ | = 1900 lb |
| Total resistance to sliding | = 9470 lb |

The factor of safety against sliding, 9470/6500 = 1.46, is less than 3 per cent below the recommended value of 1.5 and can be regarded as adequate.

These computations hold for the case that the surcharge extends from the right to point *a* above the edge of the heel. The other case of load distribution, when the surcharge is placed over the entire surface of the fill up to point *b*, evidently does not change the earth pressure on the plane *ac*. It does, however, add to the sum of the vertical forces and will consequently increase both the restoring moment $M$, and the friction along the base. Consequently, the danger of sliding or overturning is greater when the surcharge extends only to *a*, for which situation these two cases have been checked and found adequate. In view of the added vertical load, however, the bearing pressure is largest when the surface is loaded to *b*. For this case,

$$W = 13{,}158 + 100 \times 5.33 = 15{,}590 \text{ lb}$$

$$M_r = 79{,}750 + 100 \times 5.33 \times 7.09 = 94{,}850 \text{ ft-lb}$$

$$a = \frac{94{,}850 - 37{,}500}{15{,}590} = 3.68 \text{ ft}$$

which places the resultant inside the middle third. Hence, from Fig. 9.6,

$$p_1 = (4 \times 9.75 - 6 \times 3.68) \times \frac{15{,}603}{9.75^2} = 2780 \text{ psf}$$

$$p_2 = (6 \times 3.68 - 2 \times 9.75) \times \frac{15{,}603}{9.75^2} = 427 \text{ psf}$$

which is far below the allowable pressure of 4 tons per sq ft. The corresponding bearing-pressure distribution is shown in Fig. 9.8*c*.

The external stability of the wall has now been ascertained, and it remains to determine the required reinforcement and to check internal stresses.

*Arm and Key.* The moment at the bottom section of the arm has previously been determined as $M$ = 343,000 in.-lb and a thickness of

18 in. at the bottom and 8 in. at the top has been selected. With concrete covering of 3 in. clear, $d = 18 - 3.5 = 14.5$ in. The required steel area at the bottom is

$$A_s = \frac{343{,}000}{\frac{7}{8} \times 14.5 \times 20{,}000} = 1.35 \text{ sq in. per ft}$$

The required area is provided by No. 8 bars at 7 in. center to center. In order to facilitate construction, the footing is poured first and a construction joint provided at the base of the arm, as shown in Fig. 9.8*d*. The main bars of the arm, therefore, end at the top of the base slab, and dowels are placed in the latter to be spliced with them. With $u = 300$ psi, the required length of the dowels above the top of the slab is

$$l = \frac{20{,}000 \times 0.785}{3.14 \times 300} = 16.7 \text{ in.}$$

Since the entire strength of the wall depends on the strength of this splice, its length will be made 24 in.

The bending moment in the arm decreases rapidly with increasing distance from the bottom. For this reason only part of the main reinforcement is needed at higher elevations, and alternate bars will be discontinued where no longer needed. To determine the cutoff point, the moment diagram for the arm has been drawn by computing bending moments at two intermediate levels, 10 ft and 5 ft from the top. These two moments, determined in the same manner as that at the base of the arm, were found to be 160,000 and 30,000 in.-lb, respectively. The resisting moment provided by alternate bars, *i.e.*, by No. 8 at 14 in. center to center, at the bottom of the arm is

$$M = \tfrac{7}{8} \times 14.5 \times 0.675 \times 20{,}000 = 171{,}000 \text{ in.-lb}$$

At the top $d = 8.0 - 3.5 = 4.5$ in., and the resisting moment of the same bars is only $M = 171{,}000(4.5/14.5) = 53{,}000$ in.-lb. Hence, the straight line drawn in Fig. 9.8*e* indicates the resisting moment provided at any elevation by half the number of main bars. The intersection of this line with the moment diagram at a distance of 4 ft-2 in. from the bottom represents the point above which alternate bars can be discontinued. The ACI Code provides that bars shall be continued 12 diameters beyond the theoretical cutoff. Hence, alternate bars will be stopped at a distance of 5 ft-2 in. above the top of the base.

(More steel could have been saved by discontinuing, at a higher elevation, one-half of the remaining bars. However, this would have made the spacing of bars at the top equal to 28 in., which is too large. For such an arrangement it would have been necessary to provide, at the base, No. 6 bars at 4 in. center to center for main reinforcement which, with

cutoffs at two elevations, would have resulted in a spacing of 16 in. in the top portion. However, the labor cost of placing almost twice as many bars, including the bending of the correspondingly larger number of dowels, most likely would have outweighed the saving in steel.)

Since the dowels had to be extended at least partly into the key to produce the necessary length of embedment, they were bent as shown to provide reinforcement for the key. The exact force which the key has to resist is difficult to determine since probably the major part of the force acting on the portion of the soil in front of the key is transmitted to it through friction along the base of the footing. The relatively strong reinforcement of the key by means of the extended dowels is considered sufficient to prevent separation from the footing.

The sloping sides of the key were provided in order to facilitate excavation without loosening the adjacent soil. This is necessary in order to ensure proper functioning of the key.

*Base Slab.* Toe and heel are designed for moments and shears caused by the bearing pressure, acting upward, and counteracted by the superimposed weight of fill and surcharge, including the weight of the footing proper. Since both the bearing pressure and the superimposed weights are different for the two cases of surcharge extending to either *a* or *b*, both these conditions must be checked. This has been done and, for this particular wall, it was found that moments and shears were not much different for the two cases but that slightly larger values obtained with surcharge extending to *b*.

On the heel, the combined weights, including surcharge, are

$$13.5 \times 120 + 1.5 \times 150 + 400 = 2245 \text{ lb per ft}$$

The moment and shear produced by this load are counteracted by those from the bearing pressure (Fig. 9.8*c*). Hence,

$$M_d = 2245 \times 4.5^2 - 427 \times 4.5^2 - (1512 - 427) \times \frac{4.5^2}{6}$$

$$= 14{,}750 \text{ ft-lb} = 177{,}000 \text{ in-lb}$$

$$V_d = 2245 \times 4.5 - \frac{1512 + 427}{2} \times 4.5 = 5730 \text{ lb}$$

The 18-in. thickness is ample for concrete bending and shear stresses, as can be seen without computation by comparison with the corresponding values at the base of the arm whose thickness, at that point, is equal to that of the footing. The required steel area

$$A_s = \frac{177{,}000}{14.5 \times \frac{7}{8} \times 20{,}000} = 0.70 \text{ sq in. per ft}$$

is provided by No. 8 bars at 12 in. center to center = 0.78 sq in. per ft.

For the *toe*, the fill above it will be neglected as being subject to possible erosion. Hence, with only the weight of the footing

$$1.5 \times 150 = 225 \text{ lb per ft}$$

acting downward,

$$M_c = 1875 \times \frac{3.75^2}{2} + (2780 - 1875) \times \frac{3.75^2}{3} - 225 \times \frac{3.75^2}{2}$$

$$= 15{,}800 \text{ ft-lb} = 189{,}500 \text{ in.-lb}$$

$$V_c = \frac{2780 + 1875}{2} \times 3.75 - 225 \times 3.75 = 7890 \text{ lb}$$

Comparison with the moment in the arm, 313,000 in.-lb, again shows the thickness to be ample for bending. The shear stress

$$v = \frac{7890}{\frac{7}{8} \times 11.5 \times 12} = 52 \text{ psi}$$

is smaller than the allowable value $v_c = 90$ psi. The steel area

$$A_s = \frac{189{,}500}{14.5 \times \frac{7}{8} \times 20{,}000} = 0.75 \text{ sq in. per ft}$$

will again be provided by No. 8 bars at 12 in. center to center = 0.78 sq in. per ft. The required length of embedment beyond points $e$ and $d$, the same as for the No. 8 dowels, is provided as shown. The No. 4 bars indicated in Fig. 9.8*d* furnish the shrinkage reinforcement and also serve to space the main reinforcement. Drainage is to be secured by one of the methods shown in Fig. 9.1, depending on local conditions.

**9.8. Design of Counterfort Retaining Walls.** The *external stability* of a counterfort retaining wall is determined in the same manner as in the examples of Arts. 9.6 and 9.7. The *toe slab* represents a cantilever built in along the front face of the wall, loaded upward by the bearing pressure, exactly as in the cantilever wall of Art. 9.7. Reinforcement is provided by bars *a*, Fig. 9.9.

A panel of the *vertical wall* between two counterforts is a slab acted upon by the horizontal earth pressure and supported along three sides, *i.e.*, at the two counterforts and the base slab, while the fourth side, the top edge, is not supported. The earth pressure, of course, increases linearly with distance from the free surface. The exact determination of moments and shears in such a slab supported on three sides and nonuniformly loaded is rather involved.[1] It is customary in the design of such walls to disregard the support of the vertical wall by the base slab

[1] For design data on such slabs, which also occur in tanks and bins, see, "Rectangular Concrete Tanks," *Concrete Information* ST63, Portland Cement Association, 1943.

and to design it as if it were a continuous slab spanning horizontally between counterforts. This procedure is conservative, because the moments obtained by this approximation are larger than those corresponding to the actual conditions of support, particularly in the lower part of the wall. Hence, for very large installations, significant savings may be achieved by a more accurate analysis.

Slab moments are determined for strips 1 ft wide vertically, usually for the strip at the bottom of the wall and for three or four equally spaced

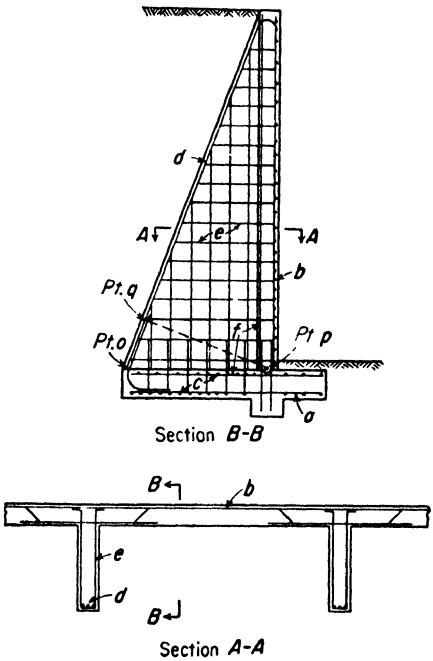



FIG. 9.9. Details of counterfort retaining wall.

additional strips at higher elevations. The earth pressure on the different strips decreases with increasing elevation and is determined from Eq. (9.1). Moments and shears cannot be determined from the recommendations of the ACI Code (see Art. 3.6), since the entire horizontal load, *i.e.*, the earth pressure, must be regarded as live load. On the other hand, an exact analysis by moment distribution or some other method, is not warranted in view of (*a*) the approximate assumptions of the design as just discussed and (*b*) the uncertain magnitude of the earth pressure, which depends, among other influences, on moisture content and may vary from one panel to the next. The authors consider it adequate to assume the positive moments midway between and the negative moments at the counterforts to be equal and to use for these moments

$$M = \frac{pl^2}{12}$$

for the two lowermost strips and

$$M = \frac{pl^2}{10}$$

for the strips at higher elevations. The smaller moment value for the bottom strips accounts for the fact that the additional support provided by the base slab decreases the moments in the bottom portions of the slab as compared with those which would obtain if this support were absent. The shear force at each support of a strip is simply computed from $V = pl/2$. Horizontal bars $b$ (Fig. 9.9) are provided as required by these moments, with increased spacing or decreased diameter at higher elevations, corresponding to the smaller moments. Alternate bars are bent up to provide for the negative moments at the counterforts, or additional straight bars are placed for them.

The *heel slab* is supported in a manner similar to that of the wall slab, *i.e.*, by the counterforts and at the wall. It is loaded downward by the weight of the fill resting on it, its own weight, and such surcharge as there may be. This load is counteracted by the bearing pressure. As in the vertical wall, a simplified analysis consists of neglecting the influence of the support along the third side and of determining moments and shears for strips parallel to the wall, each strip representing a continuous beam supported at the counterforts. The same formulas for moments and shears can be used that were given for the wall slab. With horizontal soil surface, the downward load is constant for the entire heel, whereas the upward load from the bearing pressure is usually smallest at the rear edge and increases frontward. For this reason the span moments are positive (compression on top) and the support moments negative in the rear portion of the heel. Near the wall the bearing pressure often exceeds the vertical weights, resulting in a net upward load. The signs of the moments are correspondingly inverted and steel must be placed accordingly, *i.e.*, near the top face in the span and near the bottom face at the supports. Bars $c$ are provided for these moments.

The *counterforts* are wedge-shaped cantilevers built in at the bottom in the base slab. They support the wall slab and, therefore, are loaded by the total soil pressure over a length equal to the distance center to center between counterforts. They act as a T beam of which the wall slab is the flange and the counterfort the stem. For this reason concrete stresses are always low and need not be checked. The maximum bending moment is that of the total earth pressure, taken about the bottom of the wall slab. This moment is held in equilibrium by the force in the bars $d$, and hence the lever arm of internal forces $jd$ is the perpendicular

distance $pq$, from the center of bars $d$ to the center of the bottom section of the wall slab. Since the moment decreases rapidly in the upper parts of the counterfort, part of bars $d$ can be discontinued. To find the cut-off points, it is simplest to use the same graphical method which was employed for the arm of the cantilever wall (Fig. 9.8*e*).

Shear stresses are checked in the base section so that $jd$ is here represented by the horizontal distance $op$ from the center of bars $d$ to the center of the wall slab. However, some modification of the customary shear stress computation is required for wedge-shaped members In general, such stresses, on a section parallel to the external load such as $op$, are determined by the usual formula $v_c = V_1 \, bjd$, where

$$V_1 = V \pm \frac{M}{d} (\tan \theta + \tan \phi) \tag{9.9}$$

$V$ and $M$ are the shear and moment at the section, and $\theta$ and $\phi$ are the angles which the longitudinal edges subtend with a line perpendicular to the section.[1] The minus sign is to be used when the depth increases with increasing bending moment, as is the case in counterforts For the example of Fig. 9.9, one of the two angles, say $\theta$, is zero for the front edge, while $\phi$ is the angle between the vertical and the rear edge. Usually concrete alone is sufficient to carry this shear, although bars $e$ act as stirrups and can be used for resisting excess shear.

The main purpose of bars $e$ is to counteract the pull of the wall slab, and they are thus designed for the full reaction of this slab.

The remaining bars of Fig. 9.9 serve as shrinkage reinforcement, except that bars $f$ have an important additional function. It will be recalled that the wall and heel slabs are supported on three sides. Even though they were designed as if supported only by the counterforts, they develop moments where they join. The resulting tension in and near the reentrant corner is conducive to cracking, which is counteracted by bars $f$.

[1] See "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," Report of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, *Proc. ASCE*, part 2, pp. 55, 56, June, 1940.

# SECTION 10

# ARCHES

**10.1. General.** The use of arches is the oldest structural method for bridging spans too large for straight beams. Arches were introduced by the Romans for their bridges, aqueducts, and large public buildings more than 2000 years ago. In the form of masonry, concrete, steel, and timber arches, they have been in wide use ever since. At the present time reinforced concrete arches find their application not only in bridge building but increasingly in long-span, fireproof roofs, such as in hangars, auditoriums, sports arenas, warehouses, industrial plants, and the like.

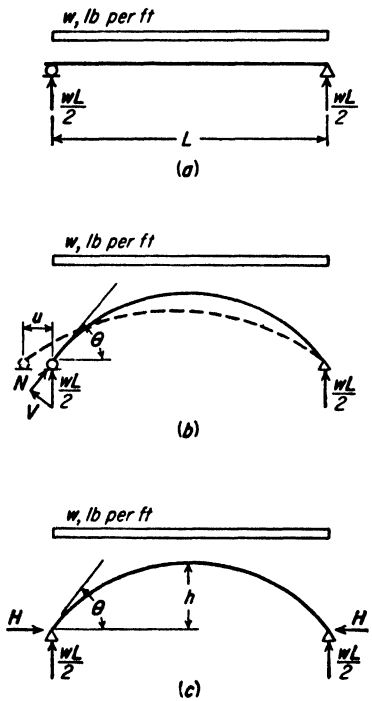


FIG. 10.1

Modern arches are of three types, the fixed arch, which is rigidly connected to the abutments, the two-hinged arch, which is supported at each end by a hinge resting on the abutment; and the three-hinged arch, which has an additional hinge at the crown. Reinforced concrete arches are mostly of the first two types, though three-hinged arches have been built very successfully, particularly the daring concrete bridges designed by Maillart in Switzerland.

The structural action and advantage of arches are easily recognized by considering a two-hinged arch. Figure 10.1*a* shows a uniformly loaded simple beam, with $M_{max} = wL^2/8$, $V_{max} = wL/2$, and axial force $N = 0$. If this beam is curved, as in Fig. 10.1*b*, $M_{max}$ is the same as before. The shear at the support, however, is decreased to $V_{max} = (wL/2)\cos\theta$ and,

in addition, an axial force is now present in the member which, at the supports, equals $N = (wL/2) \sin \theta$. This curved beam tends to straighten under load since, owing to bending, the bottom fibers are stretched, the top fibers compressed. This action tends to increase the span, as shown in Fig. 10.1*b*. To maintain the original span, as in the two-hinged arch with immovable supports (Fig. 10.1*c*), a horizontal reaction $H$ must be applied. In this two-hinged arch obviously

$$M_{max} = \frac{wL^2}{8} - Hh \qquad V_{max} = \frac{wL}{2} \cos \theta - H \sin \theta$$

$$N = \frac{wL}{2} \sin \theta + H \cos \theta$$

It is seen that, in comparison with the simple beam, the bending moment and the shear force in the arch are greatly decreased. On the other hand, an axial compression force $N$, not present in the beam, is acting in the arch. It is exactly this state of affairs which is desirable in reinforced concrete, since concrete is relatively weak in tension and shear but strong in compression. In carrying the normal tension caused by moments and the diagonal tension caused by shear, the concrete in a section is largely ineffective, while it could be fully utilized in resisting axial compression. This is exactly what is achieved in arch action, since moments and shears are greatly decreased and loads are carried primarily by axial compression. The aim of arch design, therefore, is to eliminate moments and shears as completely as possible, which can be achieved by judicious selection of the shape of the arch.

The designations of the various parts of an arch are given in Fig 10.2*a* which shows an open-spandrel fixed arch bridge. The space between the back of the arch (the curve of the extrados) and the roadway is known as the spandrel. The roadway can be carried by a beam-and-girder floor system resting on columns which, in turn, are supported on individual arch ribs, as shown in Fig. 10.2*a*. Alternatively, bridge roadways are frequently supported on barrel arches whose width is equal to that of the entire bridge, as shown in Fig. 10.2*b*. The roadway in this type is supported on earth fill which is confined laterally by spandrel walls. Structures of this type are known as spandrel-filled arches.

The choice of two-hinged or fixed arches depends on the particular conditions of design. Fixed arches require immovable abutments which prevent not only vertical and horizontal motion but also rotation of the springing line. If rock foundations are available, this is not difficult to achieve, and in this case the fixed arch is the simplest and most economical solution. In compressible soils nonyielding abutments are difficult to obtain and must often be supported on elaborate pile foundations, including batter piles. For roof arches resting on columns or walls, it

is obviously impossible to fix the springings against rotation, since the rigidity of the supporting structures is not sufficient for this purpose. Two-hinged arches, on the other hand, are insensitive to rotations and to moderate differential settlements of the abutments which, for this reason, can often be made less substantial than for fixed arches. Tie rods are

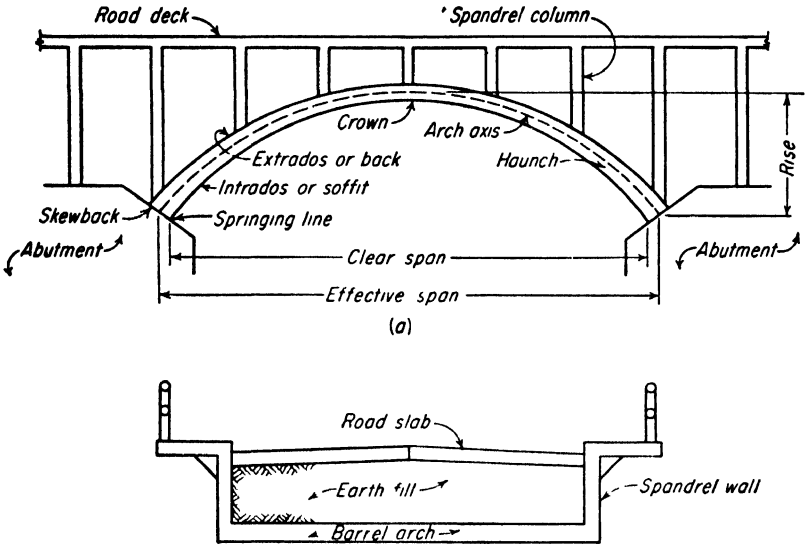



FIG 10 2 Arch bridges

often provided in two-hinged arches to resist the horizontal reaction $H$; hence the supports have to resist vertical reactions only, with a corresponding further decrease in their cost. Tied two-hinged arches are obviously advantageous in roof structures of the type shown in Fig. 10.3, where large horizontal reactions would require much heavier columns and frames. Large hangars and halls of the type of Fig. 10.4 have been

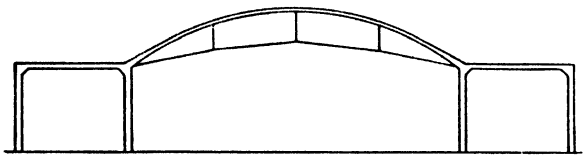


FIG 10.3. Two-hinge roof arch with cambered tie rod.

built both with fixed arches and with tied two-hinged arches; in the latter case tie rods were located below the floor. In contrast to the arch ribs of the previous examples, barrel-arch roofs of the type shown in Fig. 10.5 can be used for moderate spans of fireproof warehouses, garages, and the like. A strip of unit width can be designed as a two-hinged arch.

Horizontal beams are provided at the edges and transfer the horizontal thrust to tie rods spaced at reasonable intervals. The use of tie rods is often impossible, as in most bridges, or architecturally objectionable, as in monumental halls, or may be ruled out by vertical clearance requirements, as in some industrial and warehouse structures. Additional vertical clearance can be gained by the use of cambered tie rods, shown in

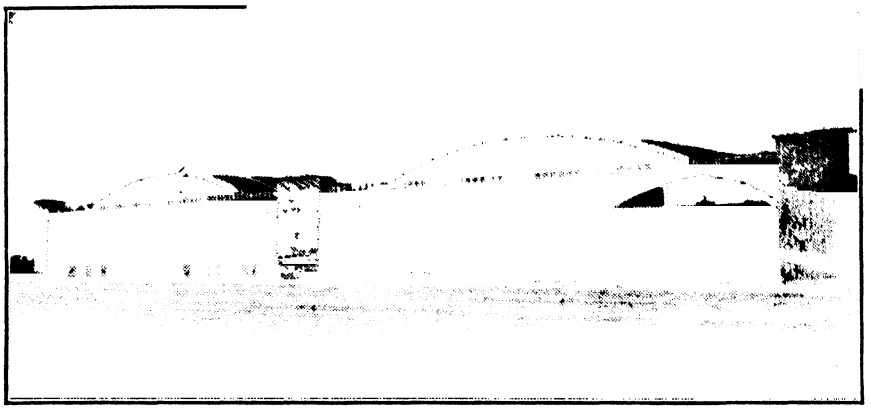


FIG. 10.4. Hangar at Chicago Municipal Airport. Fixed arches spaced at 28 ft-11½-in., clear span 257 ft, height at center 60 ft, thickness of concrete shell between arches varies from 3½ to 6 in. (*Courtesy of Ammann and Whitney.*)

Fig. 10.3. Effective hinges can be constructed at reasonable cost, provided that the reactions are not excessively large. They may be architecturally undesirable, however, and represent a maintenance problem in some cases. A well-considered decision on the type of arch to be used in a given design depends, therefore, on a balanced evaluation of factors such as those just enumerated.

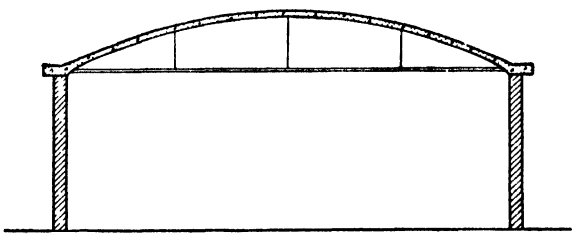


FIG. 10.5 Barrel-arch roof.

As was noted in the general discussion of arch action (Fig. 10.1), arch ribs or barrels are subject to compression, bending, and, usually small, shear. They are, therefore, designed and analyzed in the same manner as eccentrically compressed columns, with one layer of steel being located near the extrados and another near the intrados. It is economical to keep steel percentages small, on the order of 1 to 2 per cent. Lateral

ties for the reinforcement are used in arch ribs as in columns. In barrel arches, transverse reinforcement normal to the main steel is placed near both surfaces and is tied to the main longitudinal steel. This distributes localized loads and acts as temperature and shrinkage reinforcement.

**10.2. Theory of Symmetrical, Two-hinged Arches.** The two-hinged arch (Fig. 10.1*c*) differs from a curved simple beam (Fig. 10.1*b*) in that horizontally immovable hinges or tie rods prevent the spread of supports. The forces in such an arch can be analyzed by introducing a fictitious

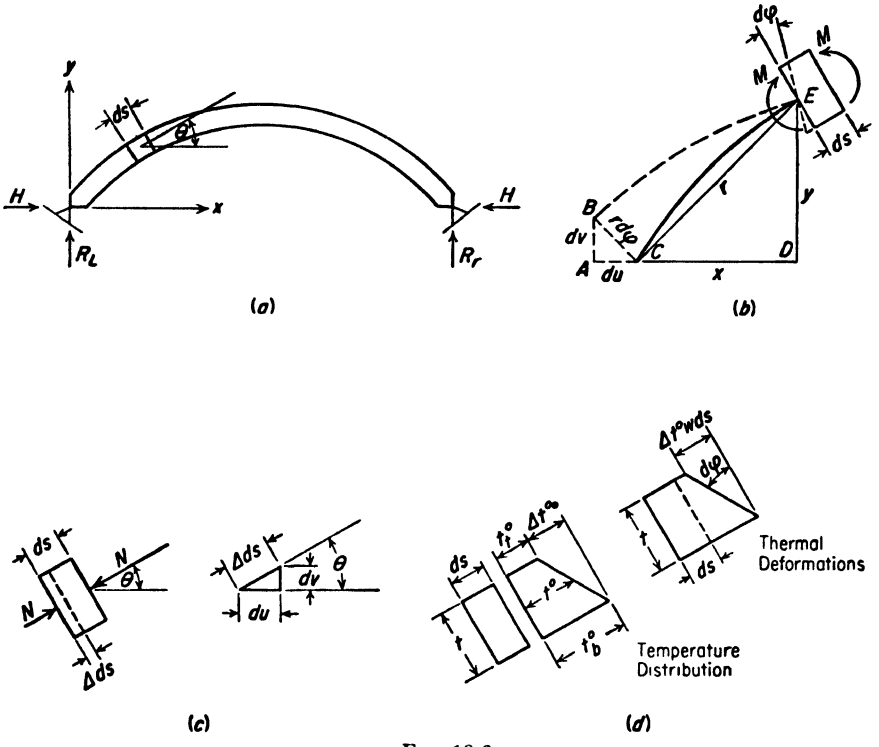



FIG. 10.6

roller support at one of the hinges which converts the arch into a simple beam. The horizontal displacement $u$ can then be determined. To convert this fictitious beam into the actual arch, a horizontal force $H$ must be introduced of sufficient magnitude to produce a displacement equal and opposite to $u$. In other words, this force brings the fictitious roller support back to its original position as it actually is in the arch.

To determine the horizontal displacement $u$, take a small element $ds$ (Fig. 10.6*a*) of the arch, and consider the effect of the deformation of this one element on the deformation of the arch as a whole. Thus all parts of the arch with the exception of this one element are considered as rigid.

The contribution of the deformation of the element $ds$ to the total displacement can be determined as follows. A bending moment $M$, considered as constant over the small length $ds$, will cause the left cross section to rotate with respect to the right one by an amount $d\phi$ which can be determined by the moment-area principle, *i.e.*, from the fact that the rotation of one section with respect to another is equal to the area of the moment curve between them divided by $EI$. Hence

$$d\phi = \frac{M\,ds}{EI} \tag{10.1}$$

The entire left portion of the arch, rigidly attached as it is to the cross section, will likewise rotate through an angle $d\phi$, as represented by the dotted position of the axis in Fig. 10.6*b*. Consequently, the left end of the arch describes an arc $r\,d\phi$. It will be noted that triangle $ABC$ is similar to triangle $CDE$. Hence, from simple proportions, $du : r\,d\phi = y : r$ and the horizontal displacement

$$du = y\,d\phi = \frac{My\,ds}{EI} \tag{10.1a}$$

This is the contribution made to the total displacement $u$ by the angular deformation due to bending of the single element $ds$. To obtain $u$ itself, it is only necessary to sum up these contributions for all elements $ds$ along the arch, or

$$u = \int_0^s \frac{My\,ds}{EI} \tag{10.2}$$

Let $M_0$ represent the bending moment due to any external arch loads caused in the fictitious simple curved beam, such as Fig. 10.1*b*. These moments, of course, are the same as those in the straight beam, Fig. 10.1*a*. Then the outward displacement due to these "simple beam moments" is

$$u_{M_0} = \int^s \frac{M_0 y\,ds}{EI} \tag{10.2a}$$

The sign convention is the same as in beam analysis; *i.e.*, moments are considered positive if they produce tension in the bottom fiber. In addition to $M_0$ the horizontal reaction $H$ causes a moment at section $x$ which is equal to

$$M_H = -Hy$$

and the inward displacement caused by these moments is

$$u_{M_H} = -H \int_0^s \frac{y^2\,ds}{EI} \tag{10.2b}$$

Since, actually, no displacement occurs at the immovable hinge,

$$u_{M_0} = -u_{M_H}$$

from which one obtains the magnitude of the horizontal reaction

$$H = -\frac{\int_0^s \frac{M_0 y\, ds}{EI}}{\int_0^s \frac{y^2\, ds}{EI}} \tag{10.3}$$

If displacements $u$ were caused exclusively by bending, Eq. (10.3) would furnish all the information necessary for analyzing a two-hinged arch. Since such an arch, as is seen from Fig. 10.1, has only one statically indeterminate reaction, $H$, it is now possible to find moments, shears, and axial forces at any point in the arch from equilibrium conditions only, since the previously unknown force $H$ is given by Eq. (10.3). However, other factors in addition to bending moments may contribute significantly to the magnitude of $u$ and must therefore be considered in determining $H$.

In addition to bending moments, shear forces and axial thrusts act on element $ds$. Deformations due to shear in arches are even smaller, relatively, than in beams and can therefore be neglected. Those caused by axial compression forces $N$, on the other hand, may affect the magnitude of $H$ significantly and must therefore be considered.

Consider an element $ds$ with a concentric force $N$ on both faces, as shown in Fig. 10.6c. The axial shortening $\Delta\, ds$ caused by the normal force $N_0$ is $N_0\, ds/EA$. $N_0$ is the component parallel to the arch axis at that point of all external forces and reactions on the fictitious curved beam to the right or left of the element $ds$. The component of $\Delta\, ds$ in the direction of $u$ is

$$du = \Delta ds \cos\theta \tag{10.4}$$

Hence

$$du_{N_0} = \frac{N_0 \cos\theta\, ds}{EA}$$

and the total displacement due to $N_0$ is

$$u_{N_0} = -\int_0^s \frac{N_0 \cos\theta\, ds}{EA} \tag{10.5}$$

The minus sign indicates that this displacement is inward, in contrast to the outward displacement $u_{M_0}$. On the other hand, the horizontal reaction $H$ results in an axial compression at any point of the arch of magnitude $H \cos\theta$; thus the inward displacement caused by this action is

$$u_{N_H} = -H \int_0^s \frac{\cos^2\theta\, ds}{EA} \tag{10.6}$$

The temperature changes which are inevitable in outdoor arches such as bridges, and which may also occur in roof arches, produce a further deformation of the arch. A uniform temperature increase of $t°$ causes the arch element $ds$ to increase in length by the amount

$$ds = \omega t° \, ds$$

where $\omega$ is the temperature coefficient of expansion. Substitution of this expression in Eq. (10.4) and integration over the entire arch gives the total support displacement due to a uniform temperature increase,

$$u_t = \omega t° \int_0^s \cos \theta \, ds = \omega t° l \tag{10.7}$$

Another factor which results in inward displacement $u$ is the inevitable shrinkage of concrete. It has been shown, however, that the unfavorable effect of shrinkage is greatly decreased by the long-time plastic flow of concrete, which allows the structure to adapt itself to those stresses which are caused by change of shape, such as shrinkage, rather than by external loads.[1]

It is proposed in this report that the limited effect of shrinkage can be accounted for by an equivalent drop in temperature of 15°F which is to be added to the actual, seasonal temperature drop for which the arch is designed. The same report suggests that temperature changes for arch bridges be taken as a rise and fall equal to 30 per cent (a total range of 60 per cent) of the difference between maximum and minimum recorded temperature at the locality. The temperature coefficient may be taken as $\omega = 0.0000055$ per degree Fahrenheit. This temperature range is applicable to outdoor arches, such as bridges. For roof arches, the temperature regime of the projected building must be considered.

In addition to the uniform temperature change just considered, a temperature differential may exist between the extrados and the intrados in roof arches, particularly if, as in the hangar of Fig. 10.4, the roof slab or shell is placed at mid-depth of the arch ribs. The total temperature changes and corresponding deformations may then be represented as in Fig. 10.6*d*. The amount by which the bottom fiber $ds$ expands more than the top fiber is $\Delta t° \, \omega \, ds$. It is seen that this results in

$$d\phi = \frac{\Delta t° \, \omega \, ds}{t} \tag{10.8}$$

If this value is substituted in the left equality of Eq. (10.1*a*) and integrated over the arch, the total displacement caused by the temperature gradient becomes

[1] See "Plain and Reinforced Concrete Arches," Report of Committee 312, C. S. Whitney, chairman, *J. ACI*, vol. 12, no. 1 (*Proc. ACI*, vol. 37) September, 1940.

$$u_{\Delta t^\circ} = \omega \, \Delta t^\circ \int_0^s \frac{y \, ds}{t} \tag{10.9}$$

These are all the types of displacements that occur in an arch with immovable hinges. In arches with tie rods, however, hinges are not immovable since the force $H$ in the tie rod causes it to stretch. For a tie rod of area $A_{tr}$ and modulus $E_{tr}$, this stretch amounts to

$$u_{tr} = H \frac{l}{E_{tr} A_{tr}} \tag{10.10}$$

For a tied arch, the sum of all other displacements must be equal to the stretch of the tie, *i.e.*,

$$u_{M_0} + u_{M_H} + u_{N_0} + u_{N_H} + u_{t^\circ} + u_{\Delta t^\circ} = u_{tr}$$

If the proper values are substituted in this equation, the unknown horizontal force $H$ may be found:

$$H = \frac{\displaystyle\int_0^s \frac{M_0 y \, ds}{I} - \int_0^s \frac{N_0 \cos\theta \, ds}{A} + E\omega \left[ t^\circ l + \Delta t^\circ \int_0^s \frac{y \, ds}{t} \right]}{\displaystyle\int_0^s \frac{y^2 \, ds}{I} + \int_0^s \frac{\cos^2\theta \, ds}{A} + \frac{El}{E_{tr} A_{tr}}} \tag{10.11}$$

This lengthy expression can usually be simplified by omitting terms which are either zero or negligibly small, depending on the particular condition. Thus, for an arch without tie rod the last term in the denominator drops out since $u_{tr} = 0$. If no temperature gradient can occur between extrados and intrados, such as in bridge arches or in roof arches located below a well-insulated roof surface, the term containing $\Delta t^\circ$ is zero. Furthermore, in many cases the displacements $u_{N_0}$ and $u_{N_H}$ caused by axial compression are very small as compared to those caused by bending moments ($u_{M_0}$ and $u_{M_H}$). For a parabolic arch, S. Timoshenko[1] has computed the amount of error obtained by neglecting the influence of axial compression in determining $H$. His results are given in the table below

| $h/l$ | $\frac{1}{12}$ | | | $\frac{1}{8}$ | | | $\frac{1}{4}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $t_c/l$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{30}$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{30}$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{30}$ |
| Error, % | 17.1 | 5.1 | 2.4 | 8 4 | 2 2 | 1 0 | 1 8 | 0 4 | 0.2 |

where $t_c$ = thickness at crown. Considering that the usual accuracy of engineering designs is hardly greater than 5 per cent on account of uncer-

[1] S. Timoshenko, "Strength of Materials," part II, p. 96, D. Van Nostrand Company, Inc., New York, 1941.

tainties of loading, materials properties, and various other factors, it is seen that the influence of axial compression can be neglected in all but very flat and/or unusually thick arches. Normally then, the second term in both the numerator and denominator can be dropped as being negligibly small. Hence, for an arch of usual dimensions without tie rod and without temperature gradient, $H$ can be determined with sufficient accuracy from

$$H = \frac{\int_0^s \frac{M_0 y}{I}\,ds - E\omega t^\circ l}{\int_0^s \frac{y^2}{I}\,ds} \qquad (10.11a)$$

Naturally the terms for temperature gradient $\Delta t^\circ$ and tie rod must be included if such are present in the structure.

The determination of the integrals in the equation for $H$ is possible, provided that $M_0$, $N_0$, $I$, $A$, $h$, and $y$ can be expressed as functions of $s$ and that these functions can be integrated. Ordinarily this is possible for only a few very simple cases with the arch axis as either a parabola or a circular arc. In most other cases it becomes necessary to approximate these integrals by a finite sum of numerical terms. The details of this procedure will be given after the theory of fixed arches is developed. Once $H$ has been found, the problem is reduced to a statically determinate one and $M$, $N$, and $V$ at any section of the arch can be found from the conditions of equilibrium.

**10.3. Theory of Symmetrical Fixed Arches.** The fixed arch shown in Fig. 10.7*a* has a total of six reactions, *i.e.*, a vertical and a horizontal reaction and a moment at each springing. There are, then, three redundant reactions to be determined. If no horizontal loads are present, the condition that the sum of horizontal forces be zero results in $H_l = H_r$.

For purposes of analysis the arch is cut through the crown and each half arch is regarded as a statically determinate cantilever fixed at its springing. In the plane of separation, in general, a horizontal and a vertical force $H_c$ and $V_c$ and a bending moment $M_c$ will act from one half onto the other. These three quantities are taken as the redundants. They are shown, in Fig. 10.7*b*, acting in the direction and sense which will be assumed as positive in the following derivations. In any other cross section of the structure, $M$, $N$, and $V$ will be assumed positive, as shown in Fig. 10.7*b*. If $M_c$, $H_c$, and $V_c$ are known, $M$, $N$, and $V$ at any point can be determined from statics.

The crown sections of the separate halves, through the action of loads, temperature, and other influences, will be displaced from their original positions. In general, there will be a horizontal displacement $u$, a vertical displacement $v$, and a rotation of the normal to the crown sec-

tion $\phi$. The directions of these displacements which are assumed positive are shown in Fig. 10.7*a*. If the structure actually consisted of two independent cantilevers, $u$, $v$, and $\phi$ would, in general, be different for the two halves. Since, actually, the arch is one continuous structure, in reality these displacements must be equal in magnitude and sign. Consequently,

$$\begin{aligned} v_l &= u_r \\ v_l &= v_r \\ \phi_l &= \phi_r \end{aligned} \tag{10.12}$$

where the subscripts $l$ and $r$ refer to the left and right half arch, respectively. These three equations permit the determination of the redundants $H_c$, $V_c$, and $M_c$. In other words, in the arch these three quantities

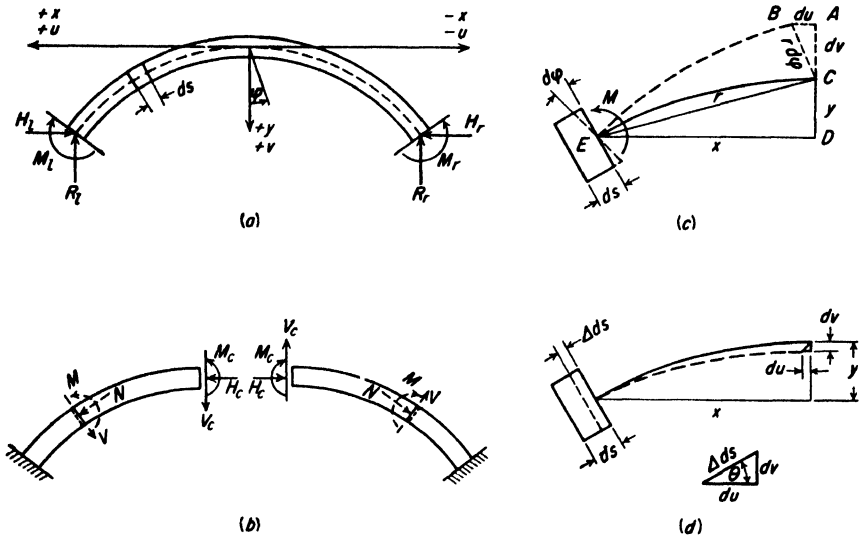



Fig. 10.7

must be of such magnitude as to join the left and the right cantilevers into one continuous structure.

Isolating an element $ds$ of the arch, Fig. 10.7*c*, one can determine the contributions $du$, $dv$, $d\phi$ made to the total displacements $u$, $v$, $\phi$ by the deformations of this single element. The bending moment $M$ causes a rotation $d\phi$ which is determined by Eq. (10.1) of Art. 10.2. The part of the arch between the section and the crown, considered rigid and represented by its axis in Fig. 10.7*c*, rotates through the same angle $d\phi$, which, consequently, also represents the rotation of the crown section caused by the deformation of the element $ds$. Furthermore, since the triangles $ABC$ and $CDE$ are similar, it is seen that

$$\frac{du}{r\,d\phi} = \frac{y}{r} \qquad \text{and} \qquad \frac{dv}{r\,d\phi} = \frac{x}{r}$$

from which

$$du = y\,d\phi \qquad dv = x\,d\phi \tag{10.13}$$

Substituting $d\phi$ from Eq. (10.1), one obtains

$$du = \pm\frac{My\,ds}{EI} \qquad dv = \mp\frac{Mx\,ds}{EI} \qquad d\phi = \pm\frac{M\,ds}{EI} \tag{10.13a}$$

In these and the following expressions, if two signs are shown, the upper refers to the left and the lower to the right half arch.

In addition to the angular rotation $d\phi$, the elements $ds$ will also be uniformly compressed or stretched by normal forces $N$, temperature variations, and shrinkage. If $\Delta ds$ is the change of $ds$ due to any of these actions, it is seen, from Fig. 10.7*d*, that

$$du = \Delta ds \cos\theta \qquad dv = \Delta ds \sin\theta \tag{10.14}$$

An axial force $N$ results in $\Delta ds = N\,ds\,/EA$, and hence in horizontal and vertical displacements

$$du = \frac{N\,ds\cos\theta}{EA} \qquad dv = \frac{N\,ds\sin\theta}{EA} \tag{10.14a}$$

At any point of the arch, $x$, $y$, the moment $M$, and the axial force $N$ are the resultant effects of the three crown quantities $M_c$, $H_c$, and $V_c$ and of the external loads acting on the half-arch cantilever in which the particular section is located. They can be calculated by statics (see Fig. 10.7*b*) and are tabulated below

**Force and Moment at Section $x$, $y$**

| Due to | $M_c$ | $H_c$ | $V_c$ | External loads |
|---|---|---|---|---|
| $M$ | $M_c$ | $H_c y$ | $-V_c x$ | $M_0$ |
| $N$ | 0 | $H_c\cos\theta$ | $V_c\sin\theta$ | $N_0$ |

The signs in this table are valid for both halves of the arch. In this connection it is important to note that for the right half arch $x$ and $\theta$ are negative, whereas $y$ is positive by the sign convention given in Fig. 10.7*a*. It should also be noted that external gravity loads result in negative $M_0$ (see Fig. 10.7*b*). $N_0$ at any section is the component in the direction of the tangent to the arch axis of all the external loads located between that section and the crown; gravity loads therefore result in positive $N_0$.

By substituting the values $M$ and $N$ from the table into Eqs. (10.13*a*) and (10.14*a*) and integrating for the left half from 0 to $+s/2$, and for the right half from 0 to $-s/2$, one obtains separately the crown displace-

ment $u$ and $v$ for either half. If these are substituted in Eqs. (10.12), they result in three simultaneous equations for $H_c$, $M_c$, and $V_c$. If any particular action, such as, for example, that of $H_c$, results in equal displacements of both halves, these can obviously be omitted from Eqs. (10.12), since equal terms on both sides of an equation cancel. For example, $H_c$ results in equal vertical displacements $v_l$ and $v_r$, as is seen directly from Fig. 10.7*c*, considering that the corresponding behavior of the right half is obtained by mirroring the entire figure about the vertical axis through the crown. Hence, the contribution of $H_c$ to the vertical displacements can be omitted in the second of Eqs. (10.12). On the other hand, the horizontal crown displacements caused by $H_c$ are equal in magnitude but opposite in sign, since the crown section of the left cantilever is displaced to the left (see Fig. 10.7*b*) and that of the right one to the right. Hence, these displacements must be entered in the first of Eqs. (10.12).

In Table 10.1 are given the crown displacements caused by the various actions, separately for both halves. They must be substituted in Eqs. (10.12) to determine the three redundants. In this table the notation "cancels" designates displacements equal in sign and magnitude for both halves and which, consequently, cancel if substituted in Eqs. (10.12). For simplicity, all values in the table are multiplied by $E$.

**TABLE 10.1. Crown Displacements of Half Arch of Fig. 10.7**

All Integrals $\int_0^{+s/2}$ for left; $\int_0^{-s/2}$ for right half; multiply by $\frac{1}{E}$

Caused by bending

| | $+M_c$ | $+H_c$ | $+V_c$ | $+M_0$ |
|---|---|---|---|---|
| $u$ | $+M_c\int \frac{y\,ds}{I}$* | $\pm H_c\int \frac{y^2\,ds}{I}$ | Cancels | $+\int \frac{M_0 y\,ds}{I}$ |
| $v$ | Cancels | Cancels | $+V_c\int \frac{x^2\,ds}{I}$ | $+\int \frac{M_0 x\,ds}{I}$ |
| $\phi$ | $\pm M_c\int \frac{ds}{I}$ | $+H_c\int \frac{y\,ds}{I}$* | Cancels | $+\int \frac{M_0\,ds}{I}$ |

Caused by arch shortening and temperature

| | $+H_c$ | $+V_c$ | $+N_0$ | $+t^\circ$ | $+\Delta t^\circ$ |
|---|---|---|---|---|---|
| $u$ | $\pm H_c\int \frac{\cos^2\theta\,ds}{A}$ | Cancels | $\pm N_0\int \frac{\cos\theta\,ds}{A}$ | $\mp \omega t^\circ E\int \cos\theta\,ds$ | Cancels |
| $v$ | Cancels | $+V_c\int \frac{\sin^2\theta\,ds}{A}$ | Cancels | Cancels | $+\Delta t^\circ\,\omega E\int \frac{x\,ds}{t}$ |
| $\phi$ | 0 | 0 | 0 | 0 | $\pm\Delta t^\circ\,\omega E\int \frac{ds}{t}$ |

* Expressions indicated by asterisk are zero when $x$ and $y$ are measured from the elastic center.

In addition to the displacements caused by the crown forces and the external loads (which include the weight of the arch), those resulting from temperature changes and shrinkage must be considered just as in the case of the two-hinged arch. In fact, these influences are of more consequence for the fixed than for the two-hinged arch, since the latter, being free to rotate at the supports, is better able to accommodate itself to changes in shape without excessive stress.

A uniform temperature increase of amount $t°$ causes an elongation $\Delta ds = \omega t° \, ds$ which, if substituted in Eqs. (10.14) and integrated over the respective half arches, gives the corresponding crown displacements. Likewise, a temperature differential between top and bottom fiber (Fig. 10.6*d* and *e*) results in a rotation $d\phi$, given by Eq. (10.8). This must be substituted in Eqs. (10.13) and similarly integrated to result in the corresponding crown displacements. The corresponding values are likewise shown in Table 10.1. The general information given in the article on two-hinged arches as to range of temperature and method of accounting for shrinkage is valid also for the fixed arch.

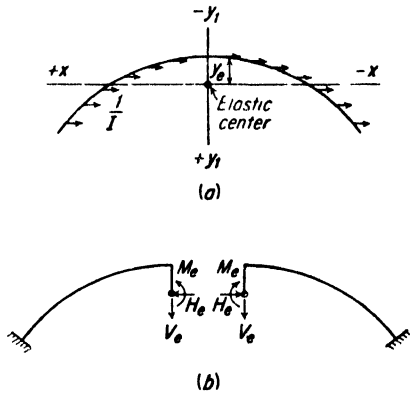


Fig. 10.8

If the displacements of the table are substituted in Eqs. (10.12), it is seen that the second equation contains only $V_e$ as an unknown and hence can be solved for this quantity. The first and third, however, contain both $H_e$ and $M_e$, resulting in two simultaneous equations which are awkward to solve and, when solved, lead to rather lengthy expressions for these two quantities. This complication can be overcome by a suitable transformation of coordinates.

Considering the first term of the first row and the second term of the third row of the displacement table, it is seen that if $\int y \, ds/I$ could be made zero, the first of Eqs. (10.12) would contain only $H_e$ as an unknown, and the third only $M_e$. Obviously, for the location of the coordinate axes, as shown in Fig. 10.7*a*, this integral is *not* zero, since $y$ is positive throughout. It is possible, however, to shift the origin to a position which will make this integral vanish. To visualize the meaning of this shift, assume the arch axis to be loaded by fictitious loads equal to $1/I$ at every point as shown in Fig. 10.8*a*. Then, since $y_1$ is the lever arm of any of these forces about the origin, $\int y_1 \, ds/I = 0$ shows that the sum of the moments of all these forces about the $x$ axis is to be zero; *i.e.*, the axis is to pass through the center of these forces, or is to be coincident with

their resultant. The coordinate origin determined in this manner is known as the *elastic center* of the arch. From the general formula for finding the location of the resultant of parallel forces, the distance $y_e$ of the elastic center from the center of the crown is then found from

$$y_e = \frac{\int_0^{s/2} \frac{y\,ds}{I}}{\int_0^{s/2} \frac{ds}{I}} \tag{10.15}$$

By symmetry these integrals need be extended only over the half arch.

Let $y_1$ designate the ordinate of any point on the arch axis measured from this new origin, as shown in Fig. 10.8*a*. Now, assume that the elastic center is joined to the crown section by a rigid, fictitious lever, as shown separately for both halves in Fig. 10.8*b*. Instead of introducing the three unknown quantities $M_c$, $H_c$, and $V_c$ at the crown, introduce three similar quantities $M_e$, $H_e$, and $V_e$ at the ends of these fictitious levers, positive as shown. Then the previously tabulated expressions for $u$, $v$, and $\phi$ refer now to the new origin (end of levers) if, in them, $y$ is replaced by $y_1$ and $M_c$, $H_c$, and $V_c$ by $M_e$, $H_e$, and $V_e$. Equations (10.12) are now valid for the ends of these levers. Since the levers are regarded as rigidly connected to their respective crown sections and the two halves are fitted together to form the complete arch, the ends of the two levers cannot separate, no matter how the arch deforms. Thus the displacements $u$, $v$, and $\phi$ must be the same for the ends of both levers.

Since the origin is now chosen such that $\int y_1\,ds/I = 0$, the first term of the first row and the second term of the third row in Table 10.1 vanish. (These terms are indicated by an asterisk in the table.) Hence, if the remaining terms of the first, second, and third row of the table are substituted, respectively, in the first, second, and third of Eqs. (10.12), they each result in one equation with one unknown, $H_e$, $V_e$, and $M_e$ in that order. Since in each individual expression the respective integral for the left half arch is added to that for the right half, integration can be extended directly from $-s/2$ to $+s/2$, rather than writing two integrals for each term. The three redundant quantities, referred to the elastic center, then become:

$$H_e = \frac{-\int_{-s/2}^{s/2} \frac{M_0 y_1\,ds}{I} - \int_{-s/2}^{s/2} \frac{N_0 \cos\theta\,ds}{A} + Et^\circ\omega l - E\,\Delta t^\circ\,\omega \int_{-s/2}^{s/2} \frac{y_1\,ds}{t}}{\int_{-s/2}^{s/2} \frac{y_1^2\,ds}{I} + \int_{-s/2}^{s/2} \frac{\cos^2\theta\,ds}{A}} \tag{10.16}$$

$$V_e = \frac{\displaystyle\int_{-s/2}^{s/2} \frac{M_0 x\, ds}{I} - \int_{-s/2}^{s/2} \frac{N_0 \sin\theta\, ds}{A}}{\displaystyle\int_{-s/2}^{s/2} \frac{x^2\, ds}{I} + \int_{-s/2}^{s/2} \frac{\sin^2\theta\, ds}{A}} \tag{10.17}$$

$$M_e = \frac{-\displaystyle\int_{-s/2}^{s/2} \frac{M_0\, ds}{I} - E\,\Delta t^\circ\,\omega \int_{-s/2}^{s/2} \frac{ds}{t}}{\displaystyle\int_{-s/2}^{s/2} \frac{ds}{I}} \tag{10.18}$$

where all integrals extend over the entire arch, as indicated.

Once these quantities are determined, it is more convenient to work with crown moments and forces (Fig. 10.7*b*) rather than with those referred to the elastic center (Fig. 10.8*b*). As is seen directly from this figure,

$$H_c = H_e \qquad V_c = V_e \qquad M_c = M_e - H_e y_e \tag{10.19}$$

Equations (10.16) to (10.18) account for all influences that act on symmetrical fixed arches. In any particular design it will often be found that certain of these influences are small as compared with others and can therefore be neglected. The analogous situation for the two-hinged arch was discussed in Art. 10.2. As in that case, the influence of rib shortening caused by axial forces $N$ is small except for unusually flat and thick arches. Except for such cases, therefore, the second term in both numerator and denominator of Eqs. (10.16) and (10.17) can be omitted. The last term in the numerator of Eq. (10.16) will often be found to be small in comparison with other terms also. Indeed, for an arch with constant cross section this term vanishes. It does so because the elastic center was chosen such that $\int y_1\, ds/I = 0$ which, for constant $I$, is the same as $\int y_1\, ds = 0$. This makes $\int y_1\, ds/t = 0$ if $t$ is constant. It is seen, therefore, that the magnitude of that term in Eq. (10.16) depends essentially on the ratio of the depths at springing and crown and will usually be negligible. Finally, the term containing $\Delta t^\circ$ in Eq. (10.18) must be included only in the rather rare case of a roof arch not insulated at the top.

Hence, the following simplified expressions are usually accurate enough for purposes of design:

$$H_e = \frac{-\displaystyle\int_{-s/2}^{s/2} \frac{M_0 y_1\, ds}{I} + Et^\circ\omega l}{\displaystyle\int_{-s/2}^{s/2} \frac{y_1^2\, ds}{I}} \tag{10.16a}$$

$$V_e = \frac{\int_{-s/2}^{s/2} \frac{M_0 x\, ds}{I}}{\int_{-s/2}^{s/2} \frac{x^2\, ds}{I}} \tag{10.17a}$$

$$M_e = \frac{-\int_{s/2}^{s/2} \frac{M_0\, ds}{I}}{\int_{-s/2}^{s/2} \frac{ds}{I}} \tag{10.18a}$$

A word of warning should be said in connection with these simplified equations. Although under the specific conditions which were enumerated, the omitted terms are "usually" negligibly small, there is no way of definitely ascertaining this fact except by computing these terms. It is therefore suggested, particularly for closely designed arches of significant spans, that the simplified equations be used for preliminary design but that in computing final stresses the influence of the omitted terms be at least estimated numerically and included in the final expressions if found significant.

**10.4. Evaluation of Integrals.** The integrals which occur in the expressions for the redundants of both the two-hinged arch [Eqs. (10.11) and (10.11*a*)] and the fixed arch [Eqs. (10.16) to (10.18) and (10.16*a*) to (10.18*a*)] can be evaluated by direct integration only in a few simple cases. For this reason it is necessary, in general, to approximate the integrals by sums of a finite number of terms.

For this purpose the arch axis (not the span) is generally divided into a number of equal intervals $\Delta s$. A maximum of 10 intervals for the half arch is generally sufficient; 6 to 8 intervals may often be used for small, secondary structures. The pertinent quantities entering a particular integral, such as $y_1$, $M_0$, and $I$ for $\int M_0 y_1\, ds/I$, are then computed for the mid-point of each individual interval $\Delta s$. Then

$$\int \frac{M_0 y_1\, ds}{I} \cong \sum \frac{M_{0n} y_{1n}}{I_n} \Delta s$$

where the subscript $n$ refers to the values $M_0$, $y_1$, and $I$ at the mid-point of the $n$th interval. The other integrals in Eqs. (10.11) and (10.16) to (10.18) are computed in similar manner by summation. Attention must be paid to the signs of the various quantities. Thus, $y_1$ is negative if measured upward from the elastic center of the fixed arch (see Fig. 10.8*a*); $x$ is negative if measured to the right of the crown of the fixed arch, but is positive throughout for the two-hinged arch, as is evident from Figs. 10.8*a* and 10.6*a*, respectively. The importance of this is evident, for instance, in connection with Eq. (10.17). If an arch is symmetrically loaded, $M_0$ and $N_0$ are equal at corresponding points of the two half

arches. On the other hand, for the left half arch, $x$ and $\sin\theta$ are positive, whereas for the right half they are negative. Each contribution to the sums for both numerator integrals made by an element on the left half is, therefore, canceled by the equal and opposite contribution made by the corresponding element on the right. Consequently, $V_c$ (and thereby $V_c$) is zero for symmetrical loads, in analogy with a simple beam which has zero shear at mid-span, if symmetrically loaded.

For fixed arches, the location of the elastic center, Eq. (10.15), is likewise calculated by replacing the integrals by corresponding sums. The computation of these sums is best arranged in tabular form, as demonstrated in the design example of Art. 10.7.

Moments of inertia can be computed for the entire concrete section, including the tension portion, without regard to reinforcement. The previously mentioned report of ACI Committee 312 suggests that $E = 4{,}000{,}000$ psi be used for calculating the redundants for all commonly used grades of concrete.

**10.5. Preliminary Arch Design.** The analysis discussed in the preceding articles can be carried out only if loads as well as all dimensions of the arch are definitely known. Before such an analysis can be made, it is therefore necessary to determine the shape and weight of the arch by a preliminary design. In general, only the span, the rise, and the loads superimposed on the arch are known. (In cases where the designer is free to select the rise it is important to consider that the horizontal thrust decreases as the ratio rise/span increases, with a consequent saving in cost of abutments.) It is then required to determine the shape of the arch axis and the dimensions of the cross sections at various points and, from these, to find the weight of the arch proper. Once these quantities are preliminarily determined, an exact stress analysis can be made. Should this analysis indicate the necessity for a considerable change of the assumed dimensions, a second, final analysis must be made, based on the corrected dimensions. Since such computations are very time-consuming, it is desirable to determine the shape of the arch by preliminary design as accurately as possible so that subsequent corrections are held to a minimum.

The axis of the arch is generally made to coincide with the equilibrium or string polygon for dead load. If this is done, bending moments are caused by live load only. This polygon can be drawn only after the weight of the arch is determined. For roof arches, where dead loads are mostly uniform along the axis, the parabola is a close approximation to the dead-load polygon and, with the designations of Fig. 10.9, the equations of the axis and the slope are

$$y = 4rlc^2 \tag{10.19a}$$

$$\tan\theta = 8rc \tag{10.19b}$$

For spandrel arch bridges, according to V. A. Cochrane[1] for open-spandrel arches,

$$y = \frac{8rl}{6 + 5r}(3c^2 + 10c^4 r) \tag{10.20a}$$

$$\tan\theta = \frac{8r}{6 + 5r}(6c + 40c^3 r) \tag{10.20b}$$

and for filled-spandrel arches

$$y = \frac{4rl}{1 + 3r}(c^2 + 24c^5 r) \tag{10.21a}$$

$$\tan\theta = \frac{4r}{1 + 3r}(2c + 120c^4 r) \tag{10.21b}$$

Next it is necessary to obtain an estimate of the maximum moments, thrusts, and shears for the purpose of determining the required cross section. To this end superimposed loads are determined, and a rough estimate is made of the weight of the arch proper. If temperature stresses need consideration, the limiting temperatures are also determined. With these data it is possible to determine $M$, $V$, and $H$ from Tables 10.2, 10.3, and 10.4. Tables 10.2 and 10.3 are set up for parabolic arches [Eqs. (10.19) mostly for roof structures]; and Table 10.4, for arches according to Eqs. (10.20) and (10.21), mostly for bridges.

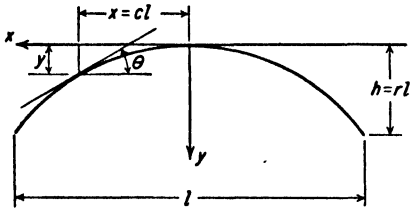
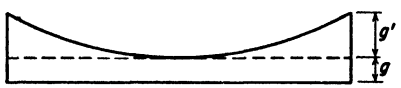



FIG. 10.9

For roof arches, the dead load is generally approximately uniformly distributed along the arch axis. The load per horizontal foot of span is therefore distributed as shown in Fig. 10.10. In order to use Tables 10.2 and 10.3, this load can be divided into a uniform load $g$ and a variable load of maximum intensity $g'$ at the springing. The tables then allow one to compute separately the influences of each of these two loads which must be added to result in the final $M$, $V$, and $H$. Live load must be so placed as to give maximum moments or forces, which will generally require partial loading of the span. The necessary formulas for partial loading are included in the tables, where the load positions, determined by $\alpha = a/l$, are as shown on the sketches. Positions of loads and values for $\alpha$ which result with good

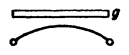
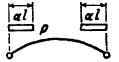
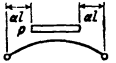
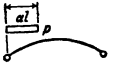
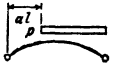
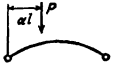



$g$ = load per foot of axis
$g/\cos\theta$ = corresponding load per foot of span

FIG. 10.10

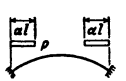
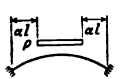
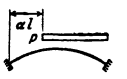
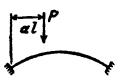
[1] "Design of Symmetrical Hingeless Concrete Arches," *Proc. Engrs.' Soc. West. Penn.*, vol. 32, p. 647, 1916.

## TABLE 10.2. Moments and Reactions of Two-hinge Parabolic Arches*

| | Case I | Case II | Case III | Case IV | Case V | Case VI | Case VII | Case VIII | Case IX |
|---|---|---|---|---|---|---|---|---|---|
| |  | |  |  |  |  |  | Uniform temperature change | Temperature differential of $\Delta t°$ |
| $R_l$ | $\frac{ql}{2}$ | $\frac{q'l}{6}$ | $p\alpha l$ | $\frac{pl}{2}(1-2\alpha)$ | $\frac{p\alpha l}{2}(2-\alpha)$ | $\frac{p\alpha^2 l}{2}$ | $P(1-\alpha)$ | None | None |
| $R_r$ | $\frac{ql}{2}$ | $\frac{q'l}{6}$ | $p\alpha l$ | $\frac{pl}{2}(1-2\alpha)$ | $\frac{p\alpha^2 l}{2}$ | $\frac{p\alpha l}{2}(2-\alpha)$ | $P\alpha$ | None | None |
| $H$ | $\frac{ql^2}{8h}$ | $\frac{q'l^2}{42h}$ | $\frac{pl^2}{8h}\alpha^2(2\alpha^2-5\alpha^2+5)$ | $\frac{-pl^2}{8h}(2\alpha^5-5\alpha^4+5\alpha^2-1)$ | $\frac{pl^2}{8h}\alpha^2(2\alpha^3+5\alpha^2-5)$ | $\frac{pl^2}{16h}\alpha^2(2\alpha^3-5\alpha^2+5)$ | $\frac{5Pl}{8h}\alpha(\alpha^3-2\alpha^2+1)$ | $\frac{15EI_c\omega t°}{8h^2}$ | $\frac{5EI_c\omega\,\Delta t°}{4t_eh}$ |
| $M_c$ | None | $\frac{q'l^2}{84}$ | $\frac{-pl^2}{8}\alpha^2(2\alpha^3-5\alpha^2+1)$ | $\frac{pl^2}{8}\alpha^2(2\alpha^3-5\alpha^2+1)$ | ..... | ... | $\frac{(1-\alpha)^2}{8}P(5\alpha^3-5\alpha^2-5\alpha+4)$ | $\frac{-15EI_c\,\omega t°}{8h}$ | $\frac{-5EI_c\omega\,\Delta t°}{4t_e}$ |
| $M_{l/4}$ | None | $\frac{23q'l^2}{5376}$ | ..... | ..... | $\frac{-pl^2}{64}(6\alpha^5-15\alpha^4+23\alpha^2-16\alpha+2)$ | $\frac{-pl^2}{64}(6\alpha^2-23\alpha^2+15)$ | $\frac{-3Pl}{32}\alpha(5\alpha^2+10\alpha+3)$ | $\frac{-45EI_c\,\omega t°}{32h}$ | $\frac{-15EI_c\omega\,\Delta t°}{16t_e}$ |

* According to Eqs. (10.19).

## TABLE 10.3. Moments and Reactions of Fixed Parabolic Arches*

| | Case I | Case II | Case III | Case IV | Case V | Case VI | Case VII | Case VIII | Case IX |
|---|---|---|---|---|---|---|---|---|---|
| | | |  |  | |  |  | Uniform temperature change | Temperature differential of $\Delta t^\circ$ |
| $R_l$ | $\frac{ql}{2}$ | $\frac{q'l}{6}$ | $pal$ | $\frac{pl}{2}(1-2\alpha)$ | $\frac{pl}{2}\alpha(\alpha^2-2\alpha^2+2)$ | $\frac{pl}{2}\frac{(1+\alpha)}{(1-\alpha)^3}$ | $P(1-\alpha)^2(1+2\alpha)$ | None | None |
| $R_r$ | $\frac{ql}{2}$ | $\frac{q'l}{6}$ | $pal$ | $\frac{pl}{2}(1-2\alpha)$ | $\frac{pl}{2}\alpha^2(2-\alpha)$ | $\frac{pl}{2}(1-2\alpha^3+\alpha^4)$ | $P\alpha^2(3-2\alpha)$ | None | None |
| $H$ | $\frac{ql^2}{8h}$ | $\frac{q'l^2}{56h}$ | $\frac{pl^2}{4h}\alpha^3(6\alpha^2-15\alpha+10)$ | $\frac{-pl^2}{8h}(12\alpha^5-30\alpha^4+20\alpha^3-1)$ | $\frac{pl^2}{8h}\alpha^3(6\alpha^2-15\alpha+10)$ | $\frac{pl^2}{8h}\frac{(1-\alpha)^3}{(6\alpha^2+3\alpha+1)}$ | $\frac{15}{4}\frac{Pl}{h}\alpha^2(1-\alpha)^2$ | $\frac{45EI_c\omega l^2}{4h^2}$ | None |
| $M_{la}$ | None | $\frac{-ql^2}{210}$ | $-\frac{pl^2}{2}\frac{\alpha^2(1-\alpha)^2}{(1-2\alpha)}$ | $\frac{pl^2}{2}\alpha^2(1-\alpha)^2(1-2\alpha)$ | $-\frac{pl^2}{2}\alpha^2(1-\alpha)^3$ | $\frac{pl^2}{2}\alpha^2(1-\alpha)^3$ | $-\frac{Pl}{2}\alpha(1-\alpha)^2(2-5\alpha)$ | $\frac{15EI_c\omega t^\circ}{2h}$ | $\frac{-EI_c\omega\,\Delta t^\circ}{t_c}$ |
| $M_{ra}$ | None | $\frac{-ql^2}{210}$ | $-\frac{pl^2}{2}\alpha(1-\alpha)^2(1-2\alpha)$ | $\frac{pl^2}{2}\alpha^2(1-\alpha)^2(1-2\alpha)$ | $\frac{pl^2}{2}\alpha^3(1-\alpha)^2$ | $-\frac{pl^2}{2}\alpha^3(1-\alpha)^2$ | $\frac{Pl}{2}\alpha^2(1-\alpha)(3-5\alpha)$ | $\frac{15EI_c\omega t^\circ}{2h}$ | $\frac{-EI_c\omega\,\Delta t^\circ}{t_c}$ |
| $M_c$ | None | $-\frac{ql^2}{560}$ | $-\frac{pl^2}{4}\alpha^3(1-2\alpha)(2-\alpha)$ | $\frac{pl^2}{4}\alpha^3(1-2\alpha)(2-\alpha)$ | $-\frac{pl^2}{8}\alpha^3(1-2\alpha)(2-\alpha)$ | $\frac{pl^2}{8}\alpha^3(1-2\alpha)(2-\alpha)$ | $-\frac{Pl}{4}\alpha^2(5\alpha^2-10\alpha+3)$ | $\frac{15EI_c\omega t^\circ}{4h}$ | $\frac{-EI_c\omega\,\Delta t^\circ}{t_c}$ |

* According to Eqs (10 19).

accuracy in maximum moments at the crown, quarter point, or springing, respectively, are given for both types of arches in Fig. 10.11.

In order to obtain simple expressions for Tables 10.2 and 10.3,

$$I = \frac{I_c}{\cos \theta}$$

was assumed. Even considerable deviation from this relationship has relatively little influence and leads to satisfactory accuracy.

For bridge arches, V. A. Cochrane has developed a series of approximate equations for determining stresses in an arch ring, which give values very close to those obtained by the exact method. While the authors' experience with these equations seems to indicate that the results obtained

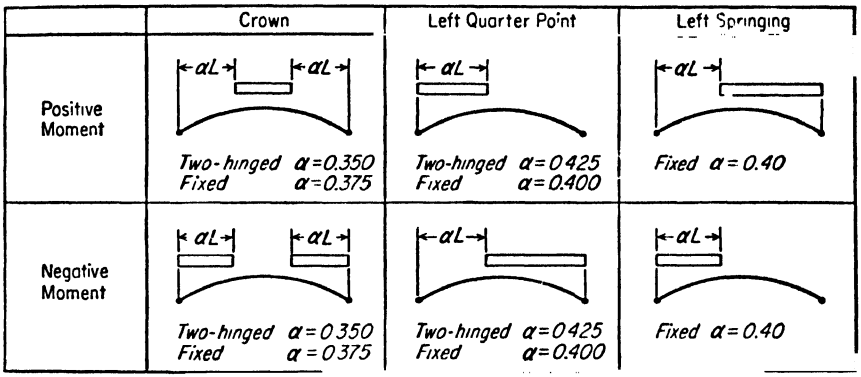



FIG. 10.11. Live-load positions for maximum moment. (Computed for parabolic arches, but satisfactorily accurate for most arches of common shapes.)

by their use are slightly smaller than those obtained by exact analysis, they are sufficiently accurate to warrant their use in preliminary calculations, and in less important structures might serve as a basis for the final designs. Some of the equations developed by V. A. Cochrane are given in Table 10.4. The following notation is used (see Fig. 10.9):

$l$ = span of arch axis, ft
$h$ = rise of arch axis, ft
$r$ = rise ratio $h/l$
$y$ = ordinate of arch axis, the abscissa of which is $cl$
$s$ = length of half axis, measured along axis from crown to springing
$s_x$ = distance along axis from crown to a given section whose abscissa is $cl = x$
$t_0$ = thickness of arch rib at crown
$t_c$ = thickness of arch rib at springing
$t_x$ = thickness of arch rib at point whose abscissa is $x$

$v = s_x/s$

$u_e$ = ratio of thickness of springing to thickness at crown
$= t_e/t_0$

$M_c$ = moment at crown, ft-lb

$T_c$ = thrust at crown, lb

$M_s$ = moment at springing, ft-lb

$T_s$ = thrust at springing, lb

$T_{m_1}$ = coefficient for $wl$ for thrust at crown corresponding to maximum positive moment at crown

$T_{m_2}$ = coefficient for $wl$ for thrust at crown corresponding to maximum negative moment at crown

$V_s$ = approximate dead-load vertical end reaction, or one-half dead weight of span, lb

$w_c$ = weight of arch at crown, plus average weight of superstructure at crown, lb per lin ft of span

$w$ = live load, lb per lin ft of span (not necessarily the same for all positions of the live load).

$\omega$ = coefficient of linear expansion due to temperature changes

$t$ = change in temperature, °F

$E$ = modulus of elasticity of concrete, psf

$I_0$ = moment of inertia of arch rib at crown, biquadratic feet.

The length of the *half axis* $s$ for parabolic arches is

$$s = \frac{l}{2}\left(1 + \frac{8}{3}r^2\right) \tag{10.22}$$

For arches according to Eqs. (10.20) and (10.21), this length can be obtained from Table 10.5.

Once approximate values of $M$, $N$, and $H$ are found for the critical sections from Tables 10.2 to 10.4, the required cross-sectional dimensions are determined in the usual manner for bending plus axial thrust. Reinforcement is usually provided symmetrically near both faces. For reasons of appearance the depth is often increased gradually from crown to springing or else is held constant, but is hardly ever decreased toward the springing. Typical thickness variations according to Cochrane are given for fixed arches in Table 10.6.

With dimensions of the cross sections determined as required by strength and appearance, the weight of the arch is computed. If the new total dead load differs materially from that assumed in the beginning, the preliminary design procedure is repeated with adjusted loads, and dimensions are correspondingly corrected.

Since, at this stage, all loads are known, it is now possible to obtain the final location of the arch axis by passing a string polygon for dead load through the mid-points of the crown and springing sections. For

## Table 10.4. Thrusts and Moments for Fixed Bridge Arches According to Cochrane

### OPEN SPANDREL ARCHES

| STRESSES PRODUCED BY | SECTION AT CROWN | SECTION AT SPRINGING |
|---|---|---|
| DEAD LOAD | $T_C = \frac{6+5r}{48r} w_C l$<br>$M_C$ assumed = 0 | $V_S = \frac{3+5r}{6} w_C l$ $\quad T_S = \sqrt{T_C^2 + V_S^2}$<br>$M_S$ assumed = 0 |
| LIVE LOAD Maximum Positive Moment | $T_C = \frac{595+315r+5r^2}{1000r} wl$<br>$M_C = +\frac{675+80r+40r^2-(20+13r)u_S+3u_S^2}{10000} wl^2$ | $T_S = \frac{933-425r+155r^2}{1000r} wl$<br>$T_C = (\frac{1}{2}T_{m2} + T_{m1}) - \frac{0.006}{u_S r} wl$<br>$M_C = +\frac{322+27r-(181+13r)(3.79-0.6r-u_S)^2}{10000} wl^2$ |
| LIVE LOAD Maximum Negative Moment | $T_C = \frac{61+2u_S+(7+2u_S)r-70r^2}{1000r} wl$<br>$M_C = -\frac{55-9u_S+0.85u_S^2}{10000} wl$ | $T_S = \frac{244+(211-2u_S)r+(120+30u_S)r^2}{1000r} wl$<br>$T_C = (\frac{1}{2}T_{m2} + \frac{0.0063}{u_S r}) wl$<br>$M_S = \frac{289-172r-(178-28r)(3.36+2.4r-u_S)^2}{10000} wl^2$ |
| TEMPERATURE (Fall of t°) | $T_C = -[184u_S-75-(205u_S-15)r-(10u_S-2)r^2] \frac{wtEI_O}{h^2}$<br>$M_C = -[4625-175u_S+244u_S^2+5r+(255-36u_S)r^2] \frac{hT_C}{100}$ | $T_S = (109-175r) T_C$<br>$M_S = M_C + hT_C$ |

### FILLED SPANDREL ARCHES

| STRESSES PRODUCED BY | SECTION AT CROWN | SECTION AT SPRINGING |
|---|---|---|
| DEAD LOAD | $T_C = \frac{1+3r}{8r} w_C l$<br>$M_C$ assumed = 0 | $V_S = \frac{2+15r}{4} w_C l$ $\quad T_S = \sqrt{T_C^2 + V_S^2}$<br>$M_S$ assumed = 0 |
| LIVE LOAD Maximum Positive Moment | $T_C = \frac{576+(189-8u_S)r-220r^2}{1000r} wl$<br>$M_C = +\frac{72+105r+220r^2-(17+10r)u_S+15u_S^2}{10000} wl^2$ | $T_S = \frac{84+66u_S+(153-51u_S)r-(500-140u_S)r^2}{1000r} wl$<br>$T_C = [\frac{1}{2}T_{m2} + T_{m1} + \frac{0.0026(u_S-2)^2}{r}] wl$<br>$M_S = +\frac{316+1370r-2100r^2-(17-20r)(3.68+7.6r-u_S)^2}{10000} wl^2$ |
| LIVE LOAD Maximum Negative Moment | $T_C = \frac{578+2u_S+(10+30u_S)r-(380+30u_S)r^2}{1000r} wl$<br>$M_C = -\frac{494-100r-(3-26r)u_S-0.75u_S^2}{10000} wl^2$ | $T_S = \frac{276+(125+6u_S)r+320r^2}{1000r} wl$<br>$T_C = [\frac{1}{2}T_{m2} - \frac{0.0026(u_S-2)^2}{r}] wl$<br>$M_S = -\frac{283.480r-9(4.22-2.8r-u_S)^2}{10000} wl^2$ |
| TEMPERATURE (Fall of t°) | $T_C = -[194u_S-75+(17u_S-31)r-140(u_S-1)r^2] \frac{wtEI_O}{h^2}$<br>$M_C = -(38.5-12.8u_S+1.6u_S^2) \frac{hT_C}{100}$ | $T_S = (113-255r) T_C$<br>$M_S = M_C + hT_C$ |

this purpose, as well as for the subsequent exact analysis, the half axis is generally divided into six to ten equal sections (see Art. 10.4) and the dead load of each section applied at its center. A smooth curve through the tangent to this string polygon then represents the final location of

**TABLE 10.5. Length $s$ of Half Axis of Fixed Bridge Arches***

| Kinds of arches | Values of $\frac{s}{l}$ for rise-ratio $r$ | | | | |
|---|---|---|---|---|---|
| | 0 10 | 0 15 | 0 20 | 0 25 | 0 30 |
| Open-spandrel arches | 0 513 | 0 529 | 0 551 | 0 577 | 0 607 |
| Filled-spandrel arches | 0 515 | 0 534 | 0 559 | | |

* V. A. Cochrane, "Design of Symmetrical Concrete Arches," *Proc. Engrs.' Soc. West. Penn.*, vol. 32, p. 647, 1916

the arch axis. Accurate values of $M$, $N$, and $V$ are now determined by elastic analysis at the crown, quarter point, and springing, and at such additional points as may be necessary. The difference between these accurate values and the approximate ones obtained from the preliminary design will require minor adjustments of cross-sectional dimensions which

**TABLE 10.6. Typical Thickness Variations of Fixed Bridge Arches***

| Value of $v$ | Values of $u_s$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 50 | 1 75 | 2 00 | 2 25 | 2 50 | 2 75 | 3 00 | 3 25 |
| | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 |
| 0 05 | 1 007 | 1 006 | 1 005 | 1 004 | 1 003 | 1 002 | 1 001 | 1 000 |
| 0 15 | 1 021 | 1 018 | 1 015 | 1 012 | 1 009 | 1 006 | 1 003 | 1 000 |
| 0 25 | 1 035 | 1 030 | 1 025 | 1 020 | 1 015 | 1 010 | 1 005 | 1 000 |
| 0 35 | 1 049 | 1 042 | 1 035 | 1 028 | 1 023 | 1 021 | 1 023 | 1 030 |
| 0 45 | 1 063 | 1 054 | 1 048 | 1 048 | 1 057 | 1 070 | 1 083 | 1 101 |
| 0 55 | 1 077 | 1 072 | 1 085 | 1 105 | 1 133 | 1 165 | 1 193 | 1 231 |
| 0 65 | 1 095 | 1 125 | 1 168 | 1 215 | 1 269 | 1 328 | 1 385 | 1 455 |
| 0 75 | 1 145 | 1 223 | 1 311 | 1 403 | 1 508 | 1 625 | 1 737 | 1 865 |
| 0 85 | 1 245 | 1 393 | 1 547 | 1 700 | 1 862 | 2 025 | 2 185 | 2 355 |
| 0 95 | 1 406 | 1 621 | 1 837 | 2 055 | 2 277 | 2 495 | 2 709 | 2 932 |
| 1 00 | 1 500 | 1 750 | 2 000 | 2 250 | 2 500 | 2 750 | 3 000 | 3 250 |

* V. A. Cochrane, "Design of Symmetrical Hingeless Concrete Arches," *Proc. Engrs.' Soc. West Penn*, vol. 32, p. 647, 1916.

can often be made merely by changing the reinforcement without alteration of over-all size. Even if the thickness and width of the arch are changed in this stage of design, the elaborate elastic analysis need not be repeated for the adjusted sections unless these adjustments are quite

sizable. This is due to the fact that the magnitudes of $M$, $N$, and $V$ caused by external loads depend primarily on the ratios of the moments of inertia along the arch to that at the crown, rather than on their absolute values. Only the stresses due to rib shortening and temperature depend directly on the absolute size of the cross section and may have to be recomputed if the dimensions change significantly. The design of a simple roof arch in Art. 10.7 illustrates the above procedure.

**10.6. Hinges in Reinforced Concrete Structures.** It is the function of hinges to transmit safely the thrust $N$ and the shear force $S$ and, at the same time, to offer little resistance to rotation. An ideal hinge should be entirely free to rotate, since only then will the bending moment at the hinge be zero, as assumed in the analysis of hinged structures. Such ideal hinges cannot be constructed, since friction or other inevitable restraining factors cannot be avoided. Practically, however, it is only necessary to ensure that the flexibility of the hinge be considerably greater than that of any other portion of the structure of which it is a part. This will result in very small moments in the hinge as compared to the moments in the structure proper, and therefore will lead only to negligible errors. The use of hinges is not restricted to arches. They are equally useful in rigid frames, in bridges as well as in buildings, particularly for long-span single-story frames. They reduce or, in the case of three-hinged spans, eliminate the stresses caused by rib shortening, by temperature, and by slight foundation movements. For these reasons a considerable reduction in the dimensions and weight of the structure and of the foundation can often be achieved by their use.

The main practical types of hinges for concrete structures are given in Fig. 10.12. The steel hinges shown there result in more perfect hinge action than the concrete hinges, but are considerably more expensive. Their use, today, is restricted to unusually heavy structures. Of the two types of concrete hinges, named after the French engineers who developed them, the Mesnager hinge ensures greater freedom of rotation and, except for very large thrusts, is the most satisfactory hinge presently in use.

In order to obtain maximum flexibility in a Mesnager hinge, it has been recommended that the hinge opening $h$ should at least be equal to the concrete thickness $t$. A value $h = 1.3t$ is recommended by some. Hinges with smaller openings, however, have been used successfully. For the same purpose of maximum flexibility, the crossed steel bars should not be too thick and short. On the other hand, if their diameter were too small in relation to their length they would be likely to buckle. For that reason the slenderness ratio $L/r$ of the hinge bars should be between the limits of 20 and 40, $L$ being the length of the inclined bar between concrete faces, that is, $h/\cos\theta$, and $r$ the radius of gyration of the bar,

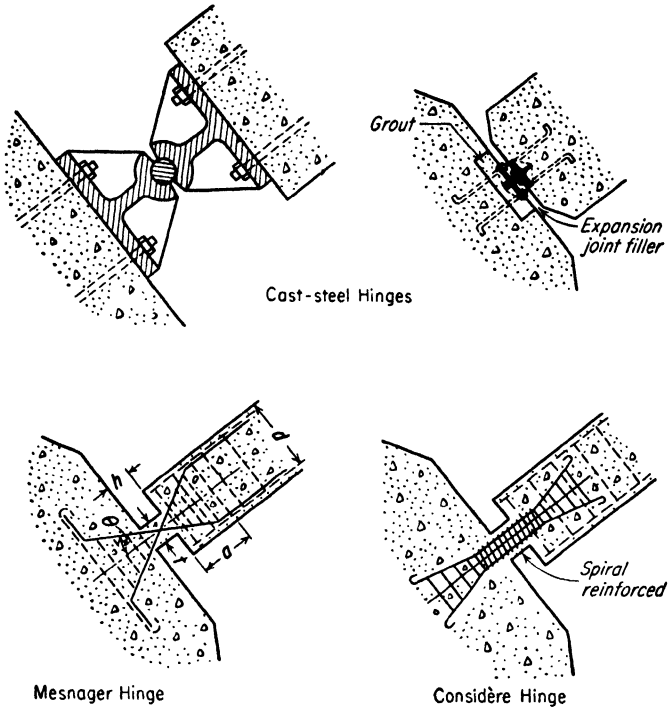



Fig. 10.12. Hinges for concrete structures.

that is, $D/4$ for round bars of diameter $D$. The length of embedment of the hinge bars on either side of the hinge should be at least equal to that prescribed for dowels connecting columns to footings.

To find the required area of the hinge bars, it is assumed that they form a triangular truss which transmits the normal and shear forces $N$ and $S$, as shown in Fig. 10.13. It is easily seen that the compression stress in bars 1 is equal to

$$f'_s = \frac{N}{A'_s \cos \theta} + \frac{S}{A'_s \sin \theta} \quad (10.23)$$

where $A'_s$ is the total combined area of all hinge bars. This stress shall not exceed 30 per cent of the yield point of the hinge bars. The large factor of safety 1/0.3 = 3.33 is introduced for the following reason: Even though, by the above procedure, the bars are assumed to be subjected only to axial compression, like members of a truss, any rotation actually occurring at the hinge bends the bars and induces corresponding flexural

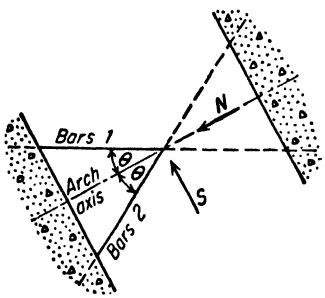



Fig. 10.13

stresses. Such rotations actually do take place during the lifetime of the structure and are caused primarily by changes in live load and in temperature. These flexural stresses superpose on the computed compression and correspondingly increase the values of the actual maximum stresses in the bars. Rather than to attempt computing these additional stresses, it is generally satisfactory to keep the compression stress $f'_s$ sufficiently low so that the bars will not be overstressed by superposed bending.

The inclined bars, transmitting their forces to the concrete by bond along the embedded length, exert a bursting force which must be resisted by lateral reinforcement (stirrups and ties), as shown in Fig 10.12. Only the part of the lateral reinforcement within a distance $a = 8D$ from the face of the concrete is considered effective in resisting the bursting force. The stress in the lateral reinforcement can be computed from

$$f''_s = \frac{(N/2)\tan\theta + (Sa/jd)}{0.005ab + A''_s} \tag{10.24}$$

where, in addition to previously defined quantities, $A''_s$ is the combined area of lateral stirrups or ties located within $a = 8D$ from the free face of the hinge, $j$ can be taken as 0.9, $d$ is shown in Fig. 10.12, and $b$ is the horizontal width of the hinge equal, in most cases, to the width of the arch rib.

More detailed information on concrete hinges can be found in a paper by G. C. Ernst;[1] most texts on steel bridges give adequate data on steel hinges of the type used for both steel and concrete arches and frames.

**10.7. Design of Two-hinged, Tied Roof Arch.** Arches of the type shown in Fig. 10.3 with 90 ft span are to be spaced at 20-ft intervals, to support a fireproof, insulated roof. The roof itself is to consist of precast slabs spanning between arches and weighing 40 psf. Four-ply tar and gravel roofing to cover the slabs weighs 5 psf. To account for the effect of both wind and snow a live load of 35 psf is assumed.

The columns of Fig. 10.3 supporting the arch cannot develop sufficient rigidity to serve as fixed supports. A two-hinged arch is therefore indicated. In addition, if the horizontal thrust were transmitted to the supporting columns and frames, they would become excessively heavy and costly. For that reason a tied arch of the type shown in Fig. 10.14 will be used. A 15-ft rise has been selected, resulting in a rise-span ratio of $\frac{1}{6}$.

The sample design that follows is abbreviated for reasons of clarity and of saving space. Computations of moments, etc., caused by small secondary loads, such as from the weight of the tie rod and the hangers, have been omitted, since they would complicate and lengthen the illustra-

[1] G. C. Ernst, "Design of Hinges and Articulations in Reinforced Concrete Structures," *Trans. ASCE*, vol. 106, p. 862, 1941.

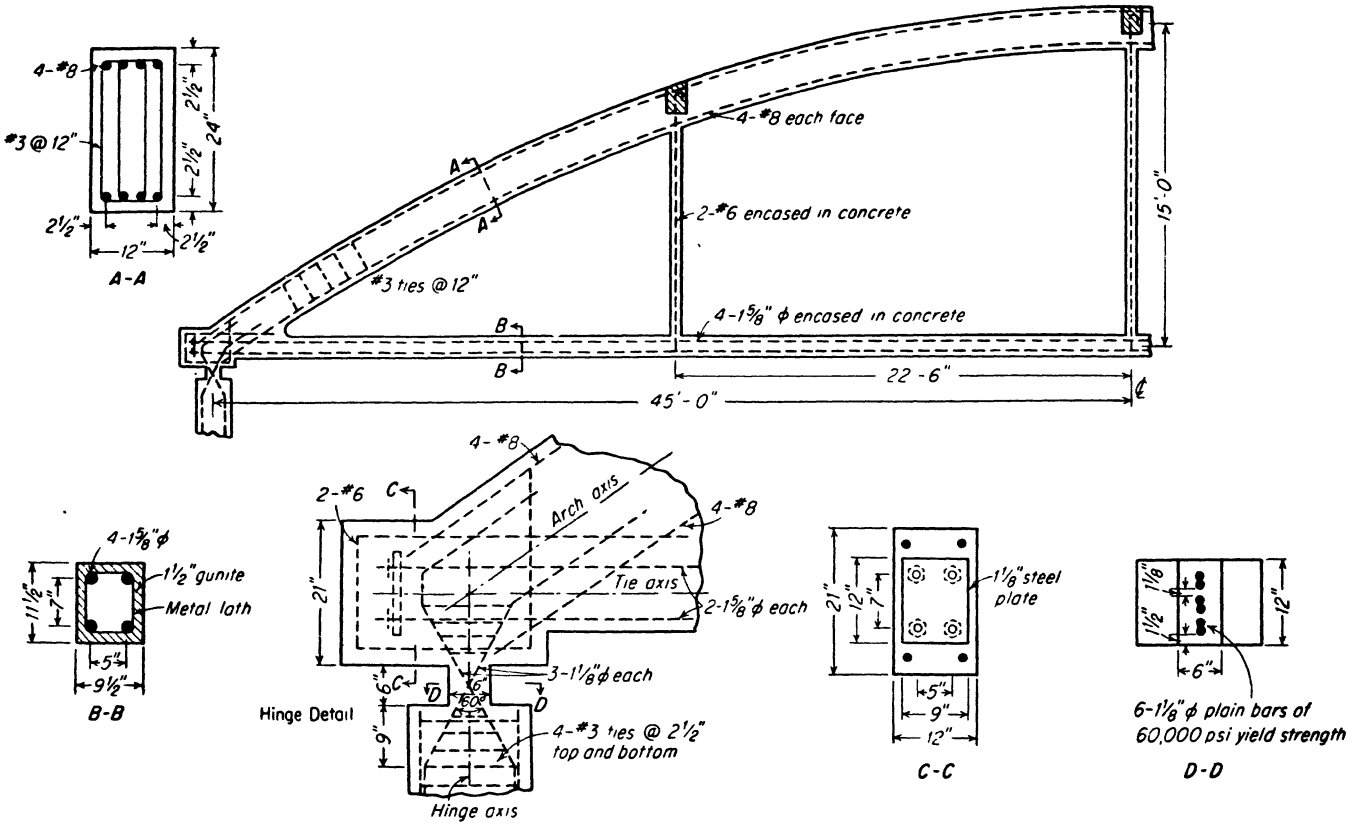



FIG. 10.14. Two-hinge, tied roof arch.

tive computations without adding new features. In actual design practice the influence of such loads should be included in the computations.

*Preliminary Design.* The loads per linear foot of arch axis are

Dead load:

| | | |
|---|---|---|
| Concrete slabs: | $40 \times 20 =$ | 800 lb per ft |
| Roofing: | $5 \times 20 =$ | 100 lb per ft |
| Arch (assumed) | $=$ | 300 lb per ft |
| Total | $=$ | 1200 lb per ft |

Live load:

Snow plus wind: $35 \times 20 =$ 700 lb per ft

The load, uniform along the axis, results in the spanwise distribution, as shown in Fig. 10.10. With $\theta$ computed from Eq. (10.19*b*), $g' = 380$ lb

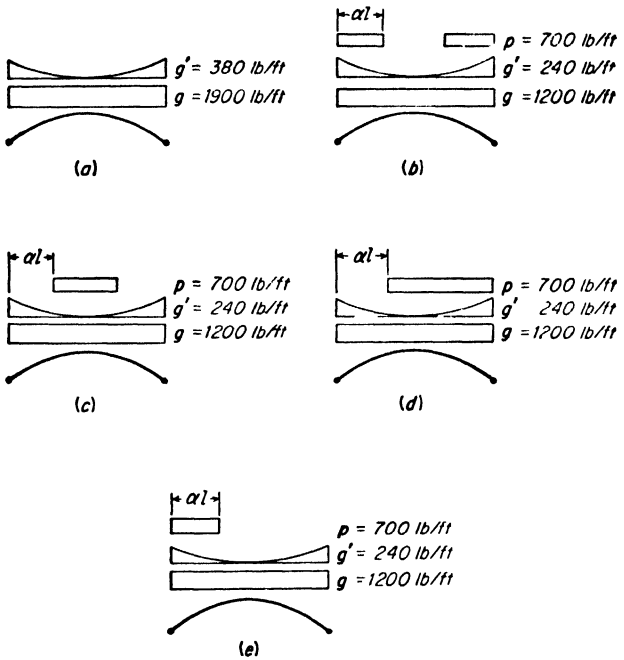



FIG. 10.15

per ft for dead plus live load, and $g' = 240$ lb per ft for dead load alone. Using the information of Table 10.2, the following moments and forces are obtained for the trial design:

Maximum horizontal reaction (loading Fig. 10.15*a*):

$$H = \frac{gl^2}{8h} + \frac{g'l^2}{42h} = 133 \text{ kips}$$

Maximum negative crown moment (Fig. 10.15*b*, $\alpha = 0.35$ from Fig. 10.11):

$$M_c = -0.00292g'l^2 - 0.0071pl^2 = -45.5 \text{ ft-kips}$$

combined with

$$H = \frac{gl^2}{8h} + \frac{g'l^2}{42h} + 0.0685\frac{pl^2}{h} = 109.7 \text{ kips}$$

Maximum positive crown moment (Fig. 10.15*c*, $\alpha = 0.35$):

$$M_c = -0.00292g'l^2 + 0.0071pl^2 = 34.5 \text{ ft-kips}$$

combined with

$$H = \frac{gl^2}{8h} + \frac{g'l^2}{42h} + 0.0564\frac{pl^2}{h} = 105.5 \text{ kips}$$

Maximum negative quarter-point moment (Fig. 10.15*d*, $\alpha = 0.425$):

$$M_{l/4} = 0.00428g'l^2 - 0.0164pl^2 = -84.2 \text{ ft-kips}$$

combined with

$$H = \frac{gl^2}{8h} + \frac{g'l^2}{42h} + 0.0760\frac{pl^2}{h} = 111 \text{ kips}$$

$$R_l = \frac{p\alpha^2 l}{2} + \frac{g'l}{6} = 75 \text{ kips}$$

$V_{l/4}$ = vertical shear at quarter point
$= 75 - 22.5 \times 1.2 - 0.240 \times {}^{90}/_{8} = 45$ kips

The inclination $\theta = 18°25'$ is determined from Eq. (10.19*b*). Then

$T_{l/4}$ = axial thrust at quarter point,
$= V_{l/4} \sin\theta + H\cos\theta$
$= 45 \times 0.316 + 111 \times 0.95 = 119.3$ kips

Maximum positive moment at quarter point (Fig. 10.15*e*, $\alpha = 0.425$):

$$M_{l/4} = 0.00428g'l^2 + 0.0164pl^2 = 100.8 \text{ ft-kips}$$

combined with

$$H = \frac{gl^2}{8h} + \frac{g'l^2}{42h} + 0.0490\frac{pl^2}{h} = 102.5 \text{ kips}$$

$$R_l = p\alpha l\left(1 - \frac{\alpha}{2}\right) + \frac{gl}{2} + \frac{g'l}{6} = 86 \text{ kips}$$

$$V_{l/4} = 86 - 22.5 \times 1.9 - 0.240 \times \frac{90}{8} = 41 \text{ kips}$$

$$T_{l/4} = 41 \times 0.316 + 102.5 \times 0.95 = 110.5 \text{ kips}$$

Maximum axial thrust at springing (Fig. 10.15*a*):

$$H = 133 \text{ kips, as computed before}$$
$$R_l = 45 \times 1.9 + 15 \times 0.380 = 92.6 \text{ kips}$$
$$T_l = R_1 \sin\theta + H \cos\theta$$
$$= 92.6 \times 0.538 + 133 \times 0.834 = 162 \text{ kips}$$

| | |
|---|---|
| Maximum force in ties: | $H = 133$ kips |
| Maximum crown moment: | $M_c = -45.5$ ft-kips |
| Accompanying crown thrust: | $T_c = 109.7$ kips |
| Maximum quarter-point moment: | $M_{l/4} = 100.8$ ft-kips |
| Accompanying quarter-point thrust: | $T_{l/4} = 110.5$ kips |
| Maximum thrust at springing: | $T = 162$ kips |

In a structure of this type, where the entire arch rib is located below a thermoinsulated roof, no temperature differential will arise in the rib. In addition, the arch and the tie are always at substantially the same temperature. Any temperature change in the enclosed space will therefore cause the arch and the tie to expand or contract by the same amount and, for this reason, will not cause any temperature stresses in the structure.

For simplicity of formwork and for good appearance, an arch of constant cross section will be used. It is consequently only necessary to design the most critical section which is seen to be that at the quarter point. A concrete with $f'_c = 3750$ psi and hard-grade steel will be used. For a 14- by 22-in. cross section with $p_g = 0.025$,

$$e/t = 100{,}800 \times 12/110{,}500 \times 22 = 0.53$$

one obtains from Diagrams 14, 18, and 19, Appendix D, the allowable stress, $f_c = 1400$ psi, and the actual stress,

$$f_c = 3.8 \times 110{,}500/22 \times 14 = 1360 \text{ psi}$$

The weight of this arch rib is 320 lb per ft, as compared with the assumed weight of 300 lb per ft. Since the final location of the arch axis obtained by means of a string polygon is likely to result in a smaller required cross section than determined in this preliminary design, 300 lb per ft will be used for the final analysis.

The tie rod must resist $H_{max} = 133{,}000$ lb. To reduce the stretch of the tie, a low unit stress of about 15,000 psi will be used. Consequently, the required tie area $A_{tr} = 133{,}000/15000 = 8.9$ sq in. Since the final design is likely to result in a smaller $H_{max}$, four 1⅝-in. plain round bars with upset threaded ends will be used, furnishing an area of $A_{tr} = 8.3$ sq in.

*Final Design.* Since the preliminary design did not result in any change of the weight of the arch rib, as originally assumed, the final design

is carried out for the same loads as before, *i.e.*, 1200 lb per ft dead load and 700 lb per ft live load.

For the purpose of drawing the dead-load string polygon and for subsequent computations, the length of the arch axis is divided into 20 equal segments, $\Delta s$. From Eq. (10.22), with $l = 90$ ft and $r = \frac{1}{6}$,

$$\Delta s = l\frac{1 + \frac{8}{3}r^2}{20} = \frac{96.66}{20} = 4.83 \text{ ft}$$

The dead load per segment is $1.2 \times 4.83 = 5.80$ kips. To draw the string polygon, it is simplest first to draw to a large scale the parabolic

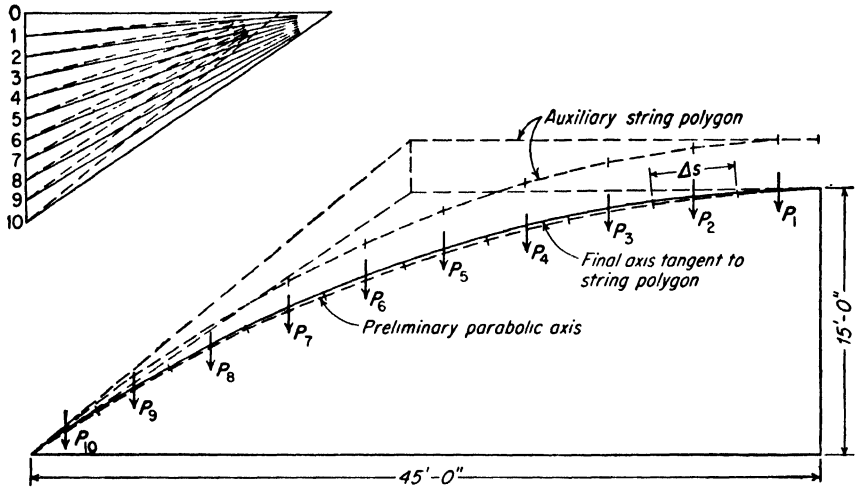



FIG. 10.16. Determination of final location of arch axis by means of dead-load string polygon.

axis used in the preliminary analysis, which is easily done by computing a sufficient number of points by means of Eq. (10.19*a*). It is sufficient to draw half the arch, as shown in Fig. 10.16. The half axis is next divided into 10 equal parts 4.83 ft each, and the weight of each such segment, 5.8 kips, is applied at its mid-point. An accurate string polygon for these loads is then drawn through the mid-points of crown and springing (not shown in Fig. 10.16 so as not to obscure the drawing). A curve tangent to this string polygon represents the final axis. Figure 10.16 shows that for this particular arch the final axis is very close to the original parabola. For the subsequent computations the coordinates $x$ and $y$ of the 10 segments are accurately scaled off the drawing.

As is seen from Eq. (10.11*a*), the quantities $y\,\Delta s/I$ and $y^2\,\Delta s/I$ must be computed for the 10 segments of the half arch. For the tentative 14- by 22-in. section, $I = 11{,}600$ in.$^4$ The pertinent quantities are

assembled in the first four columns of Table 10-7, which summarizes the numerical calculations.

For determining the dimensions of the arch, the tie rod, and the hinges, the forces and moments caused by the following loadings must be computed:

*a. Maximum vertical reaction R and maximum force in tie rod, H.* These are obtained for full live load, being placed over the entire span, so that the load per segment, dead plus live, is 9.2 kips. The corresponding reactions are $R_l = R_r = 92.0$ kips. The beam moments $M_0$ and the quantities $M_0 y\,\Delta s/I$ are computed at the mid-points of the 10 segments. They are entered in Table 10-7 in the columns headed "Case *a*." By virtue of symmetry, computations can be restricted to half the arch.

*b. Maximum positive quarter-point moment.* The preliminary design has shown that the positive quarter-point moment will govern the dimensions of the arch rib. Since a constant cross-section arch was chosen, it is sufficient for our purposes to investigate only this one loading case. For a more complete investigation, the other loading cases of the preliminary design should also be computed. This is desirable even in the case of a constant cross-section arch in view of the approximate character of the preliminary design. It becomes mandatory if the final arch axis differs considerably from the original, parabolic shape, and is also required, of course, if rib dimensions and reinforcement are varied along the span.

The exact location of the quarter point is at $x = 22.5$ ft. This point does not happen to coincide with any of the centers of the segments. Since moments and thrusts vary only very gradually in an arch subject only to distributed loads, it will be simpler and sufficiently accurate to compute the moment at that segment center located closest to the quarter point, *i.e.*, at $x = 23.5$ ft. According to Fig. 10.11, live load should be placed from the left springing up to $x = 0.125 \times 90 = 38.3$ ft. In the summation process it is quite inconvenient to place loads other than over integer segments. For this reason the first nine segments from the left springing (*i.e.*, up to $x = 40.2$ ft) will be subjected to live load. The error so introduced is entirely negligible. Hence, these nine segments are loaded by 9.2 kips each, the remaining eleven segments by the dead load of 5.8 kips. The corresponding reactions are $R_l = 82.0$ kips and $R_r = 64.4$ kips.

In view of unsymmetrical loading, beam moments $M_0$ and quantities $M_0 y\,\Delta s/I$ must be computed for both halves of the arch. They are entered in Table 10-7 in the columns headed "Case *b*."

In Table 10-7, $x$ and $y$ are given in feet, $y\,\Delta s/I$ in 1/in., $y^2\,\Delta s/I$ in 1/in.$^2$, $M_0$ in foot-kips, and $M_0 y\,\Delta s/I$ in ft-kips per in.$^2$ (It is advisable

TABLE 10-7

| | | | | Case *a* | | Case *b* | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Left half | | Right half | |
| $x$ | $y$ | $y\,\Delta s/I$ | $y^2\,\Delta s/I$ | $M_0$ | $M_0 y\,\Delta s/I$ | $M_0$ | $M_0 y\,\Delta s/I$ | $M_0$ | $M_0 y\,\Delta s/I$ |
| 1.9 | 1.30 | 0.078 | 1.2 | 174 | 13 | 156 | 12 | 123 | 10 |
| 6.0 | 3.90 | 0.235 | 10.9 | 514 | 121 | 444 | 104 | 364 | 85 |
| 10.2 | 6.20 | 0.373 | 27.6 | 823 | 306 | 713 | 269 | 580 | 216 |
| 14.5 | 8.30 | 0.498 | 49.6 | 1101 | 553 | 954 | 475 | 790 | 394 |
| 19.0 | 10.20 | 0.613 | 74.9 | 1349 | 825 | 1159 | 710 | 976 | 598 |
| 23.5 | 11.80 | 0.708 | 100.2 | 1556 | 1102 | 1321 | 935 | 1138 | 806 |
| 28.2 | 13.10 | 0.786 | 123.2 | 1728 | 1358 | 1447 | 1138 | 1276 | 1002 |
| 32.9 | 14.10 | 0.847 | 143.2 | 1862 | 1578 | 1531 | 1298 | 1390 | 1178 |
| 37.7 | 14.70 | 0.882 | 155.5 | 1947 | 1718 | 1570 | 1384 | 1475 | 1301 |
| 42.5 | 14.95 | 0.897 | 160.8 | 1991 | 1787 | 1566 | 1402 | 1533 | 1375 |
| $\Sigma$ | ..... | ..... | 848.1 | ... | 9361 | .... | 7727 | .... | 6965 |

to select units in such a manner as not to obtain unwieldy figures, such as millions on the one hand, or small decimal fractions on the other.)

From the general discussion of Eq. (10.11), Art. 10.2, it is evident that only the first term in the numerator and the first and last term in the denominator of that equation need be considered for this particular roof arch. Hence, for determining $H$, the following equation is used:

$$H = \frac{\Sigma M_0 y\,\Delta s/I}{\Sigma y^2\,\Delta s/I + El/E_{tr}A_{tr}}$$

The denominator then becomes

$$2 \times 848.1 + 4 \times 10^6 \times 90 \times 12/30 \times 10^6 \times 8.3 = 1709.5$$

The maximum force in the tie rod (Case *a*) is

$$H_{max} = \frac{2 \times 9361 \times 12}{1709.5} = 131.5 \text{ kips}$$

For the loading which results in maximum quarter-point moment (Case *b*),

$$H = \frac{(7727 + 6965)\,12}{1709.5} = 103.1 \text{ kips}$$

From Table 10-7 the simple beam moment $M_0$ at $x = 23.5$ ft is seen to be 1321 ft-kips. Hence the moment in the arch rib at that point is

$$M = M_0 - Hy = 1321 - 103.1 \times 11.80 = 103 \text{ ft-kips}$$

Since the point $x = 23.5$ ft is located at the center of the sixth arch

segment, the vertical shear at that point is

$$V = 82.0 - 9.2 \times 5.5 = 31.4 \text{ kips}$$

At that point, from Eq. (10.19*b*), $\theta = 17°42'$. The axial thrust at that point then becomes

$$T = 31.4 \times 0.303 + 103.1 \times 0.95 = 107.4 \text{ kips}$$

It will be noticed that the values of $H_{max}$, $M$, and $T$ are quite close to those obtained in the preliminary design. Such good coincidence obtains only when, as in this case, the final location of the axis is very close to that used in the preliminary design. Even then, significant differences between preliminary and final results are not infrequent; therefore the accurate analysis by summation should always be carried out.

A 12- by 24-in. section reinforced with eight No. 8 bars, as shown in Fig. 10.14, is selected for the arch rib. For $M = 103$ ft-kips and

$$T = 107.4 \text{ kips}$$

the allowable stress, from Diagram 14, is $f_c = 1365$ psi, while the maximum fiber stress, from Diagram 18, is $f_c = 1305$ psi.

With four 1⅝-in. round bars for the tie rod, as obtained in the preliminary design, the stress in the tie is 131,500/8.3 = 15,850 psi. The ends of the tie rods are upset and threaded. The long diameter of the nuts for bars of this size is 3½ in. To accommodate the wrench for tightening the nuts, a 5-in. minimum center-to-center spacing of tie bars is selected.

To transmit the pull of the tie to the concrete, an anchor plate 9 by 12 by 1⅛ in. is provided. Actually, a considerable part of the pull will be transmitted to the concrete by bond over the embedded length of the tie bars; this will be discounted, however, and the anchor designed for the entire pull. The bearing surface is $9 \times 12 - 4 \times 2.1 = 99.6$ sq in., and the bearing pressure is 131,500/99.6 = 1320 psi. As is easily computed from the dimensions of Fig. 10.14, the bearing surface represents 39.5 per cent of the cross section of the arch extension in which the anchor is located. For this percentage the ACI Code allows a bearing pressure of 1370 psi. The bending stresses in the anchor plate are investigated in the long direction, for a strip 1 in. wide. The maximum negative moment of the portion projecting beyond the center line of the anchors is $1320 \times 2.5^2/2 = 4120$ in-lb, while the maximum positive moment at the center line of the plate is

$$1320 \times 7^2/8 - 4120 = 3960 \text{ in.-lb}$$

With the section modulus equal to $1.125^2/6 = 0.210$, the maximum bending stress is 4120/0.210 = 19,600 psi, which is slightly smaller than 20,000 psi, the permissible bending stress for structural steel.

The hinges transmit only the vertical reaction $V = 92$ kips, since the horizontal component is resisted by the tie. To reduce the number and size of the hinge bars so as to accommodate them in the limited width, a steel with 60,000 psi yield point will be used, so that the allowable compression stress $f'_s = 0.3 \times 60{,}000 = 18{,}000$ psi. With an angle $\theta = 30°$ (Fig. 10.13), the required area of hinge bars is computed from Eq. (10.23), that is, $A'_s = 92{,}000/(18{,}000 \times 0.866) = 5.9$ sq in. Six $1\frac{1}{8}$-in. round plain bars are provided, furnishing an area of 5.95 sq in. With a hinge opening of 6 in., as shown in Fig. 10.14, the free length of the hinge bars is $L = 6/0.866 = 6.93$ in., while the radius of gyration is

$$r = 1.125/4 = 0.281$$

Hence the ratio $L/r = 24.7$ is within the limits recommended in Art. 10.6.

The required lateral reinforcement is obtained from Eq. (10.24) where, in this case, $a = 8 \times 1.125 = 9.0$ in., $b = 12$ in., $N = 92{,}000$ lb, and $S = 0$. With an allowable stress $f''_s = 20{,}000$ psi, the required area becomes $A''_s = 0.79$ sq in. It is seen from Fig. 10.11 that within the length $a = 9$ in. four No. 3 closed ties are provided which furnish an area of $8 \times 0.11 = 0.88$ sq in.

The required length of embedment for the hinge bars is

$$30 \times 1.125 = 34 \text{ in.}$$

beyond the faces of the concrete at either side of the hinge.

To prevent horizontal buckling of laterally unsupported arch ribs, the Special Committee of the American Society of Civil Engineering recommended[1] that horizontal braces should be provided at distances not exceeding 25 to 30 times the width of the rib. The lower figure should be used for concretes of high strength. For this purpose, as shown in Fig. 10.14, 10- by 14-in. bracing members are provided at the crown and the quarter points.

In order to avoid bending moments caused by eccentricities, the axes of the arch, the tie, and the hinge must be made to intersect at one common point, as shown in the hinge detail of Fig. 10.14.

[1] "Concrete and Reinforced Concrete Arches," Final Report, *Trans. ASCE*, vol. 100, p. 1429, 1935.

# SECTION 11

# SLAB, BEAM, AND GIRDER BRIDGES

**11.1. Types of Bridges.** Reinforced concrete is particularly adapted to use in short- to moderate-span highway bridges, because of its durability, rigidity, and economy, as well as the comparative ease with which a pleasing architectural appearance can be secured. The most widely used general types of concrete bridges for short spans are the slab bridge

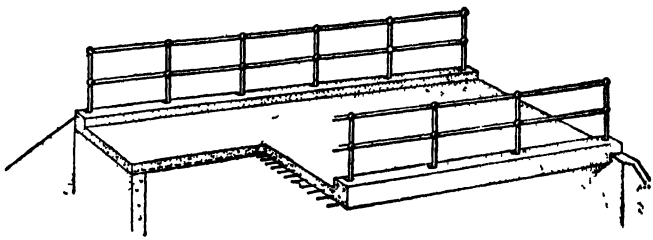


FIG. 11.1. Slab bridge.

(Fig. 11.1), the T-beam or deck-girder bridge (Fig. 11.2), and the through-girder bridge (Fig. 11.3). Concrete slab bridges are economical for very short spans, about 10 to 25 ft. Concrete girder spans are normally economical only for spans up to about 70 to 90 ft. They have been used, however, for spans well over 100 ft, but in most of these bridges economy

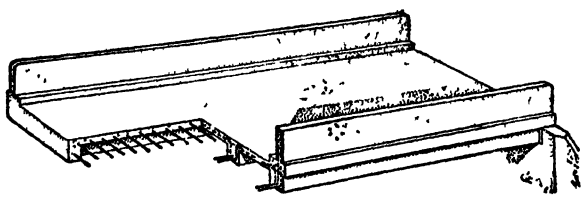


FIG. 11.2. Deck-girder bridge.

has been sacrificed for other considerations which were of more importance in each specific case.

Two relatively recent engineering developments have led to the increased use of structural concrete in bridges of medium span length—100 to 250 ft. One has been the perfection of devices—commonly called

shear connectors—designed to make a concrete deck act integrally with supporting steel stringers. Although such a bridge is composite in nature and not strictly a reinforced concrete structure in the usual sense of the

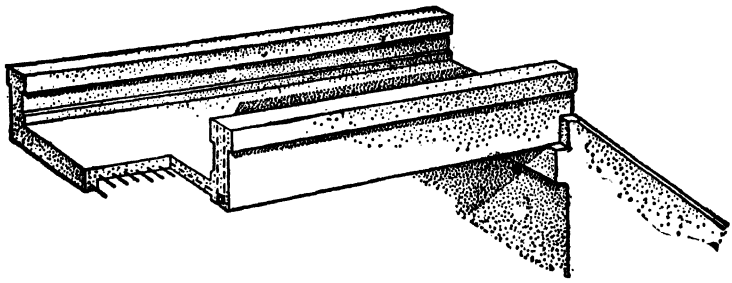


FIG. 11.3. Through-girder bridge.

term, the design of this type of bridge will be discussed in some detail in this section because of its widespread use in present day highway overpass construction. Figure 11.4 shows the steel stringers and shear connectors of a composite bridge prior to placement of the reinforcement and

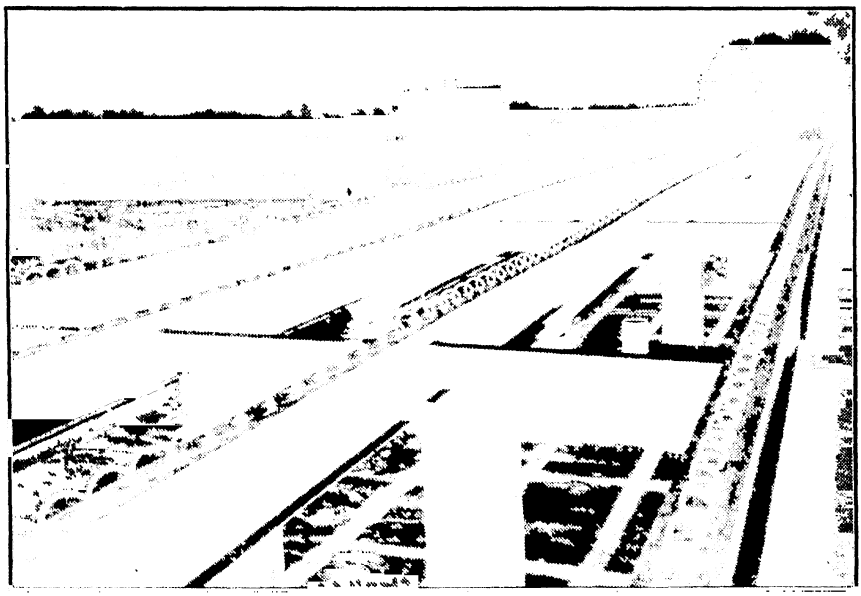


FIG. 11.4. Composite bridge during construction.

concrete. The other development, prestressed concrete, is discussed in Sec. 12. A number of prestressed concrete bridges have been, and are being, built in this country; but we still have not used the medium to the

extent that it has been used in Europe. Figure 11.5 shows a French prestressed concrete highway bridge with a span of 180 ft. Similar bridges have been built with spans up to 240 ft.

Structural steel is used more generally than reinforced concrete for bridges which are intended primarily for railroad traffic, except the shortest spans, for which reinforced concrete slab bridges are frequently constructed. This section will therefore be devoted to the discussion of the details and design of highway bridges only.

**11.2. Live Loads.** *Truck Loadings.* The live loads specified by the American Association of State Highway Officials—whose specifications

FIG. 11.5. The Luzancy bridge, 180-ft span, prestressed concrete.

control the design of most American highway bridges—consist of standard, idealized trucks or of live loads which are equivalent to a series of trucks. Two systems of loadings are provided: the *H* loadings, as shown in Fig. 11.6, and the *H-S* loadings, as shown in Fig. 11.7. The *H-S* loadings are heavier than the corresponding *H* loadings.

As shown in the figures, the highway loadings are divided into several classes. The number of the loading indicates the gross weight in tons of the truck or tractor. The gross weight of the truck or tractor is divided between the front and rear axles as shown in Figs. 11.6 and 11.7. For the *H-S* loadings, a semitrailer with rear-axle load equal to the rear-axle load on the tractor is added.

Lane loadings, which are to be used when they produce greater stress then the corresponding truck loadings, are shown in Fig. 11.8. In general, the lane loadings, which are designed to approximate the effect of a series of trucks, govern the design of the longer-span bridges.

*Other Roadway Loadings.* The possibility that the bridge may be required to carry electric railways, railroad freight cars, military vehicles, or other extraordinary vehicles should be considered in design. The class of such traffic shall determine the loads to be used.

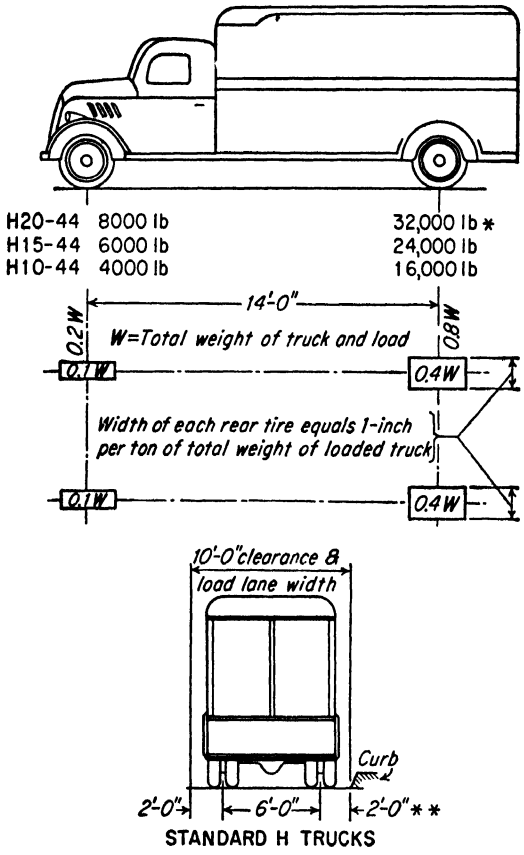



FIG 11.6 Standard *H* truck loadings

* In the design of floors (concrete slabs, steel grid floors and timber floors) for *H*20 or *H*20-*S*16 loading one axle load of 24 000 lb or two axle loads of 16 000 lb each spaced four ft apart may be used, whichever produces the greater stress instead of the 32 000 lb axle shown

** For slab design the center line of wheel shall be assumed to be 1 ft from face of curb

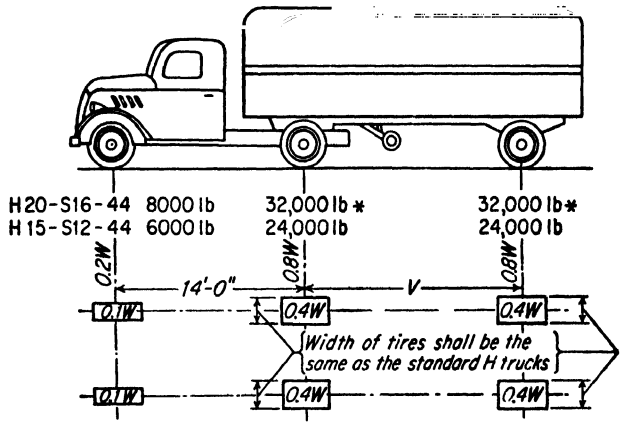
*Sidewalk Loadings.* Sidewalk floors, stringers, and their immediate supports are usually designed for a live load of at least 85 psf of sidewalk area.

*Selection of Loadings.* The AASHO specifications state that for truck highways, or for other highways which carry heavy truck traffic, the minimum live load shall be the *H*15-*S*12 loading. It is common practice at present to design bridge spans on major highways for the *H*20-*S*16 load-

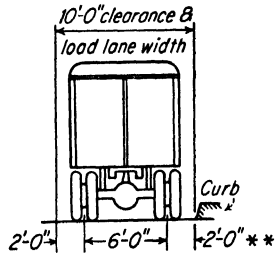
ing, using the lighter loadings only for structures on secondary roads where the occurrence of heavy traffic will be negligible.

**11.3. Application of Loadings.** Some of the more important rules for applying the selected AASHO loading follow.

1. Where the spacing of main supporting members exceeds 10.5 ft for concrete or steel-grid floors, the lane loading or standard trucks shall be




*W= Combined weight on the first two axles which is the same as for the corresponding H truck*
*V= Variable spacing-14 feet to 30 feet inclusive. Spacing to be used is that which produces maximum stresses.*




STANDARD H-S TRUCKS

FIG. 11.7. Standard *H-S* truck loadings.

* In the design of floors (concrete slabs, steel grid floors, and timber floors) for *H*20 or *H*20-*S*16 loading, one axle load of 24,000 lb or two axle loads of 16,000 lb each, spaced four ft apart, may be used, whichever produces the greater stress, instead of the 32,000-lb axle shown.

** For slab design the center line of wheel shall be assumed to be 1 ft from face of curb.

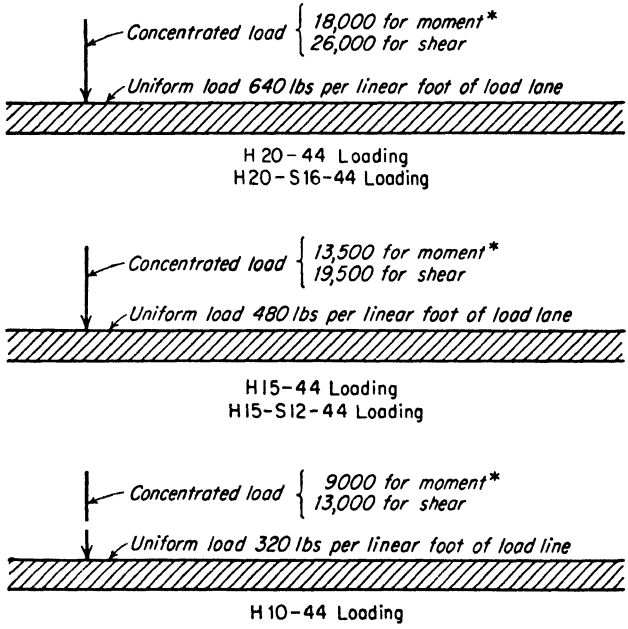
assumed to occupy a width of 10 ft. They shall be assumed to occupy any position within their design traffic lane which will produce the maximum stress. The width of the design traffic lanes shall be determined from the formula

$$W = \frac{W_c}{N}$$

where $W_c$ = roadway width between curbs exclusive of median strip
$N$ = number of design traffic lanes as shown in the following table
$W$ = width of design traffic lane

| $W_c$, ft | $N$ |
|---|---|
| 20–30 | 2 |
| Over 30–42 | 3 |
| Over 42–54 | 4 |
| Over 54–66 | 5 |
| Over 66–78 | 6 |
| Over 78–90 | 7 |
| Over 90–102 | 8 |
| Over 102–114 | 9 |
| Over 114–126 | 10 |

2. Each 10-ft lane loading or single standard truck shall be considered as a unit, and fractional load lane widths or fractional trucks shall not be used.




H LANE AND HS LANE LOADINGS

FIG. 11.8. Equivalent lane loadings.

* For continuous spans another concentrated load of equal weight shall be placed in one other span in the series in such position as to produce maximum negative moment.

3. The number and position of the lane loading or truck loading shall be as specified above. The type of loading to be used—lane loading or

truck loading—shall be such as to produce maximum stress subject to the reduction specified below.

*Reduction in Load Intensity.* Where maximum stresses are produced in any member by loading any number of traffic lanes simultaneously, the following percentages of the resultant live-load stresses shall be used in view of improbable coincident maximum loading:

| | *Per cent* |
|---|---|
| One or two lanes | 100 |
| Three lanes | 90 |
| Four lanes or more | 75 |

**11.4. Impact.** Live-load stresses due to truck loading (or equivalent lane loading) are increased to make allowance for vibration and the sudden application of the load. The increase is computed by the formula

$$I = \frac{50}{l + 125}$$

where $I$ = impact fraction of live-load stress
$l$ = loaded length, ft

The maximum impact allowance to be used shall be 30 per cent.

**11.5. Distribution of Loads.** When a concentrated load is placed on a reinforced concrete slab, the load is distributed over a larger area than the actual contact area. For example, if a concentrated load with a bearing area of 1 sq ft were placed on the slab of a through-girder bridge, it is reasonable to assume that, on account of the stiffness of the slab, the strips of slab at right angles to the girder and adjacent to the 1-ft strip in direct contact with the load would assist in carrying the load. Similarly, with the beams of a T-beam bridge, spaced fairly close, a concentrated load placed directly over one of the beams would not be carried entirely by that beam, for the concrete slab is sufficiently rigid to transfer part of the load to adjacent beams.

No distribution is assumed in the direction of the span of the member. The effect of any such distribution would be comparatively small.

The following recommendations for the distribution of loads are taken from the specifications of the American Association of State Highway Officials.

**11.6. Distribution of Wheel Loads on Concrete Slabs.** The pertinent rules for the distribution of wheel loads on concrete slabs and some additional design requirements are as follows.

*Span Lengths.* For simple spans, the span length shall be the distance center to center of supports but not to exceed clear span plus thickness of slab.

The following effective span lengths shall be used in calculating distribution of loads and bending moments for slabs continuous over more than two supports:

Slabs monolithic with beam (without haunches): $S$ = clear span
Slabs supported on steel stringers: $S$ = distance between edges of flanges plus ½ of the stringer flange width
Slabs supported on timber stringer: $S$ = clear span plus ½ thickness of stringer

*Edge Distance of Wheel Load.* In designing slabs, the center line of wheel load shall be assumed to be 1 ft from the face of the curb.

*Bending Moment.* Bending moment per foot width of slab shall be calculated according to methods given under Cases *A*, *B*, and *C*.

In Cases *A*, *B*, and *C*:

$S$ = effective span length as defined under "Span Lengths"
$E$ = width of slab over which a wheel load is distributed
$N$ = number of lanes of traffic on bridge
$W$ = width of roadway between curbs on bridge
$W$ = width of graded roadway across culverts
$Q$ = uniform lane load per linear foot of lane
$P$ = load on one wheel
$P1$ = load on one wheel of single axle
$P2$ = load on one wheel of tandem axle
$P'$ = concentrated lane load per lane

**Case *A*. Main Reinforcement Perpendicular to Traffic**
(See note under Case *B*)

| Distribution of wheel loads | Formulas for moments | |
|---|---|---|
| | Freely supported spans | Continuous spans |
| **Single axle:** | | |
| Spans 2 to 7 ft, $E = 0.6S + 2.5$ | $M = +0.25\frac{P1}{E}S$ | $M = \pm 0.2\frac{P1}{E}S$ |
| Spans over 7 ft, $E = 0.4S + 3.75$ | $M = +0.25\frac{P1}{E}S$ | $M = \pm 0.2\frac{P1}{E}S$ |
| **Tandem axles:** | | |
| Spans 2 to 7 ft, $E = 0.36S + 2.58$ | $M = +0.25\frac{P2}{E}S$ | $M = \pm 0.2\frac{P2}{E}S$ |
| Spans over 7 ft, $E = 0.063S + 4.65$ | $M = +0.25\frac{P2}{E}S$ | $M = \pm 0.2\frac{P2}{E}S$ |

NOTE: In the design for *H*20 or *H*20-*S*16 loads, the single 24,000-lb axle governs for spans under 10.5 ft and the two 16,000-lb axles govern for spans of 10.5 ft or over.

**Case *B*. Main Reinforcement Parallel to Traffic, Spans 2 to 12 Feet***

| Distribution | Formulas for moments | |
|---|---|---|
| | Freely supported spans | Continuous spans |
| Spans 2 to 12 ft, $E = 0.175S + 3.2$ | $M = +0.25\frac{P}{E}S$ | $M = \pm 0.2\frac{P}{E}S$ |

NOTE: The formulas for distribution and moment in Cases *A* and *B* utilize a single wheel load which has been found to simplify calculations and to give essentially the values obtained by the more exact method of placing all wheels of a truck in positions which produce maximum moment.

**Case *C*. Main Reinforcement Parallel to Traffic, Spans over 12 Feet***

Distribution:

(*a*) Wheel loads: $E = \frac{10N + W}{4N}$; load per foot width of slab $= \frac{P}{E}$

(*b*) Lane loads:

$$\text{Uniform load} = \frac{NQ}{0.5W + 5N} \text{ per square foot of slab}$$

$$\text{Concentrated load} = \frac{NP'}{0.5W + 5N} \text{ per foot width of slab}$$

The loads as here determined shall be placed on the span or spans in position to cause maximum positive or negative moments, following standard practice for design of simple or continuous spans.

The portions of the spans between inflection points may be designed as simple spans and the portions over piers designed as cantilevers supporting the simple spans.

*Edge Beams* (*Longitudinal*). Edge beams shall be provided for all slabs having main reinforcement parallel to traffic. The beam may consist of the curb section reinforced, of a beam support, or of additional slab width. It shall be designed to resist a live-load moment of $0.10PS$, where $P$ = the wheel load and $S$ = span length.

The moment as stated is for a freely supported span. It may be reduced 20 per cent for continuous spans unless a greater reduction results from a more exact analysis.

*Distribution Reinforcement.* Reinforcement shall be placed in all slabs transverse to the main steel reinforcement, to provide for lateral distribution of the concentrated live loads, except that reinforcement will not be required for culverts or bridge slabs when the depth of the fill over the

* In the design of *H*20 and *H*20-*S*16 loads, the single 24,000-lb axle shall be used for spans under 18 ft in accordance with the formulas of Cases *B* and *C*. For spans 18 ft and over the two 16,000-lb axles govern and the moments obtained from the formulas of Cases *B* and *C* using a single axle load of 24,000 lb shall be increased by the percentage expressed by the formula 1.6 (*S*-18).

slab exceeds 2 ft. The amount shall be the percentage of the main reinforcement steel required for positive moment as given by the following formula:

$$\text{Percentage} = \frac{100}{\sqrt{S}} \qquad \text{maximum 50 per cent}$$

where $S$ equals the effective span of the slab in feet.

*Shear and Bond Stress in Slabs.* Slabs designed for bending moment in accordance with the foregoing shall be considered satisfactory in bond and shear.

**11.7. Distribution of Wheel Loads to Longitudinal Beams and Transverse Floor Beams.** The following rules govern the distribution of wheel loads affecting longitudinal and transverse concrete beams supporting concrete slabs. The AASHO specification contains additional recommendations for steel and timber flooring which are not reproduced here.

*Position of Loads for Shear.* In calculating end shears and end reactions in transverse floor beams and longitudinal beams and stringers, no lateral or longitudinal distribution of the wheel shall be assumed for the wheel or axle load adjacent to the end at which the stress is being determined. For loads in other positions on the span, the distribution for shear shall be determined by the method prescribed for moment.

*Bending Moment in Stringers.* In calculating bending moments in longitudinal beams or stringers, no longitudinal distribution of the wheel loads shall be assumed. The lateral distribution shall be determined as follows:

1. Interior stringers. Interior stringers supporting concrete floors shall be designed for loads determined in accordance with the following:

| *One traffic lane, fraction of a wheel load to each stringer* | *Two or more traffic lanes, fraction of a wheel load to each stringer* |
|---|---|
| $\frac{S^*}{6.0}$ | $\frac{S^*}{5.0}$ |
| If $S$ exceeds 6.0 ft, see footnote † | If $S$ exceeds 10.5 ft, see footnote † |

* $S$ = average spacing of stringers in feet.

† In this case the load on each stringer shall be the reaction of the wheel loads, assuming the flooring between stringers to act as a simple beam.

2. Outside stringers. The live load supported by outside stringers shall be the reaction of the truck wheels, assuming the flooring to act as a simple beam between stringers.

3. Total capacity of stringers. The combined load capacity of the beams in a panel shall not be less than the total live and dead load in the panel.

*Bending Moment in Floor Beams (Transverse).* In calculating bending moments in floor beams, no transverse distribution of the wheel loads

shall be assumed. If longitudinal stringers are omitted and the concrete floor is supported directly on floor beams, the fraction of wheel load allotted to each floor beam is $S/6$, where $S$ equals the spacing of beams in feet. If $S$ exceeds 6 ft, the load on the beam shall be the reaction of the wheel loads, assuming the flooring between beams to act as a simple beam.

**11.8. Abutments.** Bridge abutments serve to transmit the load from the superstructure to the foundation and they also act as retaining walls to hold back the earth fill behind them.

*Bridge Seats.* The bridge seats of abutments which support the fixed ends of concrete highway bridges are quite frequently built as horizontal

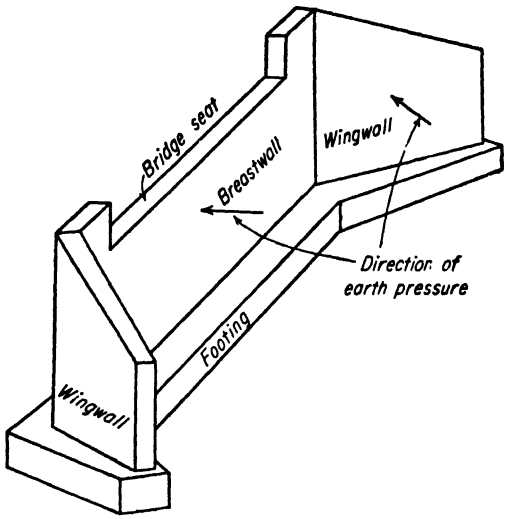



FIG. 11.9. Details of abutment without backwall.

surfaces without parapets or backwalls, as shown in Fig. 11.9. The slab or the beams of the deck rest directly on the bridge seat; in deck-girder bridges, transverse diaphragm walls are constructed between the beams to hold back the earth above the bridge seat. These diaphragms are thin walls, 6 or 8 in. in thickness, with a nominal amount of reinforcement.

Figure 11.10 shows a modified form of the construction described above, in which a very low backwall is used, primarily for the purpose of counteracting the tendency of the abutment to move inward under the deck. The dowels $b$ and $c$ which are shown in the abutment and in the deck construction serve to tie the deck, the abutment, and the approach slab together.

Another type of bridge seat is shown in Fig. 11.11. This type is particularly adapted to the fixed ends of deck-girder bridges. The bridge seat is constructed with notches into which the bridge beams are built. Fig. 11.11 is a diagrammatic sketch, and does not show the final relative

position of the deck and abutment, the two parts of the figure actually fit together as shown by the dotted lines Ledges are provided in the abutment and at the top of the deck beams to support the approach slab, as shown in the figure Dowels are placed in the abutment in such a manner as to project up into the beams and deck slab.

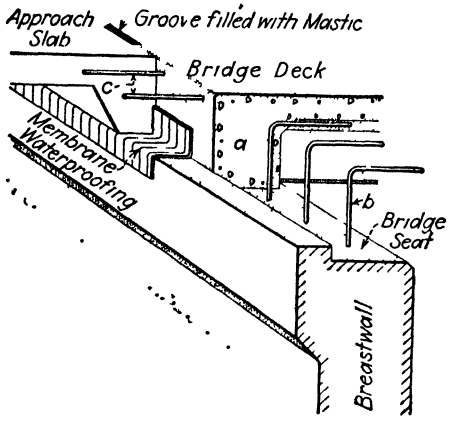



FIG. 11 10 Details of bridge seat with low backwall

A third type of bridge-seat construction is shown in Fig 11 12. This is the usual type with backwall, and is used at the expansion end of the bridge. Two arrangements of joints are shown in the figure The backwall is extended upward to the roadway surface in *a*, while it stops at the bottom of the roadway slabs in *b* Type *b* is usually preferred, in

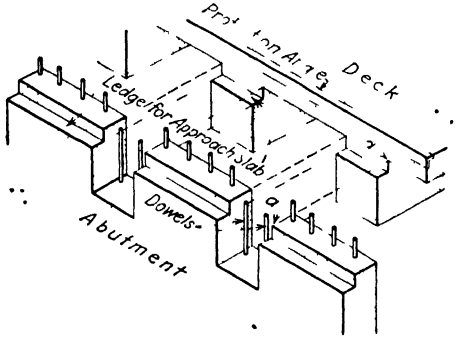



FIG 11 11 Details of bridge seat with pockets for beams

that only one joint is required in the roadway surface, as compared with two joints in type *a*.

*Breastwalls.* The two most common types of breastwalls are the cantilever retaining wall of reinforced concrete and the type that acts as a vertical slab supported horizontally by the deck and by the foundation.

Buttressed breastwalls are rarely used in short-span concrete bridges Gravity walls of plain concrete are sometimes used especially where the height of the abutment is not great

The cantilever breastwall is generally used at the expansion end of the bridge The type that is supported horizontally by the deck and the

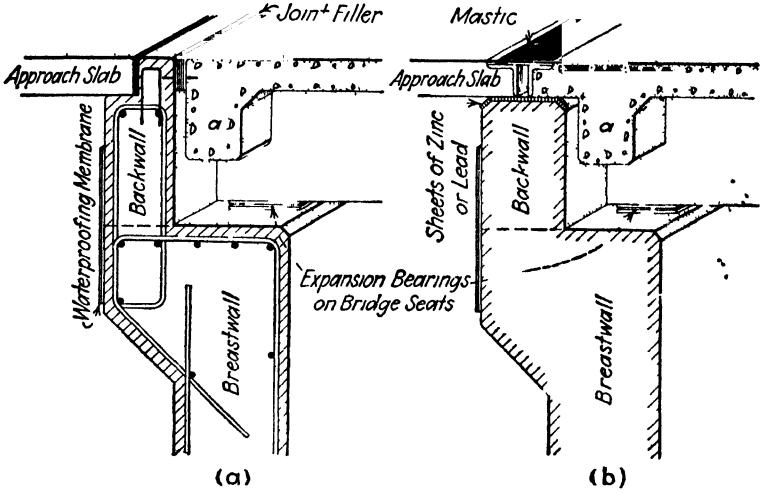



FIG 11 12 Details of bridge seats with backwalls

footing can be used only at the fixed end, in such cases it is always advisable to have at least a low parapet or backwall as shown in Fig 11 10, in order to obtain the direct horizontal resistance which is assumed to be furnished by the deck

*Wingwalls* Gravity or cantilever type wingwalls may be used These are topped with a simple coping which is built parallel to and slightly above the earth fill

It is advisable to construct vertical expansion joints at the junction of the wingwalls and breastwall in order to avoid unsightly cracks which are apt to form along these planes Sometimes instead of an expansion joint, merely a vertical groove is constructed at the joint, any crack that may be developed will then be inconspicuous The expansion joint or the groove should preferably be placed at the end of the bridge seat, as shown in Fig 11 13

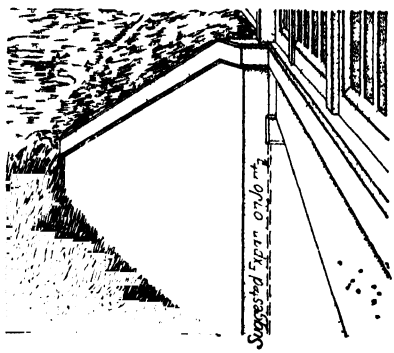


FIG 11 13 Vertical expansion joint in breastwall

**11.9. Miscellaneous Construction Details.** *Diaphragms.* In deck-girder bridges, thin transverse walls called diaphragms are constructed between the beams at the ends of the bridge, as shown at *a* in Figs. 11.10 and 11.12. These diaphragms serve two purposes: first, they furnish lateral support for the beams, and second, when no parapet or backwall is built above the bridge seat they prevent the earth fill from spilling out on the bridge seat. At the fixed ends, the diaphragms may rest directly on the bridge seat, as shown in Fig. 11.10; at the expansion ends, the diaphragms should be kept clear of the bridge seat, as shown in Fig. 11.12, so as not to develop friction which would interfere with the freedom of longitudinal movement. The diaphragms are usually from 6 to 8 in. thick, with a nominal amount of reinforcement.

Intermediate diaphragms should be constructed between the beams of deck-girder bridges with spans greater than 40 ft; the intermediate diaphragms are placed at the center of the span, or at the third points of the span.

*Fixed Bearings.* Fixed bearings can safely be used at both ends of spans up to 40 ft. Two such bearings are illustrated in Figs. 11.10 and 11.11. The dowels which serve to tie together the deck construction and the abutment are not designed to resist any appreciable amount of bending moment, and hence no joint rigidity can be assumed in the design of the deck or abutment.

*Expansion Bearings.* For spans greater than 40 ft, an expansion bearing should be provided at one end of the bridge. A suitable form of expansion bearing for spans up to 50 ft is shown in Fig. 11.12. A pair of steel plates is placed under each girder; the lower plate is anchored to the concrete in the abutment, and the upper plate is fastened to the under side of the girder. Frequently, a thin layer of graphited asbestos, zinc, or copper is placed between the plates to reduce the friction.

For the shorter spans and light loadings, frequently the only expansion device is a layer of tar paper inserted between the bridge seat and the deck. Obviously this detail is not desirable where provision for expansion is of any importance, because of the questionable efficiency of the construction.

For spans greater than 50 ft, the deflection of the deck may rotate the ends of the girders so much that the bearing pressure may become concentrated on a narrow strip along the front edge of the bridge seat and may cause local damage if flat bearing plates are used. In order to counteract this tendency, one plane and one curved plate should be used, as shown in Fig. 11.14.

*Deck Joints.* The number of expansion joints in a bridge deck should be kept as low as possible. In single-span bridges of the type under discussion, one such joint at most is usually necessary; that joint is at

one end of the bridge A suitable detail for the joint is shown in Fig 11.12*b*, in this detail an angle is anchored to the end of the deck slab, and a similar angle is anchored to the end of the approach slab. A tread

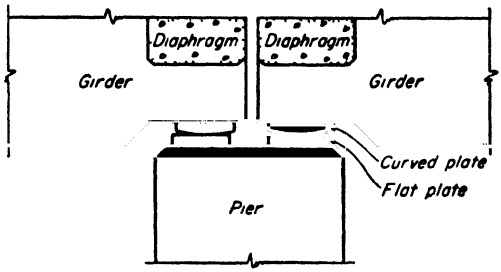



FIG 11 14 Expansion bearings

plate is fastened, by means of rivets or tap screws, to the top of the angle at the end of the deck, and this plate bears on the angle at the end of the approach slab The tread plate does not cover the latter angle completely, sufficient provision for expansion of the deck slab being made The groove between the end of the tread plate and the edge of the approach slab is usually filled with mastic, as shown in Fig 11 12*b*, in order to make the joint watertight The mastic is soft enough to permit movement of the tread plate, but firm enough to resist displacement

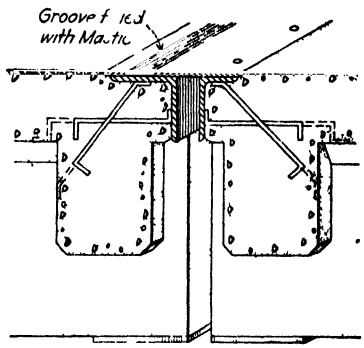



FIG 11 15 Details of expansion joint

An enlarged detail of a similar joint is shown in Fig 11 15, this detail applies specifically to the joint between the ends of two adjacent spans in a multispan bridge

Thin copper or zinc strips, bent in the form of the letter T as shown in Fig. 11.16, are sometimes placed across a joint, and embedded in the

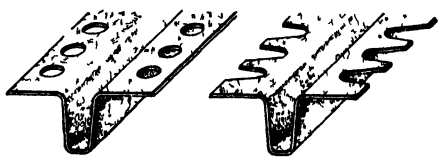


FIG. 11 16. Metal strips for expansion joints

concrete on either side of the joint, in order to add to the watertightness. The joint is filled with a mastic compound. A detail of this type is shown in Fig. 11.12*a*.

Joints at the fixed end of a span are made watertight by simply filling a groove at the top of the joint with mastic, as shown in Fig. 11.10. The membrane waterproofing which is shown on top of the low parapet in this figure is furnished to ensure watertightness in the horizontal joint.

*Drainage.* Surface water on bridge decks should be disposed of as quickly and as directly as possible. This is accomplished by crowning the roadway about $\frac{1}{8}$ in. per ft, and pitching the gutters to drain into inlets. In single-span bridges which are built on a grade, no special provision for longitudinal drainage is ordinarily necessary. The water is carried by the transverse crown to the curbs or gutters and thence to the low end of the bridge, where it can easily be deflected away from the roadway.

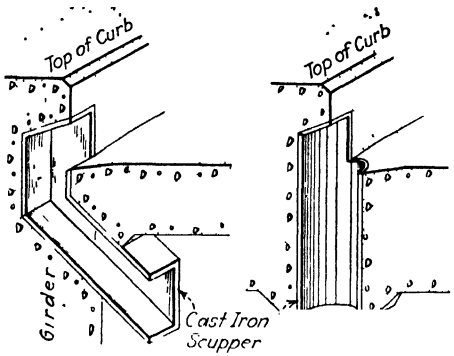



FIG. 11.17 Typical scupper details.

If the bridge is horizontal longitudinally, a longitudinal camber can be built into the bridge deck by elevating the middle portion of the span when the forms are erected. This camber will serve to carry the water in the gutters to either end of the span. A camber of about $\frac{1}{10}$ in. per ft is satisfactory for concrete decks.

When necessary, special drain inlets can be constructed by building cast-iron scuppers in the gutters. Two types of scuppers are shown in Fig. 11.17. The scuppers should be so designed as to prevent the drain water from touching the concrete. Drain inlets are generally necessary in multispan bridges, because of the impracticability of carrying the surface water to the ends of the bridge.

**11.10. Design of a Slab Bridge.** A slab bridge similar to that shown in Fig. 11.1, but with a concrete railing, is to be designed according to the following data:

| | |
|---|---|
| Clear span | 15 ft-0 in. |
| Clear width | 26 ft-0 in. |
| Live loading | *H*20-*S*16 |
| Wearing surface | 30 psf |

$f'_c = 3000$; $f_s = 18{,}000$; $n = 10$. Following the AASHO specifications an allowable concrete stress, $f_c$, of $0.40 \times f'_c = 1200$ psi will be used.

The effective span of the slab is assumed as 15 ft + 1 ft = 16 ft and a total thickness of slab of 11 in. is selected for trial. The total dead load per square foot is then 167 lb and the dead-load moment

$$\tfrac{1}{8}wl^2 = \tfrac{1}{8} \times 167 \times (16)^2 = 5350 \text{ ft-lb.}$$

The load on each rear wheel is 12,000 lb.

$$E = \frac{10N + W}{4N} = \frac{10(2) + 26}{4(2)} = 5.75$$

since Case *c* of Art. 11.6 applies.

The load on a unit width of slab is therefore

$$\frac{12{,}000}{5.75} = 2080 \text{ lb}$$

and the live-load moment is

$$\frac{2080}{2} \times \frac{16}{2} = 8320 \text{ ft-lb}$$

The impact coefficient computed by the formula of Art. 11.4 is

$$I = \frac{50}{16 + 125} = 0.355$$

which is greater than the specified maximum of 30. The latter will therefore be used giving impact moment of

$$0.30 \times 8320 = 2500 \text{ ft-lb}$$

The total moment due to dead, live, and impact effects is 16,170 ft-lb.

For the unit stresses specified, $K = 207$ (Table 5) and

$$d = \sqrt{\frac{16{,}170 \times 12}{208 \times 12}} = 8.8 \text{ in.}$$

Taking $d = 9$ in. and $t = 11$ in. as assumed, the protective covering under the main reinforcement will be about $1\frac{1}{2}$ in., which is greater than the minimum of 1 in. specified by the AASHO for slabs. The assumed thickness is therefore adequate and will be used.

The required main reinforcement is

$$A_s = \frac{16{,}170 \times 12}{18{,}000 \times 0.867 \times 9} = 1.38 \text{ sq in. per ft}$$

which is furnished by No. 7 bars 5 in. center to center ($A_s = 1.44$ sq in. per ft). As specified in Art. 11.6, the amount of transverse reinforce-

ment required for proper distribution of the concentrated loads (also to provide for shrinkage) is $A_s/\sqrt{S} = 1.44/\sqrt{16} = 0.36$ sq in.; No. 5 bars 10 in. center to center are placed directly on top of the longitudinal reinforcement. The railing is poured monolithically with the slab and tied to it with No. 5 bars, 12 in. center to center. These bars extend horizontally into the slab and vertically into the railing. The crown of the roadway is obtained by varying the thickness of the bituminous wearing surface.

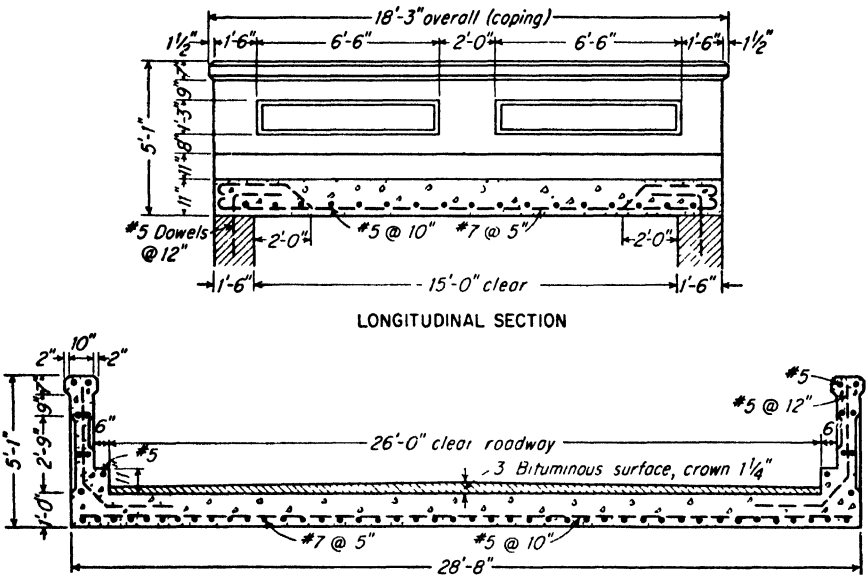



FIG. 11.18. Details of slab bridge.

No expansion bearings are necessary for a span of this length, and a fixed bearing is constructed at each end by placing vertical dowels in the abutment so that they will project up into the concrete deck slab. The bridge seat is to be coated with a bituminous mixture before the slab is poured. Details are shown in Fig. 11.18.

## DESIGN OF A T-BEAM OR DECK-GIRDER BRIDGE

**11.11. Data and Specifications.** It is required to design a T-beam or deck-girder bridge for the following conditions:

| | |
|---|---|
| Clear span | 48 ft-0 in. |
| Clear width | 26 ft-0 in. |
| Live loading | *H20-S16* |

$f'_c = 3000$; $f_s = 20{,}000$; $f_v = 20{,}000$; $n = 10$. The bridge is to consist of four intermediate beams and two outside beams supporting a floor slab. The outside beams are to project at least 9 in. above the floor slab to act as a curb.

**11.12. Slab.** Since the outside beams will be rather heavy in comparison with the floor slab and since all beams and slabs will be poured monolithically and reinforced by diaphragms, the slab may be considered as fully restrained by the outside beams and the span taken as the clear distance between supports. Assuming that the intermediate beams will be 14 in. in width, the maximum clear span of the slab will be 4 ft-4 in. (see Fig. 11.19). The outside beams will be assumed to require a width of 20 in. The AASHO specifications for design moments will be followed (see Art. 11.6).

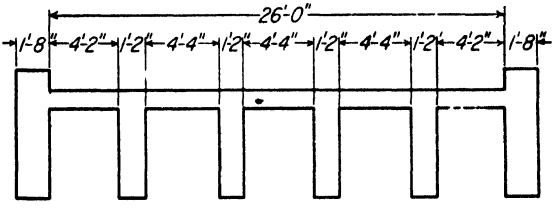



FIG. 11.19

Assuming a total thickness of slab (including a $\frac{3}{4}$-in. wearing surface) of 6 in. and allowing 15 psf for possible future protective covering, the total dead load per square foot is 90 lb. A coefficient of $\frac{1}{10}$ for positive and negative dead-load moments will be assumed in the absence of definite specification values. The dead-load moment is

$$\tfrac{1}{10} \times 90 \times (4.33)^2 \times 12 = 2{,}020 \text{ in.-lb}$$

For live-load moment computations, Case *A* of Art. 11.6 applies.

$$E = 0.6S + 2.5 = 0.6 \times 4.33 + 2.5 = 5.1 \text{ ft}$$

$$M_L = \pm 0.2 \frac{P1}{E} S = \pm 0.2 \times \frac{12{,}000}{5.1} \times 4.33 \times 12 = 24{,}200 \text{ in.-lb}$$

Since the loaded length is small, the impact coefficient is 0.3, the impact moment is 7260 in.-lb, and the total moment is 33,480 in.-lb. Then

$$d = \sqrt{\frac{33{,}480}{12 \times 197}} = 3.76 \text{ in.}$$

Taking $d = 3\frac{3}{4}$ in. with $1\frac{1}{2}$ in. of insulation below the center of the bars, and with the $\frac{3}{4}$-in. wearing surface, which is not considered structurally effective, the total thickness is 6 in., as assumed.

$$A_s = \frac{33{,}480}{20{,}000 \times 0.875 \times 3.75} = 0.51 \text{ sq in.}$$

which is furnished by No. 5 bars 7 in. center to center. In order to avoid excessive bending of bars, straight bars will be used in the slab. Straight No. 5 bars 7 in. on centers top and bottom will provide a surplus of steel in some areas but the cost of the additional steel will probably be offset by the savings in fabrication and handling of the bent bars which would otherwise be required. Temperature and distribution stresses in the direction of the span are provided for by placing four No. 4 bars in the top and bottom of each slab panel, parallel to the beams, at about 12-in. centers. Complete details of the slab are shown in Fig. 11.24.

**11.13. Intermediate Beams.** The intermediate beams are T beams with a flange width equal to the distance center to center of beams. The required dimensions are governed by either the maximum moment or maximum shear. The bridge seats will be assumed 2 ft-0 in. in width and the effective span length center to center of bearings taken as 50 ft-0 in.

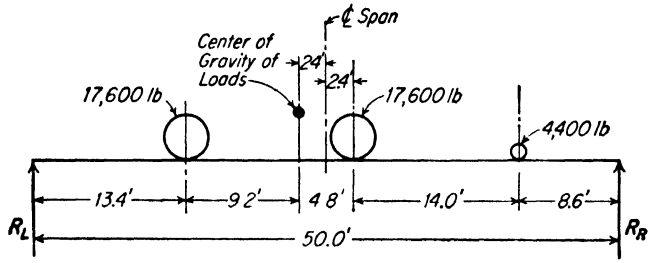



FIG. 11.20

*Dead-load Moments.* The weight from the slab per foot of beam is $90 \times 5.50 = 495$ lb. The cross section of the beam below the slab is assumed as 14 in. $\times$ 28 in., which adds an additional weight of 409 lb per ft, making the total weight 904 lb per ft. Then the dead-load moment at the center of the span is

$$M_D = \tfrac{1}{8} \times 904 \times 50^2 \times 12 = 3{,}390{,}000 \text{ in.-lb}$$

In order to determine the points at which some of the horizontal steel may be bent up, it is necessary to compute the moment at some sections between the point of maximum moment and the support. At 10 ft from the support, the dead-load moment is 2,170,000 in.-lb, and at 20 ft, 3,260,000 in.-lb.

*Live-load Moments.* The absolute maximum live-load moment will occur with an *H*20-S16 truck on the bridge in the position[1] shown in Fig. 11.20. With the distribution of loads as specified in Art. 11.7, each intermediate beam must support $5.50/5.0 = 1.10$ wheel loads per wheel.

[1] For a discussion of the position of moving loads for absolute maximum moment, see any elementary text in structural theory.

Therefore, the load from the rear wheel is $1.1 \times 16{,}000 = 17{,}600$ lb and from the front wheel $1.1 \times 4000 = 4400$ lb.

$$R_L = [17{,}600(36.6 + 22.6) + 4400 \times 8.6] \div 50 = 21{,}600 \text{ lb}$$

and

$$M_L = [21{,}600(27.4) - 17{,}600(14)] \times 12 = 4{,}150{,}000 \text{ in.-lb}$$

At 10 ft from the left support, the maximum live-load moment occurs with the rear trailer wheel at that point and the front wheel 38 ft from the left support. With this position of the loads,

$$R_L = 17{,}600 \times {}^{40}\!/_{50} + 17{,}600 \times {}^{26}\!/_{50} + 4400 \times {}^{12}\!/_{50} = 24{,}400 \text{ lb}$$

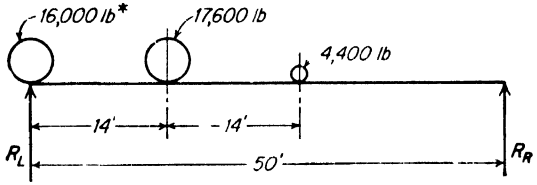
and

$$M_{10} = 24{,}400 \times 10 \times 12 = 2{,}930{,}000 \text{ in.-lb}$$

Similarly,

$$M_{20} = 3{,}960{,}000 \text{ in.-lb}$$

*Impact Moments.* In computing impact moments, it is usual, for a beam such as this, to consider the whole span as the loaded length.




*See Art 11.7 for prescribed distribution of loads for shear computations.

FIG. 11.21

The impact coefficient is therefore $50/(50 + 125) = 0.285$, and the impact moments are as follows:

$$M_{max} = 1{,}183{,}000 \text{ in.-lb}$$
$$M_{20} = 1{,}130{,}000 \text{ in.-lb}$$
$$M_{10} = 835{,}000 \text{ in.-lb}$$

*Maximum Total Moments.* The sum of the maximum dead-load, live-load, and impact moments is 8,723,000 in.-lb. The total moments at the 10-ft and 20-ft points are 5,935,000 in.-lb and 8,350,000 in.-lb, respectively.

*Dead-load Shears.* The maximum dead-load shear at the end of the beam is $904 \times 25 = 22{,}600$ lb. At 10 ft from the support, the shear is 13,560 lb and at 20 ft from the support, 4520 lb.

*Live-load Shears.* The maximum shear in the center beam occurs with the truck on the span in the position shown in Fig. 11.21.

The maximum live-load shear is 30,600 lb. The shears at the 10-ft, 20-ft, and 25-ft points are 24,300 lb, 16,400 lb, and 12,700 lb, respectively.

*Impact Shears.* Using loaded lengths of 50 ft, 40 ft, 30 ft, and 25 ft, respectively, the impact shears are as follows:

| | |
|---|---|
| End shear | 8700 lb |
| 10-ft section | 6900 lb |
| 20-ft section | 4700 lb |
| Mid-span | 3600 lb |

*Total Shears.* The total shears are as follows:

| | |
|---|---|
| End shear | 61,900 lb |
| 10-ft section | 44,800 lb |
| 20-ft section | 25,600 lb |
| Mid-span | 16,300 lb |

*Determination of Cross Section and Steel Area.* The area $b'd$ required to sustain the maximum shear is $61{,}900/(\frac{7}{8} \times 180) = 394$ sq in. With $b' = 14$ in., the required $d = 28.1$ in. With $d$ taken as 29 in. and the main reinforcement placed in three rows 3 in. center to center, with $2\frac{1}{2}$ in. of insulation for the lowest row, the depth of beam below the slab is $29 + 5\frac{1}{2} - 5\frac{1}{4} = 29\frac{1}{4}$ in. This is sufficiently close to the assumed value that the dead-load moments need not be revised.

$$A_s = \frac{8{,}723{,}000}{20{,}000[29 - (5.25/2)]} = 16.5 \text{ sq in.}$$

which is furnished by eleven No. 11 bars.

A check of the maximum stress in the concrete, using Diagram 3, Appendix D, shows a maximum compression of 1210 psi. This is practically equal to the allowable of 1200 psi and is permissible.

For bond,

$$\Sigma_0 = \frac{61{,}900}{300 \times \frac{7}{8} \times 29} = 8.1 \text{ in.}$$

This is furnished by two of the No. 11 bars. The lower row of four bars, approximately one-third of the total steel, will be continued straight into the support and hooked. The remainder will be bent up to assist in resisting the diagonal tension stresses.

*Web Reinforcement.* The portion of the beam over which web reinforcement is required is determined by computing the unit shears at various points on the beam. This is shown graphically in Fig. 11.22. The concrete resists a unit shear of 60 psi[1] and the remainder must be cared for by bent-up bars or stirrups. The upper row of bars will be bent up at an angle of 45°; a single bar will be bent first, and then the remaining

[1] The AASHO specifications provide for a shear stress of $0.02f'_c$ in beams in which the longitudinal bars are not specially anchored.

two. The bars in the middle row will be bent up in pairs. In the moment diagram of Fig. 11.22*c*, the points *c*, *d*, *e*, and *f* represent the points of which totals of one, three, five, and seven bars may be bent up.

The maximum distance over which inclined bars may be considered effective in resisting diagonal tension is equal to 29 in., in accordance with

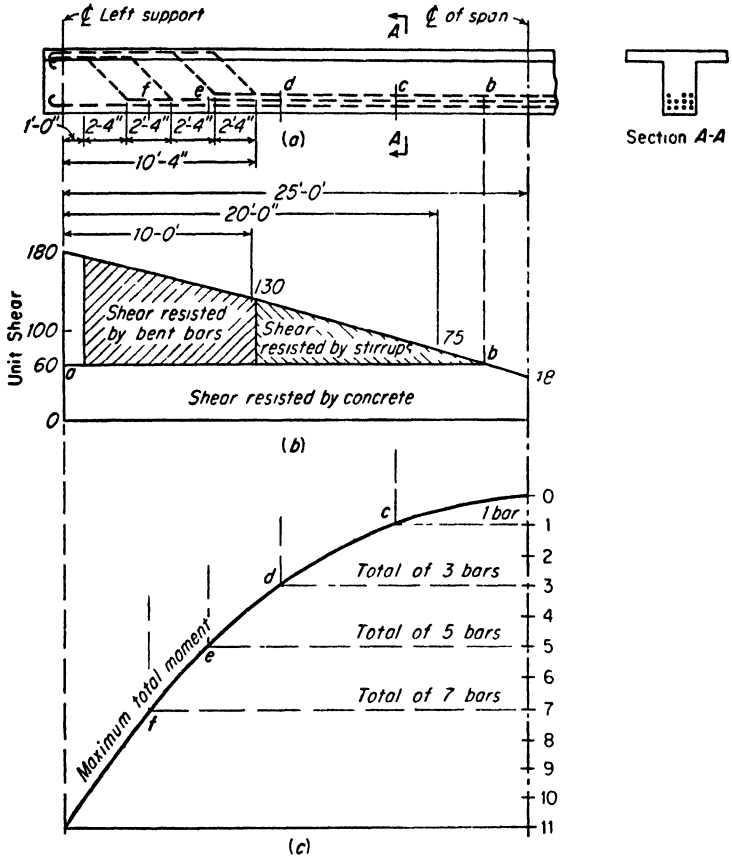



FIG. 11.22

the ACI Code. The pair nearest the support will be bent up 2 ft-4 in. from the support and the others as shown in Fig. 11.22*a*.

At a distance of 10 ft-4 in. from the center of the support the unit shear is 128 psi and the required spacing of No. 5 U stirrups is

$$s = \frac{20\,000 \times 2 \times 0.307}{(128 - 60)14} = 12.90 \text{ in.}$$

The maximum allowable spacing of stirrups is $\frac{1}{2} \times 29 = 14.5$ in. (see Art. 3.34).

The final arrangement of stirrups will be as follows: with the first stirrup 4 in. from the edge of the support, 8 at 15 in., 4 at 10 in., and 8 at 15½ in. Theoretically, stirrups are not needed over the first 10 ft-4 in. from the support where the inclined bars resist the diagonal tension or from the point $b$ to the center of the span, but the arrangement here given ensures the proper tying of the flange to the web and gives an additional safeguard against diagonal tension cracks.

**11.14. Exterior Beams.** Since the exterior beams are to project a minimum of 9 in. above the slab to form curbs for the protection of the railings, they must be designed as rectangular beams.

*Dead-load Moments.* The weight from the slab per foot of beam is 90 × 2 = 180 lb. The weight of the railing, details of which are shown in Fig. 11.21, is assumed as 300 lb per ft. The cross section of the beam is assumed as 20 × 50 in. which weighs 1040 lb per ft. The total dead

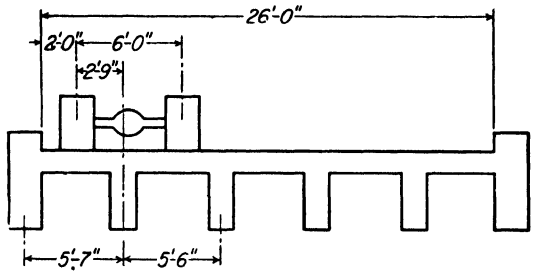



FIG. 11.23

load is 1520 lb per ft, and the maximum dead-load moment at the center of the span is

$$M_D = \tfrac{1}{8} \times 1520 \times 50^2 \times 12 = 5{,}710{,}000 \text{ in.-lb}$$

*Live-load and Impact Moments.* A portion of each wheel load which rests on the exterior slab panel is supported by the exterior beam. That portion is obtained by placing the wheels as close to the curb as the clearance diagram will permit and treating the exterior slab panel as a simple beam. The position is shown in Fig. 11.23 and the proportion of the load is 2.75/5.58 = 0.493. The longitudinal position of the load which will produce the absolute maximum bending moment is the same as for the intermediate beams, as shown in Fig. 11.20. The moments are directly proportional, and the absolute maximum moment in the exterior beam is

$$M_L = \frac{0.493}{1.10} \times 4{,}150{,}000 = 1{,}860{,}000 \text{ in.-lb}$$

The impact moment is 0.285 × 1,860,000 = 530,000 in.-lb.

*Total Maximum Moment.* The total maximum moment is 8,100,000 in.-lb.

*Shears* The maximum dead-load shear at the end of the beam is

$$V_D = 1520 \times 50_2 = 38{,}000 \text{ lb}$$

and the shear at the 10-ft section is 22,800 lb.

The maximum live-load shear is proportional to the maximum live-load shear in the intermediate beams and is

$$V_L = \frac{0.483}{1.10} \times 30{,}600 = 13{,}400 \text{ lb}$$

Similarly the shear at the 10-ft section is 10,900 lb.

The impact shears are 3900 lb and 3100 lb, respectively, and the total shears are:

| | |
|---|---|
| End shear..... | 55,300 lb |
| 10-ft section . . | 36,800 lb |

*Determination of Cross Section and Steel Area.* For the assumed width of 20 in. the required depth is

$$d = \sqrt{\frac{8{,}100{,}000}{197 \times 20}} = 45.4 \text{ in.}$$

Taking $d = 45\frac{1}{2}$ in. and assuming two rows of steel 3 in. center to center with $2\frac{1}{2}$ in. insulation for the lower row, the total height of the cross section is $15\frac{1}{2} + 1\frac{1}{2} + 2\frac{1}{2} = 19\frac{1}{2}$ in., approximately as assumed.

$$A_s = \frac{8{,}100{,}000}{20{,}000 \times 0.875 \times 45.5} = 10.2 \text{ sq in.}$$

which is furnished by eight No. 10 bars, of which four bars are placed in each row. Two bars are sufficient to develop the bond stress, so that the remaining six bars can be bent up to assist in resisting diagonal tension if desired.

*Web Reinforcement.* At the support, the unit shear is

$$\frac{55{,}300}{20 \times \frac{7}{8} \times 45.5} = 69 \text{ psi}$$

and at the 10-ft section, 45 psi, so that web reinforcement is required for a distance of only $\frac{69 - 60}{69 - 45} \times 10 = 3.8$ ft from the support. Bending up two bars 36 in. from the edge of the support and two bars 36 in. farther toward the center, the diagonal tension is theoretically provided for (1 ft-0 in. + 3 ft-0 in. + 3 ft-0 in. = 7 ft-0 in.), but in order to reinforce the beam against shrinkage and to tie it together, ½-in. round

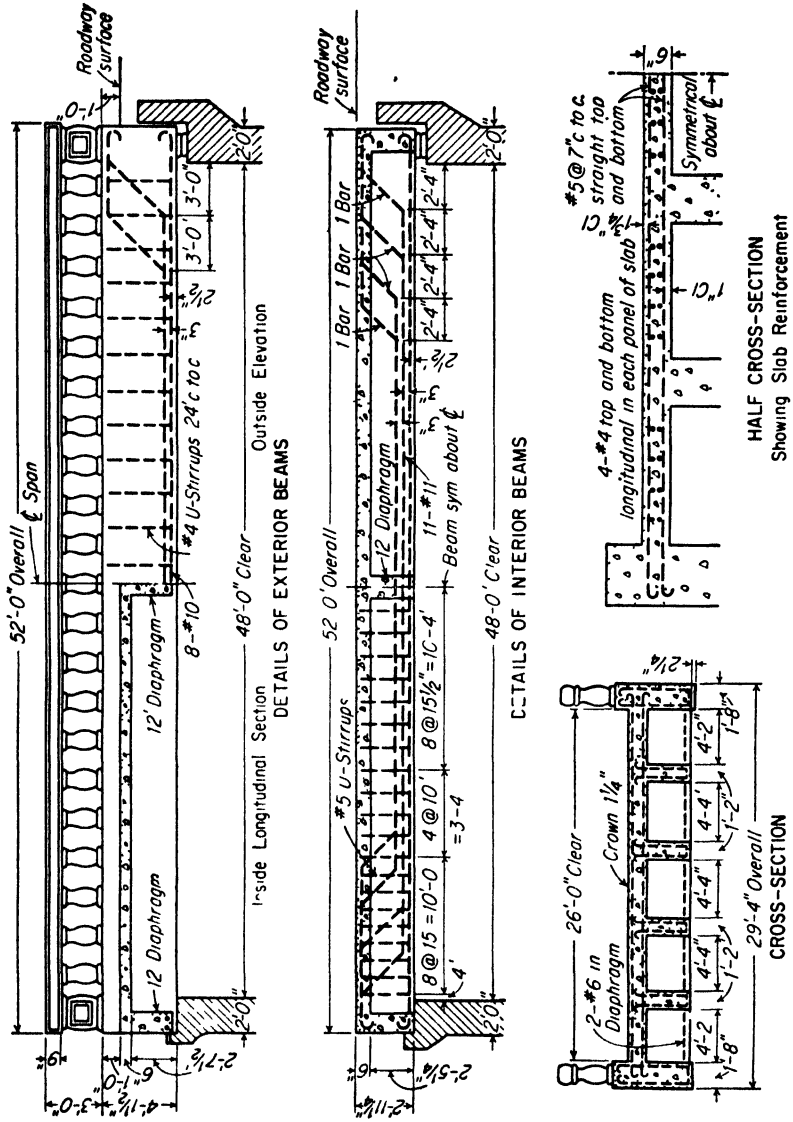



Fig. 11.24 Details of a T-beam bridge.

U stirrups will be placed about 2 ft-0 in. center to center throughout its length. A diagram similar to Fig. 11.22 could be constructed for this beam, but it is not necessary, for it is evident from a study of that figure that ample steel is provided in the bottom of the beam to care for the positive moment.

**11.15. Miscellaneous Details.** *Diaphragms.* A transverse diaphragm *a*, Figs. 11.10 and 11.12, will be built between the beams at either end of the bridge. The chief function of these diaphragms is to furnish lateral support to the beams; with some abutment details, the diaphragms also serve to prevent the backfill from spilling out on the bridge seats. A similar diaphragm will be built between the beams at mid-span. Such intermediate diaphragms are recommended for all spans in excess of 40 ft-0 in.

*Fixed Bearing.* A fixed bearing similar to Fig. 11.10 is provided at one end. Vertical dowels *b* are placed in the breastwall of the abutment and are bent so as to project into the longitudinal beams or into the deck slab; horizontal dowels *c* are embedded in the deck and approach slabs. The diaphragm *a* may rest directly on the bridge seat.

*Expansion Bearing.* An expansion bearing is provided at one end of the span. Its details are similar to Fig. 11.12*b*. The diaphragm *a* is flush with the bottoms of the intermediate beams and not in contact with the bridge seat.

*Waterproofing and Drainage.* All joints are to be filled with a mastic compound to prevent water from seeping through the joints.

Removal of surface water will be accomplished by crowning the roadway $1\frac{1}{4}$ in. and pitching the gutters toward the ends by providing a camber of $2\frac{1}{2}$ in. at the center. This latter is accomplished by raising the forms at the center. Besides facilitating drainage, this camber serves to prevent the appearance of sag that would be evident if the girders were at the same level throughout the span. Full details are shown in Fig. 11.24.

## DESIGN OF A THROUGH-GIRDER BRIDGE

**11.16. Data and Specifications.** It is required to design a through-girder bridge for the following conditions:

| | |
|---|---|
| Clear span | 48 ft-0 in. |
| Clear width | 26 ft-0 in. |
| Live loading | *H*20-*S*16 |

$f'_c = 3750$; $f_s = 20{,}000$; $f_v = 20{,}000$; $n = 8$.

The bridge is to consist of a slab supported between two girders.[1]

[1] With this width of roadway it might be more economical to use cross beams, in order materially to reduce the thickness of the floor slab. For greater widths, there is no doubt of the economy of the use of such beams.

**11.17. Slab.** Although the slab is poured so as to act monolithically with the girders, the girders themselves are unrestrained, and since the slab is in junction with the girders near the bottom of the latter, a torsional moment of considerable amount is brought to the girders, so that they may not be considered to fully restrain the ends of the slab. Tests have shown a point of inflection in the slab about 1 ft from the edge of the girders; but many designers prefer to consider the slab as simply supported, and that practice will be followed in the accompanying design. Slabs may be designed for tensile steel only or for both tensile and compressive steel. The latter design allows a thinner slab but requires more steel. Both types are used. The present design will consider tensile steel only.

Assuming an effective depth of slab of 11 in. with $1\frac{1}{2}$ in. insulation, $\frac{3}{4}$ in. for wearing surface, and 15 lb for future protective covering, the dead load per square foot is 181 lb. The span of the slab is taken as the clear span plus the structural thickness or $26 + (12.5/12) = 27.0$ ft. The maximum dead-load moment per foot of slab is

$$M_D = \tfrac{1}{8} \times 181 \times (27.0)^2 \times 12 = 198{,}000 \text{ in.-lb}$$

Since the main reinforcement in the slab is perpendicular to the traffic, Case *A*, Art. 11.6, governs the computation of maximum positive live-load moment. The width (parallel to traffic) over which a wheel load is distributed is

$$E = 0.063S + 4.65 = 0.063 \times 27.0 + 4.65 = 6.35 \text{ ft}$$

The maximum live-load moment is

$$M_L = 0.25 \frac{P2}{E} S = 0.25 \times \frac{8000}{6.35} \times 27.0 \times 12 = 102{,}000 \text{ in.-lb}$$

Since the loaded length is extremely short, the impact coefficient is 0.30 and

$$M_I = 0.30 \times 102{,}000 = 30{,}600 \text{ in.-lb}$$

The total maximum positive moment is

$$M = 198{,}000 + 102{,}000 + 30{,}600 = 330{,}600 \text{ in.-lb}$$

$$d = \sqrt{\frac{330{,}600}{12 \times 246}} = 10.6 \text{ in.}$$

Adopting an effective depth of $10\frac{3}{4}$ in., the total slab thickness is 13 in. The required steel area is

$$A_s = \frac{330{,}600}{20{,}000 \times 0.875 \times 10.75} = 1.75 \text{ sq in.}$$

which is furnished by No. 8 bars at $5\frac{1}{2}$ in. on centers ($A_s = 1.72$ sq in. per ft, which is within 2 per cent of the required area). The required distribution reinforcement (see Art. 11.6) is $1.72\sqrt{27.0} = 0.33$ sq in. This will be provided by placing No. 5 bars at 12-in. centers, directly on top of the main reinforcement.

Since the slab and girders are to be poured monolithically, some negative moment will be developed at their junction. Therefore one-third of the main reinforcing bars will be bent up near the supports and continued near the top of the slab to within about 2 in. of the outside face of each girder, where they will be anchored by means of hooks. In addition, No. 8 bars at 15-in. centers will be placed in the top of the slab near the girders and bent up into the girders to within about 3 in. of the top of the girders as shown in Fig. 11.26.

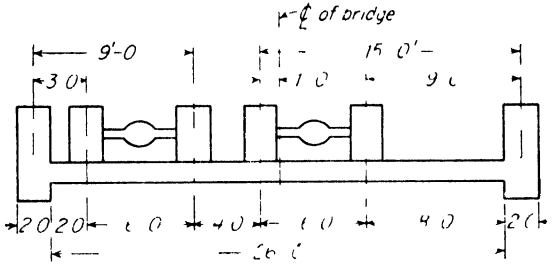



FIG. 11.25

**11.18. Main Girders.** The weight from the slab per foot of girder is $181 \times 13 = 2350$ lb. The cross section of the girder is assumed as $21 \times 74$ in. over-all, which would give a weight of 1850 lb per ft, but since the sides are paneled in the architectural treatment, the weight will be assumed as 1700 lb per ft. The dead-load moment is therefore

$$M_D = \tfrac{1}{8} \times 4050 \times 50^2 \times 12 = 15{,}200{,}000 \text{ in.-lb}$$

In order to obtain the maximum live-load moment, one line of trucks is placed as close to the face of one girder as the clearance diagram will permit, as shown in Fig. 11.25. Then treating the slab as a simple beam and taking moments about the right-hand girder, the proportionate part of one wheel load $P$, carried by the left-hand girder, is

$$P(9.00 + 15.00 + 19.00 + 25.00) \div 28.00 = 2.43P$$

The position of the loads longitudinally is the same as for the deck-girder bridge previously designed (see Fig. 11.20) and the moments are proportional. Therefore,

$$M_L = \frac{2.43}{1.10} \times 4{,}150{,}000 = 9{,}150{,}000 \text{ in.-lb}$$

The impact moment is

$$M_I = 0.285 \times 9{,}150{,}000 = 2{,}610{,}000 \text{ in.-lb}$$

and the total maximum moment is

$$M = 15{,}200{,}000 + 9{,}150{,}000 + 2{,}610{,}000 = 26{,}960{,}000 \text{ in.-lb}$$

$$d = \sqrt{\frac{26{,}960{,}000}{24 \times 246}} = 67.6 \text{ in.}$$

Adopting an effective depth of 68½ in., to reduce slightly the required steel, assuming two rows of bars 3 in. center to center with the center of the lower row 3 in. from the bottom of girder, the total depth is

$$68\tfrac{1}{2} + 1\tfrac{1}{2} + 3 = 73 \text{ in.}$$

approximately as assumed.

$$A_s = \frac{26{,}960{,}000}{20{,}000 \times 0.875 \times 68.5} = 22.5 \text{ sq in.}$$

Fourteen No. 11 bars provide 21.87 sq in., which is close enough to the required area to be satisfactory. All the bars will be continued at the bottom from one end of the girder to the other.

The shears are computed in a manner similar to that used for the intermediate beams of the deck-girder bridge (see Art. 11.13). The values are as follows:

| | Total shear, lb | Unit shear, psi |
|---|---|---|
| Center of support | 193,000 | 169 |
| 5-ft section | 164,000 | 144 |
| 10-ft section | 130,000 | 114 |
| 15-ft section | 95,000 | 83 |

In computing the above unit shears, the minimum width, which is $24 - (2 \times 2\tfrac{1}{2}) = 19$ in., was used (see Fig. 11.26).

Using No. 5 U stirrups,

$$\frac{f_s A_v}{b} = \frac{20{,}000 \times 2 \times 0.307}{19} = 646$$

and the required spacings are as follows:

Center of support: $646 \div (169 - 75) = 6.9$ in.
5-ft section: $646 \div (144 - 75) = 9.4$ in.
10-ft section: $646 \div (114 - 75) = 16.6$ in.
15-ft section: $646 \div (83 - 75) = 81.0$ in.

The maximum allowable spacing is $\frac{1}{2} \times 68.5 = 34.2$ in.

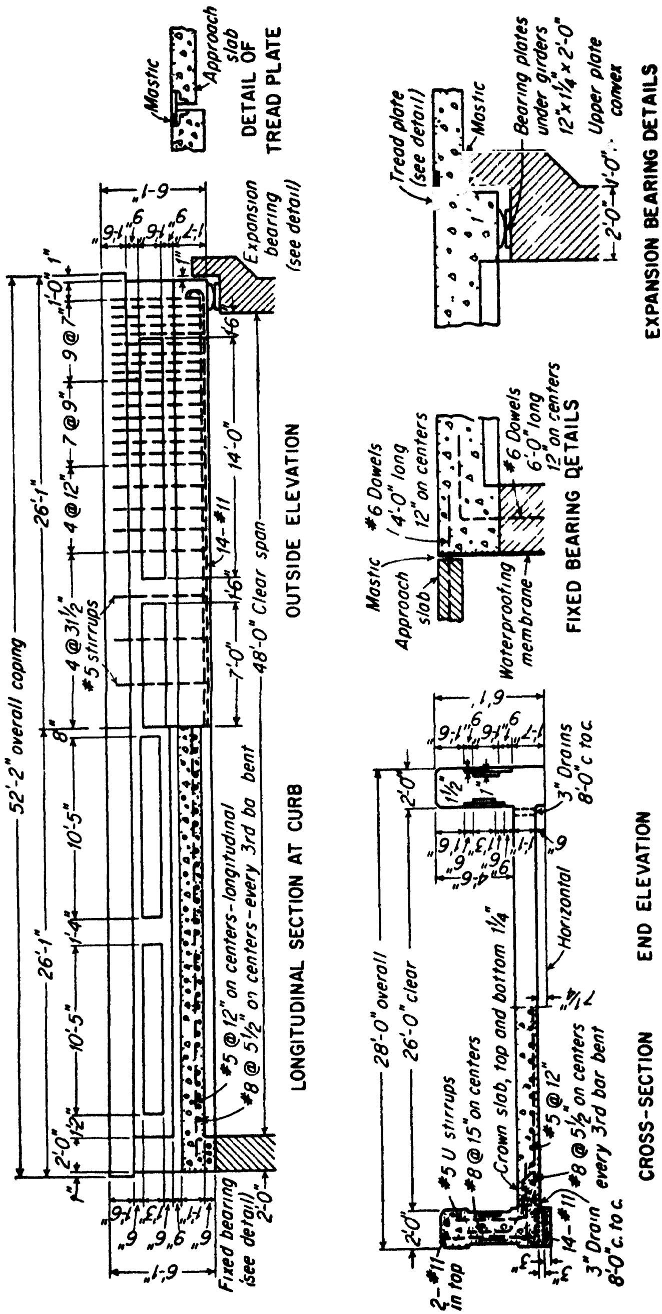



FIG. 11 26. Details of through-girder bridge

The first stirrup will be placed at the center of the support and the spacings thereafter will be as follows: 9 at 7 in., 7 at 9 in., 4 at 12 in., and 4 at 31½ in.

**11.19. Miscellaneous Details.** The details of the architectural treatment of the girders are shown in Fig. 11.26. The floor slab is framed into the girders 6 in. above the bottom of the girders in order to avoid the necessity of bending the slab reinforcement to clear the bars in the girders.

A fixed bearing will be constructed at one end by placing No. 6 dowels in the breastwall of the abutment, at 12-in. centers, and bending these dowels into the slab and girders as shown in Fig. 11.26. Horizontal dowels will also be used. The slab will be deepened so that the bottom of the slab will rest on the bridge seat and serve as a backwall for the abutment.

An expansion bearing at the other end of the bridge is similar to the type described in Art. 11.9 and illustrated in Fig. 11.14. The details are shown in Fig. 11.26. The required area of the bearing plates is

$$193{,}000 \div 700 = 276 \text{ sq in.}$$

Transverse drainage is provided by a 1¼-in crown and longitudinal drainage by sloping the gutters toward 3-in. drains which are placed close to each girder at 8-ft centers

Abutments are designed according to the principles outlined in Sec. 9.

## COMPOSITE BRIDGE CONSTRUCTION

**11.20. Description of General Type.** In addition to girder bridges entirely constructed of reinforced concrete, a frequently used type of construction consists of longitudinal steel girders with a reinforced concrete deck. If the deck merely rests on the girders, it does not represent an integral part of the transverse cross section. That is, since no provision is made to transfer longitudinal shears from the girder to the deck, the latter does not assist in carrying longitudinal bending moments. In contrast, steel-concrete composite construction as used in many present-day bridges consists essentially of three elements: (1) longitudinal steel beams, usually called stringers or girders; (2) a reinforced concrete deck; and (3) devices, usually called shear connectors, which are welded to the top flanges of the beams and protrude into the slab to tie slab and beam together and to force them to act as a unit in resisting girder moments and shears. Figure 11.4 shows steel stringers with spiral shear connectors attached. Figure 11.27 is a typical example of a completed bridge of this type. Simple spans up to about 125 ft and continuous spans up to about 175 ft have been built in this manner.

The principal advantage of this type of construction is that it utilizes

the concrete deck, which in normal construction does not contribute to longitudinal strength, to assist in resisting longitudinal bending. The resulting composite section is usually shallower and lighter--but none

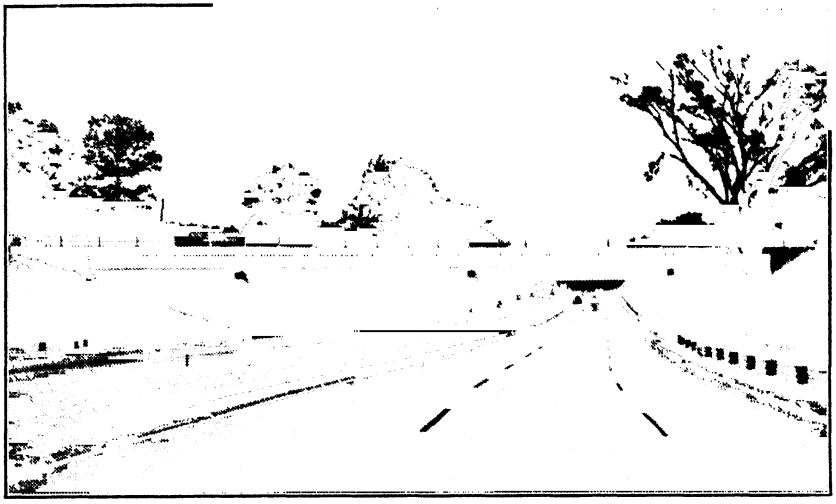


FIG. 11.27. Continuous bridge of composite construction

the less stiffer—than a comparable section of ordinary concrete slab steel stringer construction in which there is no provision for composite action.

Some of the different types of shear connectors which have been used

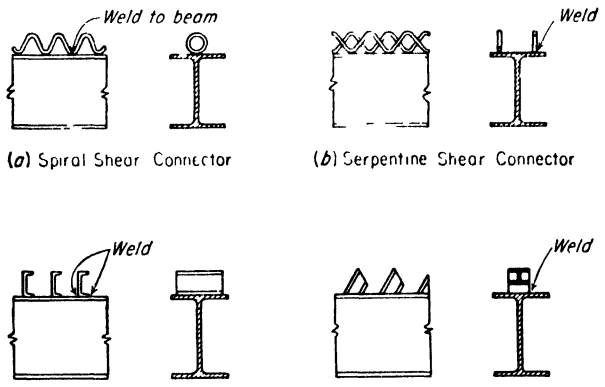



FIG. 11.28. Typical shear connectors

successfully are shown in Fig. 11.28. The spiral connector, known as the Alpha spiral, is perhaps the most commonly used connector. Whatever type of shear connector is used, it should extend at least halfway

into the slab, but not closer than 1½ in. to the top of the slab. In the transverse direction the concrete deck is designed as a one-way slab using the principles outlined in Art. 11.12. In simple composite spans the concrete deck forms part or all of the compression region of the composite section. In continuous spans over the supports the concrete is in tension, and additional longitudinal reinforcement or steel cover plates must be added.

The steel stringers alone must support their own dead weight as well as the dead weight of the formwork and fresh concrete of the deck unless sufficient intermediate shoring is used. The usual principles of steel design, which are covered in elementary steel-design textbooks, apply to the design of the stringers to resist dead loads.

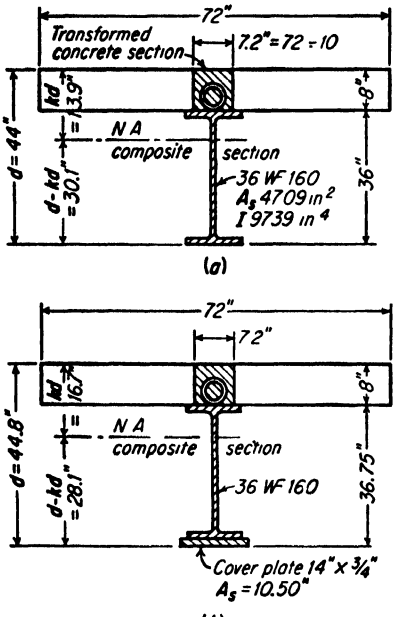



FIG. 11.29

The live load is resisted by the composite section. The resisting moment of the composite section is usually computed from the properties of the transformed section obtained by substituting for the concrete slab an equivalent amount of steel.

In designing a composite section, the bending moment for dead load and that for live load plus impact are determined. For a trial section a beam or girder with a resisting moment of about one-half of that required for dead load, live load, and impact may be selected, but in the usual case this must not be less than that required for dead load alone. In making such a selection the deepest beam satisfying clearance requirements should usually be selected in order that the deflection may be kept within the specified limits. The method of computing the resisting moment of the composite section is illustrated in Art. 11.21.

**11.21. Resisting Moment of a Composite Section.** In Fig. 11.29*a* there is shown the cross section of a composite section. In this case the steel stringers are spaced 6 ft-0 in. on centers. With $f'_c = 3000$, $f_c = 1200$, $f_s = 20{,}000$, and $n = 10$, the width of the transformed area of concrete is 7.2 in. Taking moments about the top of the slab, the distance $kd$ to the neutral axis of the transformed section is

$$kd = \frac{47.09 \times 26 + 7\,2 \times 8 \times 4}{47.09 + 7.2 \times 8} = 13.9 \text{ in.}$$

The moment of inertia of the transformed section about its neutral axis is

$$\begin{aligned} 36 \text{ WF } 160 &= 9{,}739 \text{ in.}^4 \\ +47.09(12.1)^2 &= +6{,}894 \text{ in.}^4 \\ \text{Concrete: } +7.2\,\frac{(13.9)^3}{3} &= +6{,}445 \text{ in.}^4 \\ -7.2\,\frac{(5.9)^3}{3} &= -493 \text{ in.}^4 \\ I &= 22{,}585 \text{ in.}^4 \end{aligned}$$

Resisting moment as governed by concrete:

$$M_c = \frac{I}{kd} \times f_c \times n = \frac{22{,}585}{13.9} \times 1200 \times 10 = 19{,}498{,}000 \text{ in.-lb}$$

Resisting moment as governed by steel:

$$M_s = \frac{I}{d - kd} \times f_s = \frac{22{,}585}{30\,1} \times 20{,}000 = 15{,}007{,}000 \text{ in.-lb}$$

The latter value would govern the design in this case. As mentioned before, in areas of negative moment of continuous beams where tension would occur at the top of the section, the concrete is considered ineffective, and longitudinal bars or cover plates on the beams are added to resist the moment. Such reinforcement must, of course, be extended far enough to be firmly anchored in a region of concrete in compression.

It will be noted that the above section is somewhat unbalanced, that is, the resisting moment of the concrete is considerably higher than that of the steel. A closer balance can usually be obtained by adding a cover plate to the bottom flange of the beam. Since this arrangement is fairly common in simple-span composite construction, the effect of adding a 14- by ¾-in cover plate to the bottom of the above section will be illustrated. The cross section of all elements will be assumed as being effective (see Fig. 11.29*b*) since most construction of this type is welded.

Taking moments about the top of the slab, the distance *kd* to the neutral axis of the transformed section is

$$kd = \frac{47.09 \times 26 + 10.50 \times 44.37 + 7.2 \times 8 \times 4}{47.09 + 10.50 + 7\,2 \times 8} = 16.7 \text{ in.}$$

The moment of inertia of the transformed section about its neutral axis is

$$\begin{aligned}
36\text{ WF }160 &= 9{,}739\text{ in.}^4\\
+47.09(9.3)^2 &= +4{,}073\text{ in.}^4\\
\text{Cover plate: } +10.50(27.7)^2 &= +8{,}056\text{ in.}^4\\
\text{Concrete: } +7.2\,\frac{(16.7)^3}{3} &= +11{,}178\text{ in.}^4\\
-7\,2\,\frac{(8.7)^3}{3} &= -1{,}580\text{ in.}^4\\
I &= 31{,}466\text{ in.}^4
\end{aligned}$$

Resisting moment as governed by the concrete:

$$M_c = \frac{31{,}466}{16\,7} \times 1200 \times 10 = 22{,}610{,}000\text{ in.-lb}$$

Resisting moment as governed by the steel:

$$M_s = \frac{31{,}466}{28.1} \times 20{,}000 = 22{,}396{,}000\text{ in.-lb}$$

It is seen that, with the addition of the cover plate, $M_c$ and $M_s$ are practically equal, *i.e.*, a balanced design has been obtained. Also, by this addition the resisting moment has been increased from 15,007,000 to 22,396,000 in.-lb, *i.e.*, by 19 per cent. To achieve this larger resisting moment by a design without a cover plate, a much heavier wide-flange beam would have to be used whose weight would be considerably greater than that of the 36 WF 160 section plus cover plate and shear connectors.

Although it is impossible to make a conclusive economic comparison of composite versus ordinary construction on the basis of the one example presented here, some indication of the possible saving in steel can be obtained. In order to furnish a resisting moment equivalent to that supplied by the composite section of Fig. 11.29*b*, a simple wide-flange beam would have to have a section modulus = 2,396,000/20,000 = 1120 in.$^3$ The largest wide-flange beam, a 36 WF 300, has a section modulus of 1105 in.$^3$

The weight of the steel per foot in the composite section shown in Fig. 11.29*b* is

| | |
|---|---|
| WF beam | 160.0 lb |
| Cover plate | 35.7 lb |
| Shear connectors (approx.) | 3.3 lb |
| | 199.0 lb |

While this represents a saving in steel weight of about 33 per cent, this is not a true comparison as far as cost is concerned, because to the cost of this lighter section must be added the cost of attaching the connectors to the upper flange of the beam and the cost of attaching the cover plate

to the lower flange of the beam. However, these operations are usually considerably less in cost than the additional weight of fabricated steel required when the composite section is not used.

Shearing stresses in the beam web may be computed by the customary formulas from strength of materials, again keeping in mind that the dead-load forces are resisted by the girder alone and the live-load forces by the composite section. The maximum shear stress at any cross section is then

$$v_{max} = \frac{V_{DL}}{t_s h_s} + \frac{V_{LL} Q_c}{I_c t_s}$$

where $v_{max}$ = maximum unit shearing stress in web
$V_{DL}$ = dead-load shear at section
$V_{LL}$ = live-load shear at section
$t_s$ = web thickness
$h_s$ = depth of steel beam
$I_c$ = moment of inertia of composite section about its neutral axis
$Q_c$ = statical moment of composite section on one side of the neutral axis, about the neutral axis.

In bridges of ordinary proportions shearing stresses are seldom critical.

**11.22. Design of Shear Connectors.** The shear connectors must be strong enough to transmit the horizontal shear on the plane between the steel and concrete. Theoretically the horizontal shear per unit length of beam may be computed from the usual formula: $v = VQ/I_c$

where $v$ = horizontal shearing stress per unit length of beam between concrete slab and steel beam
$V$ = shear at section
$Q$ = statical moment of slab about neutral axis of composite section
$I_c$ = moment of inertia of composite section

Since most shear connectors are of rather unusual shapes, their actual strength is usually determined by tests. Several manufacturers present safe shear-load tables for their particular connectors. Alpha spirals, for example, are made of $^1/_2$-, $^5/_8$-, and $^3/_4$-in.-diameter bars with spiral diameters from $3\frac{1}{2}$ to 11 in. As an example of carrying capacity, the Porete Company, manufacturers of Alpha spirals, recommend an allowable shear per running foot of 22.08 kips for a $^5/_8$-in.-diameter spiral with a 6-in. spiral pitch in cases where the basic steel working stress is 18,000 psi.

# SECTION 12

# MODERN DEVELOPMENTS IN CONCRETE DESIGN

## ULTIMATE OR PLASTIC DESIGN

**12.1. Introduction.** Most current methods of structural design are based on the assumption that stress and strain are proportional, *i.e.*, that the material performs in a purely elastic manner. In Secs. 1 to 5 it was pointed out in various connections that this assumption is rather far from the truth, especially for concrete. In particular, in Sec. 4 it was shown that current procedures of column design are no longer based on the supposition of elasticity but rather on the actual behavior of concrete under load. This behavior was shown in Fig. 4.2 which indicates that stresses and strains in concrete are proportional only at relatively low stresses but that at higher stresses the strain increases at a faster rate than the stress.

This fact was well known in the early days of reinforced concrete research and design. In fact, A. N. Talbot, one of the pioneers in this field, pointed out as early as 1905 that a curved stress-strain relationship must be used for an accurate determination of strength of reinforced concrete members. For the sake of simplicity, however, later reinforced concrete theory was exclusively based on the assumption of elastic behavior. In many cases this method led to uneconomical design; *i.e.*, many reinforced concrete members proved to be much stronger than indicated by the elastic theory, so that smaller dimensions could have been used had a more accurate design method been available. This discrepancy became so glaring in column design that it finally led to abandoning the concept of elasticity for such members in favor of the procedure discussed in Art. 4.2.

Present concrete codes, therefore, represent a rather inconsistent compromise, since the assumption of perfect elasticity is retained for the design of beams, while the actual, partly plastic behavior of concrete serves as a basis for the design of axially loaded columns. The practical difficulties of this situation are realized when one considers that

beams on the one hand and axially loaded columns on the other are but the two extremes of the more general case of bending plus axial stress (Sec. 5), in which the eccentricity $e$ can vary all the way from zero (axially loaded columns) to infinity (simple bending of beams). To make matters even more involved, present codes do take partial account of the plasticity of concrete in regard to beams with compression reinforcement, as was briefly indicated in Art. 3.46.

The only way to avoid this inconsistency and to achieve design of maximum economy is to base the design of all types of concrete members on the actual performance of concrete, as depicted schematically in Fig. 4.2. This step has already been taken in the concrete design codes of several foreign countries. In the United States development has been somewhat slower, chiefly because it is realized that the state of knowledge of these matters is as yet incomplete. To eliminate the gaps in available information, a considerable amount of research is being carried on at present.

A number of American and foreign engineers and investigators have proposed more or less different new methods of design and analysis of this type. In the following articles the two methods developed in this country will be briefly discussed, since it is likely that one of them, possibly with suitable modifications, will eventually be adopted as standard design procedure. In view of this prospect, a comprehension of these methods is essential at this time, even though they have not yet become a part of everyday engineering practice.

In design work the adequacy of a structure is ensured by the introduction of a factor of safety, usually applied to the stress. For example, the ACI Code specifies 45 per cent of the concrete strength $f'_c$ as the allowable concrete stress in beams and 50 per cent of the yield point as the allowable steel stress; *i.e.*, a factor of safety of about 2 is stipulated. If stresses exceed the proportional limit, loads, strains, and stresses are no longer proportional. This can be seen from the numerical discussion of Fig. 4.2, Art. 4.2. For this reason, factors of safety cannot be applied to stresses under such conditions. One proceeds, therefore, to determine the ultimate strength of a member, *i.e.*, the load or moment at which it will begin to fail. The permissible design load or moment is then the ultimate load or moment divided by the factor of safety.

For this reason the discussion that follows will deal exclusively with the actual carrying capacity, *i.e.*, with ultimate loads or ultimate moments, which should not be confused with the design loads or moments, obtained in all other sections of this book by using permissible stresses. Once this ultimate load or moment is determined for a given member or structure, its safe, or design, load is simply obtained by dividing the capacity load by a factor of safety.

**12.2. Stresses in Reinforced Concrete Beams.** In the elastic theory of reinforced concrete it is assumed that stresses in a beam are distributed as shown in Fig. 3.2. It is further implied that the beam will fail if either the steel stress $f_s$ becomes equal to the yield point $f_y$, or the concrete stress $f_c$ equal to the concrete strength $f'_c$, or both. Current design procedures, such as those of the ACI Code, are based on these simplified assumptions. It is well known, however, that actual concrete stresses are distributed in a different manner in a beam.

Figure 12.1 shows a typical stress-strain diagram for a concrete cylinder with $f'_c = 3000$ psi (see also Fig. 4.2). The details of the shape of this curve vary within certain limits, depending on the particular cements

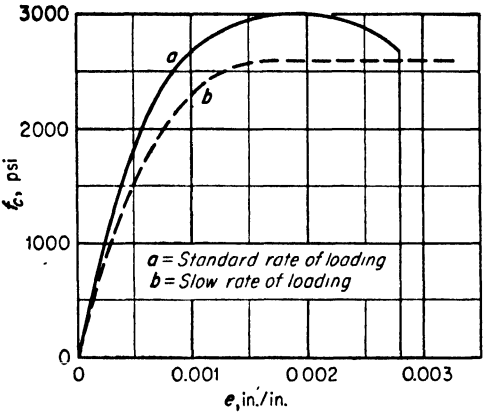



FIG. 12.1. Typical stress-strain curves for a concrete with $f'_c = 3000$ psi.

and aggregates used. It is seen that, for standard rate of loading, the full strength of 3000 psi is reached on a gradually curving line; that the test cylinder then continues to compress under nearly constant and, later, under gradually decreasing load; and that it finally breaks at a strain, for this particular example, of 0.0028 in. per in. and at a stress considerably below $f'_c$. To each strain there corresponds a definite stress, though the exact pairs of values are seen to depend also on the rate of loading.

This information is obtained from compression tests of standard concrete cylinders in which, for any particular load, the strain is the same at all points of the cross section. In beams, on the other hand, since originally plane sections remain plane during the process of bending, strains increase in direct proportion to the distance from the neutral axis. This is seen from Fig. 12.2*a*, where a portion of a beam of unit length is shown; if the left end of that portion is held immovable, the right section remains plane and rotates as shown. If the strain at the top fiber is, say, 0.0024 in. per in., the strains in other fibers are smaller

in proportion to the ratio of their distances from the neutral axis. If this beam is made from the concrete of Fig. 12.1, it is reasonable to assume that to a given strain the same stress will correspond both in the beam and in the cylinder. Hence, for standard rate of loading, the stress at the top fiber will be 2900 psi, while at a distance of $kd/4$ from the top fiber it is 3000 psi (as obtained from Fig. 12.1 for a strain of 0.0018), and so on down toward the neutral axis. The corresponding distribution of compression stress is shown in Fig. 12.2*b* and is seen to be of exactly the same shape as Fig. 12.1. In the same figures are also shown the corresponding distributions for a slow rate of loading.

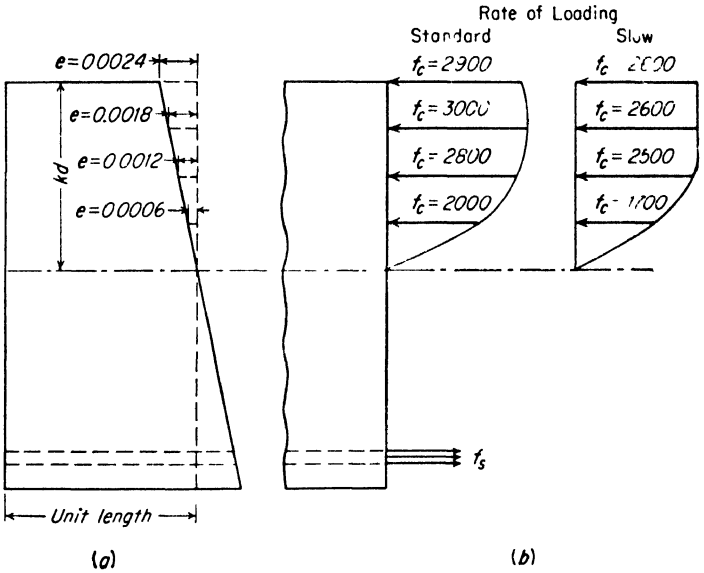



FIG. 12.2. Typical strain and stress distributions in a beam, $f'_c = 3000$ psi.

As the load is increased, the strains increase and the beam will fail when either the steel begins to yield or when the strain in the concrete becomes so large that crushing results, or both. (Actually, failure does not ensue immediately when the steel begins yielding; but at that stage the tension cracks in the concrete become so large as to make the beam practically useless, and actual failure takes place at a load only very slightly larger than that which caused first yielding.)

An accurate prediction of the strength of beams, as required for economical design, can be made only by taking account of the actual stress distribution of Fig. 12.2*b*, in contrast to the fictitious straight-line distribution assumed in the elastic theory. If this were done accurately for the rather complicated stress distribution curves of Fig. 12.2*b*, formulas would result which would be too lengthy and tedious for practical

design work. The proponents of various plastic theories of design, therefore, have replaced these curves by simpler, appropriately chosen straight-line figures. This can be achieved in the following manner.

For a rectangular beam, the area under the stress distribution curve, multiplied by the width of the beam, represents the total compression force in the concrete, which acts at the centroid of that area. This force, multiplied by its distance from the center of the tension steel, is equal to the bending moment. Hence, the same bending moment will be obtained if the actual distribution curve is replaced by a simplified outline, provided (*a*) the area under both of them is the same and (*b*) their respective centroids have the same distance from the tension steel. The difference between the two methods discussed in the following sections consists chiefly in the choice of this simplified shape of the stress-distribution figure.

**12.3. Jensen-PCA Method.** In 1943, V. P. Jensen[1] developed a method for analyzing the carrying capacity of beams by taking account

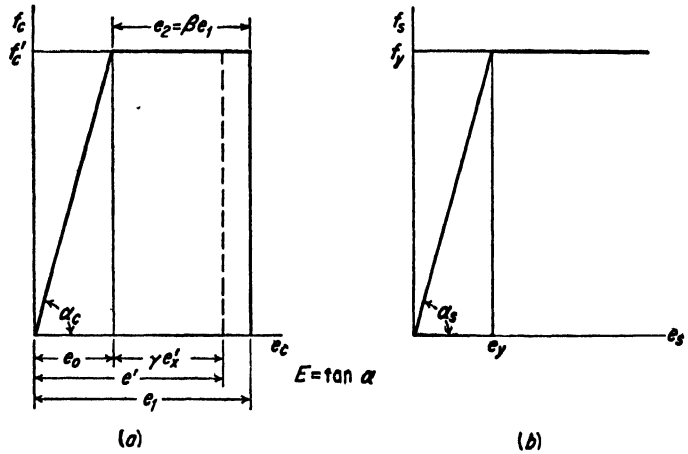



FIG. 12.3. Simplified stress-strain curves of the Jensen-PCA method: (*a*) for concrete; (*b*) for steel.

of the plastic deformation of concrete. This method was subsequently elaborated and slightly modified by the Portland Cement Association (PCA).[2]

In this method the stress-strain curve for a standard rate of loading, which is of the general shape of Fig. 4.2 or 12.1, is replaced by a trapezoid, as shown in Fig. 12.3*a*. The stress-strain diagram for steel (see

[1] V. P. Jensen, "The Plasticity Ratio of Concrete and Its Effect on the Ultimate Strength of Beams," *J. ACI*, vol. 14, p. 565, 1943.

[2] "Ultimate Design of Reinforced Concrete," *Concrete Information* St68, Structural Bureau, Portland Cement Association, 1944.

Fig. 4.2) is of the same general shape, and is very nearly identical with that of Fig. 12.3*b*.

It is generally observed that concretes of higher strength $f'_c$ behave in a more brittle manner than low-strength concretes; *i.e.*, cylinders of such concretes fracture at smaller total strains $e_1$ than do cylinders of lower strength. V. P. Jensen took account of this observation by defining as the *plasticity ratio*

$$\beta = \frac{e_2}{e_1} = \frac{1}{1 + (f'_c/4000)^2} \tag{12.1}$$

That is, $e_2 = \beta e_1$ is that portion of the ultimate strain $e_1$ which is plastic *i.e.*, which occurs without increase in test load. It is seen that the plasticity ratio $\beta$ decreases with increasing $f'_c$, indicating more brittle behavior. Up to the smaller strain $e_0$, the concrete, in the simplified diagram, is assumed to behave elastically, so that Young's modulus $E_c$ is the slope of the inclined portion of that diagram. On the basis of test results Jensen proposed for this modulus the expression

$$E_c = \frac{E_s}{(5 + 10{,}000/f'_c)} \tag{12.2}$$

where $E_s = 30{,}000{,}000$ psi is Young's modulus for steel. Hence, the ratio of the elastic moduli is

$$n = 5 + \frac{10{,}000}{f'_c} \tag{12.3}$$

If a beam is loaded in such a manner that the strain in the outer fiber $e'$ is smaller than $e_1$, as shown in Fig. 12.3*a*, then, by the same reasoning as in Art. 12.2, the corresponding stress distribution is that of Fig. 12.4. If the amount of reinforcement is such that the steel will start yielding ($f_s = f_y$) before the strain $e'$ in the top fiber has become equal to the ultimate strain $e_1$ at which crushing starts, then the total tensile force in the steel at incipient failure is

$$T = A_s f_y = pbdf_y \tag{12.4}$$

as shown in Fig. 12.4.

The resisting moment of the beam is then determined as follows: The total compression force $C$ in the concrete is equal to the area of the stress trapezoid times the width of the beam, *i.e.*,

$$C = \frac{1 + \gamma}{2} bf'_c kd \tag{12.5}$$

Since the sum of the horizontal forces must be zero, one obtains from $T = C$

$$k = \frac{2pf_y}{(1 + \gamma)f'_c} \tag{12.6}$$

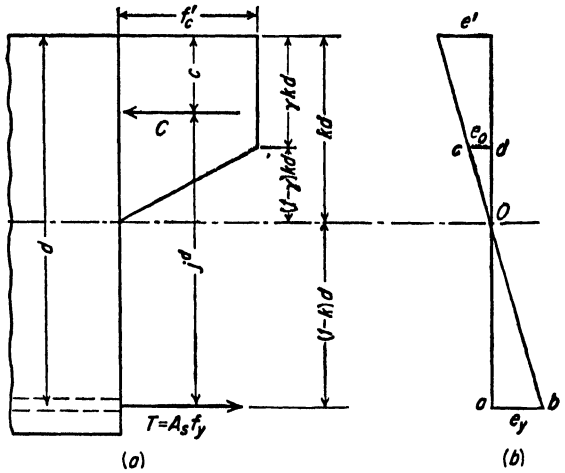



FIG. 12.1. Stress and strain distributions in a beam (Jensen-PCA method).

The lever arm of internal forces is $jd = d - c$, where $c$, as in Fig. 12.4, is the distance from the top fiber to the line of action of $C$, that is, the distance to the centroid of the trapezoid. With

$$c = \frac{kd}{3}\left(1 + \frac{\gamma^2}{1+\gamma}\right)$$

one obtains

$$j = 1 - \frac{k}{3} - \frac{\gamma^2}{1+\gamma}\frac{k}{3} \tag{12.7}$$

On the other hand, considering that strains are proportional to distances from the neutral axis, one obtains from the similar triangles *oab* and *ocd* of Fig. 12.4*b*

$$\frac{e_u}{(1-k)d} = \frac{e_0}{(1-\gamma)kd}$$

As can be seen from Fig. 12.3, $e_0 = f'_c/E_c$ and $e_u = f_u/E_s$. With these values substituted in the above equation

$$\frac{k}{1-k} = \frac{nf'_c}{(1-\gamma)f_u} \tag{12.8}$$

Equations (12.6) and (12.8) contain the common unknowns $\gamma$ and $k$. From Eq. (12.6),

$$\gamma = \frac{2pr}{k} - 1 \tag{12.9}$$

where $r = f_u/f'_c$. If this value is substituted in Eq. (12.8), one obtains for the position of the neutral axis

$$k = \frac{2pr^2 + n}{2r + n} \tag{12.10}$$

Finally, for the lever arm of internal forces, by substituting $\gamma$ from Eq. (12.7),

$$j = 1 - \frac{k}{3} - \frac{(2pr - k)^2}{6pr} \tag{12.11}$$

The resisting moment of the beam is then

$$M = Tjd = pf_w jbd^2 = Kbd^2 \tag{12.12}$$

where $K = pf_w j$. The similarity with the customary formula for review of beams [Eq. (3.5*a*)] will be immediately recognized. To be sure, the expressions for $k$ and $j$ [Eqs. (12.10) and (12.11)] are more complicated than the corresponding formulas of the elastic theory [Eqs. (3.2) and (3.3)]. These values, however, can be tabulated once and for all similarly to Table 6 (Appendix D), and thus the practical use of this method is no more involved than customary procedures. (Such tables are given in the quoted PCA publication.)

It is seen from Eq. (12.9) that when $k = 2pr$, $\gamma = 0$; in this case the stress distribution is that of Fig. 12.5*a*, *i.e.*, the same as in the customary elastic theory. If $k = 2pr$ is substituted in Eq. (12.10), one obtains that value of the reinforcement ratio $p_c$ at and below which the stress distribution is of the type of Fig. 12.5*a*, namely,

$$p_c = \frac{n}{2r(n + r)} \tag{12.13}$$

For beams with $p$ smaller than $p_c$, Eqs. (12.10) and (12.11) do not apply; the stress distribution is that of the elastic theory, and $k$ and $j$ are determined from Eqs. (3.2) and (3.3). It is seen, therefore, that this method contains the customary procedures as a special case for small percentages of reinforcement.

On the other hand, if the concrete strain $c'$ in the extreme fiber becomes equal to the ultimate strain $c_1$ (Fig. 12.3*a*), it is assumed that the concrete begins to crush. If the steel begins to yield simultaneously, both materials are used to their full capacity. In this case Fig. 12.3*a* indicates that $\gamma = \beta$, and consequently the stress distribution in the beam is that of Fig. 12.5*b*. Substituting $\gamma$ for $\beta$ in Eqs. (12.6) and (12.8) and eliminating $k$, one obtains that steel percentage

$$p_0 = \frac{n(1 + \beta)}{2r[n + (1 - \beta)r]} \tag{12.14}$$

at which both materials are used to capacity. Equations (12.10) to (12.12) are then used, as before, to determine the resisting moment of a beam so reinforced.

If a steel percentage larger than $p_0$ were used, a significant increase in carrying capacity could be obtained only by using compression reinforcement.

Figure 12.5*a* and *b* illustrate graphically the difference between the elastic and the plastic theory for ordinary grades of material ($f_y$ = 40,000, $f'_c$ = 3000 psi). The elastic methods limit the usable compression force $C$ developed by the concrete to that corresponding to the triangular area of Fig. 12.5*a*. The plastic theory recognizes that the concrete is actually capable of developing a much larger compression force before crushing. In the Jensen-PCA method the maximum force $C$ that the concrete can develop corresponds to the trapezoidal area of Fig. 12.5*b*.

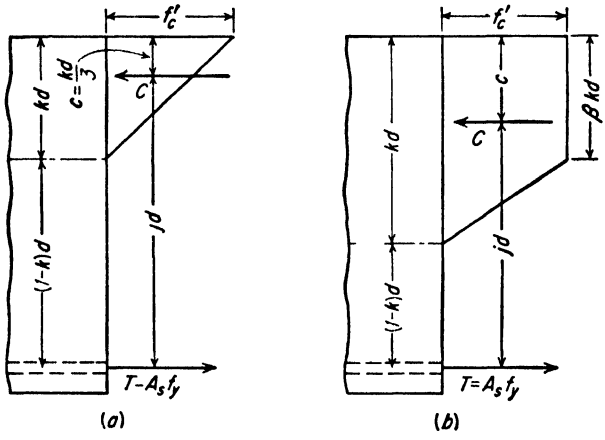



FIG. 12.5. Limiting stress distributions of the Jensen-PCA method: (*a*) for small steel ratios ($p \leqq p_c$), (*b*) for large steel ratios ($p \geqq p_0$)

*Illustrative Examples.* The practical significance of this method is easily realized from the two following examples.

I. A 12- by 24-in. beam is reinforced with eight No. 8 bars in two rows. Intermediate-grade steel ($f_y$ = 40,000 psi) and $f'_c$ = 3000 psi are used. If the customary factor of safety of 2 is stipulated, determine the allowable moment by the Jensen-PCA method, and compare with the value obtained according to the ACI Code.

$$A_s = 6.28 \text{ in.}^2 \qquad b = 12 \text{ in.} \qquad d = 20.5 \text{ in.} \qquad p = 0.0256$$

For the materials used, according to Eq. (12.3), $n$ = 8.33, and $r$ = 13.33. From Eq. (12.13), $p_c$ = 0.0144. Since $p$ is larger than $p_c$, Eqs. (12.10) to (12.12) must be used. They give $k$ = 0.496, $j$ = 0.817, and the ultimate moment $M$ = 4,230,000 in.-lb. The allowable moment is half this value, that is, $M$ = 2,115,000 in.-lb.

By the ACI Code, $k$ = 0.510, $j$ = 0.830. Since $p$ is larger than the

ideal percentage of reinforcement, the concrete stress governs, and the allowable moment is $M$ = 1,440,000 in.-lb.

II. The same beam is reinforced with four No. 8 bars in one row. The identical determinations are required.

Since now $d$ = 21.5 in., $p$ = 0.01215. This percentage is smaller than $p_c$, and consequently the customary beam formulas must be applied. Using the values $n$ and $p$, as given above, from Eqs. (3.2), (3.3), and (3.5), respectively, $k$ = 0.360, $j$ = 0.830, and the ultimate moment $M$ = 2,380,000 in.-lb so that the allowable moment $M$ = 1,190,000 in.-lb.

By the ACI Code, $k$ = 0.385, $j$ = 0.872, and since $p$ is smaller than the ideal percentage so that the steel stress governs, $M$ = 1,180,000 in.-lb. (The small difference in results, despite the fact that the same formulas are used here for both methods, comes from the different value of $n$, Eq. (12.3), as compared to that used in ACI Code.)

From these examples it is seen that the customary method and the Jensen-PCA method give practically identical results for relatively small percentages of reinforcement. However, for larger steel ratios, the resisting moment indicated by the plastic theory far exceeds that determined from the elastic theory, by almost 50 per cent in the case of Example I.

This situation is well borne out by numerous tests which show that the customary elastic method is in reasonable agreement with experimental results only for values of $p$ smaller than $p_c$. For beams without compression reinforcement, but with $p$ in excess of $p_c$, the elastic theory indicates carrying capacities smaller than those obtained from tests, increasingly so as $p$ increases. On the other hand, the Jensen-PCA method has been checked against some 70 beam tests, carried out in three different laboratories, with steel percentages from rather low values up to $p$ = 0.056 and with $f'_c$ varying from 1100 to 5800 psi. The agreement between theory and test was excellent throughout; differences between predicted and experimental strength rarely exceeded 5 per cent and were considerably smaller in most cases.

It is apparent from this discussion that in many cases of relatively large $p$, where the customary theory would call for compression reinforcement to provide the required resisting moment, the beam is actually capable of developing the same moment without such compression reinforcement. This situation arises very frequently at the supports of continous T beams, one of the most frequent types of members. Since the large amount of compression reinforcement required by the elastic theory in such instances is not only wasteful but makes for difficulties in placing the excessive amount of steel, the present ACI Code makes partial allowance for the actually larger strength by permitting for the compression steel twice the stress computed by the elastic theory (see Art. 3.46).

Instead of this rather arbitrary stipulation the plastic theory provides one single, consistent method of computing beam strengths for any value of $p$ and, in addition, is in uniformly good agreement with test results.

**12.4. Whitney Method.** In a series of papers C. S. Whitney[1] has proposed a method of ultimate design different from and somewhat simpler than the Jensen-PCA method. Like the latter, Whitney's method is derived from and checked against numerous test results. In their practical results, *i.e.*, the calculated strength of beams of various proportions and made from various grades of concretes and steels, the two methods differ but little. In contrast to the Jensen-PCA method, the greater simplicity of Whitney's method enables it to be applied without difficulty not only to beams but also to members subject to bending and axial stress.

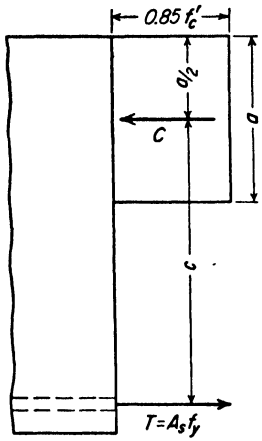



FIG. 12.6. Stress distribution in a beam (C. S. Whitney method).

Whitney assumes the stress distribution in a beam to be of the type of the "slow loading" rather than the "standard loading" curve (see Figs. 12.1, 12.2*b*, and a so 4.2). It is seen that according to this distribution a considerable part of the compression portion of a beam is subject to practically constant stress of intensity equal to, or only slightly greater than, $0.85f'_c$. For purposes of simplification, he replaces this distribution diagram by a rectangle of width $0.85f'_c$ and depth $a$ (Fig. 12.6). It should be noted that $a$ is not equal to the distance to the neutral axis $kd$. In fact, the location of this axis remains undetermined in this method. As was pointed out before, the real stress distribution can be replaced by one of any shape, provided it represents correctly the total compression in the concrete and the lever arm of internal forces. Whether these requirements are satisfied can be established conclusively only by comparison with a wide range of test results, as was done by both Jensen and Whitney.

Stresses in a beam are then assumed to be distributed as shown in Fig. 12.6, where $a$ is not equivalent to the usual $kd$, but the lever arm of internal forces $c$ has the same meaning as the usual $jd$. For a rectangular beam, since the sum of horizontal forces is zero, that is, $T = C$,

$$A_s f_y = ab0.85f'_c$$

from which, with $p = A_s/bd$ and $m = f_y/0.85f'_c$, one obtains the depth of

[1] See *J. ACI*, vol. 8, p. 183, 1937; *Trans. ASCE*, vol. 107, p. 251, 1942; *J. Boston Soc. Civil Engrs.*, vol. 35, no. 1, p. 29, 1948.

the rectangular stress block

$$a = pmd \tag{12.15}$$

and the lever arm

$$c = d\left(1 - \frac{pm}{2}\right) \tag{12.16}$$

If yielding of the steel governs the strength of the beam, its resisting moment is

$$M_s = A_s f_y c = pf_y\left(1 - \frac{pm}{2}\right)bd^2 \tag{12.17}$$

which can be written in the usual form

$$M_s = Kbd^2 \tag{12.18}$$

where

$$K = pf_y\left(1 - \frac{pm}{2}\right) \tag{12.19}$$

On the other hand, the moment of the concrete compression force about $T$ is

$$M_c = 0.85f'_c abc = 0.85f'_c ab\left(d - \frac{a}{2}\right)$$

This can be written as

$$M_c = 0.85f'_c \frac{a}{d}\left(1 - \frac{a}{2d}\right)bd^2 \tag{12.20}$$

which is seen to be an equation of the form $M = Kbd^2$.

In the Jensen-PCA method the maximum moment which the concrete is capable of developing was determined by assuming that crushing begins when the concrete strain becomes equal to $\epsilon_1$. No such assumption was made by Whitney. Instead, by evaluating a large number of beam tests he found that for large reinforcement percentage $p$, that is, when concrete rather than steel governed, the resisting moment was

$$M_c = \frac{f'_c}{3}bd^2 \tag{12.21}$$

From Eqs. (12.20) and (12.21) it is seen that

$$0.85f'_c \frac{a}{d}\left(1 - \frac{a}{2d}\right) = \frac{f'_c}{3}$$

Solving this equation, one obtains the maximum value of $a/d$ at the moment when compression failure starts which is

$$\frac{a}{d} = 0.537$$

Since, from Eq. (12.15), $p = (a/d)(1/m)$ the steel percentage at which the steel starts yielding at the same time at which the concrete begins crushing is

$$p_0 = \frac{0.537}{m} \tag{12.22}$$

For reinforcement ratios $p$ smaller than $p_0$, Eq. (12.17) determines the strength of the beam. If $p$ is equal to, or larger than, $p_0$, the strength of the beam is governed by that of the concrete and is given by Eq. (12.21); hence the maximum value which $K$ can reach, no matter how large the value of $p$, is

$$K_{max} = \frac{f'_c}{3} \tag{12.23}$$

Analysis of a large number of beam tests with $p$ larger than $p_0$ showed Eq. (12.21) to be very accurate for the range of concrete strengths $f'_c$ between 3000 and about 5000 psi. For values below 3000, the actual test strengths of beams proved to be larger than indicated by Eq. (12.21), the more so, the smaller the $f'_c$. This discrepancy is undoubtedly due to the fact that the Whitney method takes no account of the greater plasticity of low-strength concretes as compared to the more brittle concretes of higher strength. The Jensen-PCA method accounts for this difference by means of the plasticity ratio $\beta$ [Eq. (12.1)], which, however, results in rather unwieldy design formulas. The discrepancy in the Whitney method, for low-strength concretes with $p$ larger than $p_0$, is of no particular practical consequence since (1) with modern methods of concrete manufacture and control, strengths below $f'_c = 3000$ psi occur quite rarely; (2) the steel percentage is usually below $p_0$, in which case Eq. (12.17) holds, rather than Eq. (12.21), (3) the error, if any, is on the safe side except possibly for concretes of extremely high strength.

To show how the two methods compare, the problems of Art. 12.3 will now be recomputed using Whitney's method.

*Illustrative Examples.* I. With the same data as Example I, Art. 12.3, $p = 0.0256$ and $m = 15.7$. From Eq. (12.22),

$$p_0 = 0.537/15.7 = 0.0342$$

Since $p$ is smaller than $p_0$, the steel strength governs and the ultimate moment, from Eq. (12.17), is 4,110,000 in.-lb. With a factor of safety of 2, the allowable moment becomes $M = 2{,}055{,}000$ in.-lb. This compares with 2,115,000 in.-lb by the Jensen-PCA method and 1,440,000 in.-lb by presently current codes.

II. With the same data as Example II, Art. 12.3, $p = 0.01215$ and $m = 15.7$. From Eq. (12.17), the ultimate moment is 2,430,000 in.-lb, and the allowable moment $M = 1{,}215{,}000$ in.-lb. This compares with

1,190,000 in.-lb by the Jensen-PCA method and 1,180.000 in.-lb by current codes.

It is seen that the two new methods agree with each other very closely. They are in good agreement with the customary elastic theory only for relatively low $p$. They represent correctly the higher empirical strength of beams with larger $p$ as compared to the erroneously low strength predicted by the elastic analysis.

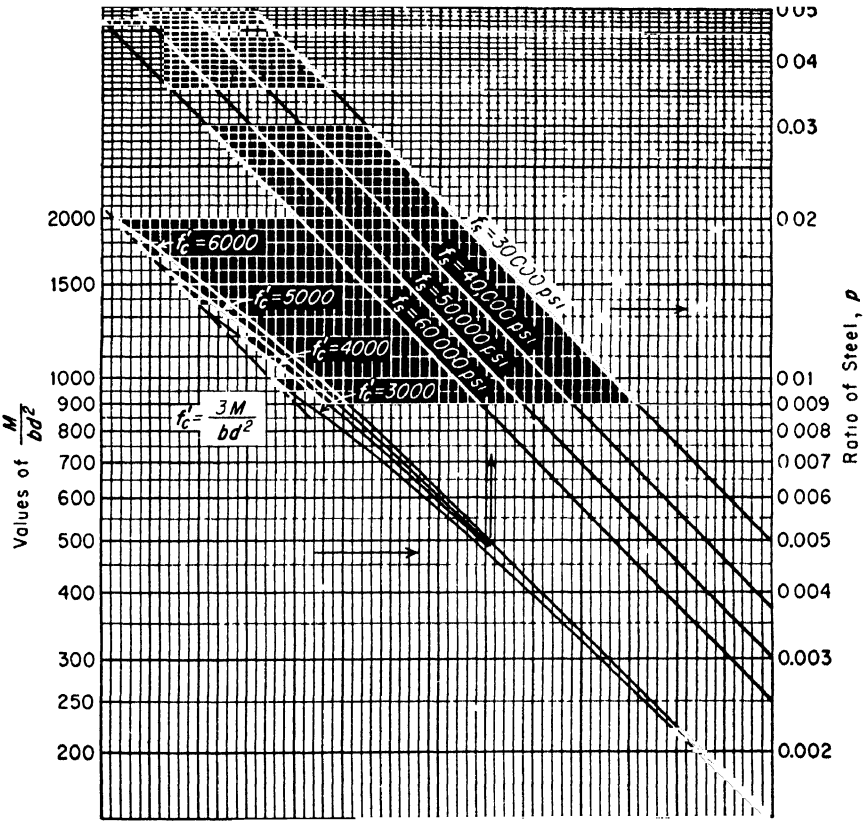



FIG. 12.7. Diagram for design and review of beams (C. S. Whitney method).

To simplify the routine use of his method, C. S. Whitney has prepared the diagram of Fig. 12.7. This graph is used for design in the manner indicated by the arrows and can also be used for review by reading it in the direction opposite to that of the arrows. The diagram determines ultimate moments which must be divided by an appropriate factor of safety.

In addition to beams C. S. Whitney has also applied his method to eccentrically loaded columns, *i.e.*, to the case of bending plus axial com-

pression. It is again assumed that the actual stress distribution can be replaced by a rectangular stress block. It is further assumed that Eq. (12.21) also holds for eccentrically loaded columns, although it was derived empirically from tests on beams. Finally, the method is so adjusted that in the limit, *i.e.*, with zero eccentricity, it results in the well-established formula for the strength of axially loaded columns, Art. 4.2. The resulting design equations for rectangular sections, though naturally more lengthy than those for beams, have been incorporated by Whitney in simple design graphs.[1]

Test evidence on the strength of eccentrically compressed members has been inadequate until recently. An extensive new series of tests on 120 eccentrically loaded columns, of square sections with ties as well as of circular section with spirals, with eccentricity ratios $e/d$ from 0 to 1.25, has filled this gap.[2] The results of these tests are in good agreement with the predicted strength as computed both by the Whitney and the Jensen-PCA method. In conjunction with the previously available test results on concentrically loaded columns on the one hand, and on beams in simple bending on the other, they confirm the validity of the ultimate theory of reinforced concrete design for all types of members.

**12.5. Practical Significance of Ultimate or Plastic Design.** It has been shown in Arts. 12.3 and 12.4 that the ultimate design methods do not differ significantly from the elastic theory for values of $p$ smaller than the "ideal" percentage $p_i = n/2r(n + r)$ of the elastic theory. At that value this theory assumes the resisting moments of steel and concrete to be equal. The corresponding ultimate bending moment is $M = Kbd^2$, where $K = p_i f_u j$. This bending moment, by the elastic theory, can be increased either by increasing the tension steel only or by increasing the tension steel and also furnishing compression steel. Both these measures represent uneconomical use of steel, since any amount of steel in excess of $p_i$, by the elastic theory, provides only half as much or less additional strength than does the same amount in an underreinforced beam, *i.e.*, with $p$ less than $p_i$. This is illustrated by the lower curve of Fig. 12.8, where for low steel ratios below $p_i$ an increase in $p$ of 0.005 furnishes an increase in moment of about $200bd^2$, while if $p$ is larger than $p_i$, the same amount of additional steel adds only about $40bd^2$ to the carrying capacity. All moment values quoted in this article are ultimate moments which must be divided by an appropriate factor of safety to result in allowable design moments. Had compression reinforcement been used in addition to added tension reinforcement the situation would not have been quite as unfavorable but even then any steel, compression

[1] See the third reference in the footnote on page 434.

[2] E. Hognestad, "A Study of Combined Bending and Axial Load in Reinforced Concrete Members," *Univ. Ill. Eng. Exp. Sta. Bull.* 399, 1951.

or tension, beyond $p_i$ is only half or less as effective than the same amount of steel in an underreinforced beam of the same dimensions.

Tests and ultimate design methods based on tests have shown that this consequence of the elastic theory is incorrect. The actual behavior is depicted in Fig. 12.8 by the upper curves. They show that additional steel increases the strength of the beam at a practically constant rate until the much larger critical percentage $p_0$ is reached [Eq. (12.14) or (12.22)]. For example, to provide $M = 600bd^2$, the elastic theory would call for $p = 0.0215$, while actually only $p = 0.014$ is required to develop

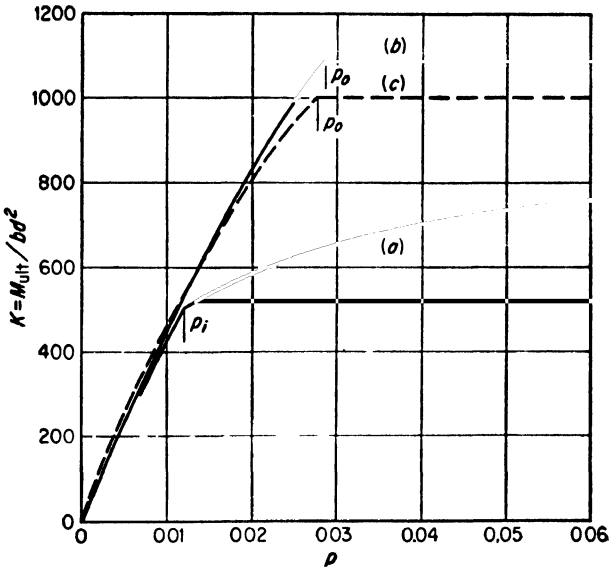



FIG. 12.8. Ultimate strength of beams by (*a*) the elastic theory; (*b*) the Jensen-PCA method; (*c*) the C. S. Whitney method. $f'_c = 3000$ psi; $f_y = 50{,}000$ psi

this moment. For $M = 700bd^2$, the respective steel percentages are 0.040 as compared to 0.0165, a difference of about 150 per cent.

This large strength of beams with comparatively high steel ratios, if correctly reflected in design, would allow practically all the beam steel to be used in its most economical manner, *i.e.*, as simple tension reinforcement at steel ratios below $p_0$. Conversely, in the case of long-span beams (high moment with low shear) it would allow much shallower beams to be used merely by using relatively high ratios of tension reinforcement, with a consequent saving in concrete volume, over-all height, and dead load of the structure.

In recent years a definite tendency has developed to use expensive high-strength concretes in conjunction with standard-strength steels.

This combination is advantageous where high shears combine with relatively low moments (heavily loaded short-span beams). However, the seeming advantage of this combination for moment capacity (long-span beams) is entirely based on the erroneous conclusion of the elastic theory, as can be seen from Table 12.1. In this table, for various combinations of steel and concrete strengths, are shown the critical percentages $p_t$ and $p_0$ of the elastic and the ultimate theories, respectively, and the corresponding moment coefficients $K$.

TABLE 12.1

| $f_y$ | $f'_c$ | Elastic theory | | Jensen-PCA method | | Whitney method | |
|---|---|---|---|---|---|---|---|
| | | $p_t$ | $K$ | $p_0$ | $K$ | $p_0$ | $K$ |
| 40,000 | 3,000 | 0 016 | 514 | 0 039 | 1,149 | 0 034 | 1,000 |
| 40,000 | 5,000 | 0 027 | 925 | 0 051 | 1 601 | 0 057 | 1,667 |
| 60,000 | 3,000 | 0 0083 | 423 | 0 022 | 1 024 | 0 023 | 1,000 |
| 60,000 | 5,000 | 0 014 | 747 | 0 028 | 1,391 | 0 038 | 1,667 |

From this table it is seen that for the standard combination 40,000 3000 uneconomical excess steel would be required by the elastic theory for bending moments larger than $514bd^2$. If high-strength concrete with $f'_c = 5000$ psi is used, the elastic theory indicates that moments up to $925bd^2$ can be developed with economical steel percentages smaller than $p_t$. Actually, however, as indicated by the plastic design methods, moments up to about $1000bd^2$ can be developed with the lower-strength concretes without adding uneconomical steel in the form of either compression reinforcement or excess tensile steel. Hence, the use of high-strength concrete in this case has no advantage as far as moment capacity is concerned.

On the other hand, reinforcing steel of higher than average strength can be produced at little extra cost. The use of such steels in combination with average-strength concretes is penalized by the elastic theory in consequence of the incorrectly low $p_t$ indicated for this combination. In fact, for the combination 60,000/3000, by that theory, the maximum moment that can be developed with economical use of steel is $423bd^2$, which is less than that which can be achieved using the combination 40,000/3000, *i.e.*, $514bd^2$. In contrast, the plastic theory indicates that both these combinations give about the same maximum economical carrying capacity, $1000bd^2$, but that this strength is achieved by using considerably smaller amounts of the high-strength steel than would be required for standard steel ($p_0 = 0.023$ as against 0.034, by the Whitney method). This saving in steel is desirable not only because it more than

makes up for the small price difference between the two grades. More important, the large amount of standard-strength steel which would be required to develop such high moments makes for very considerable difficulties in placing both steel and concrete. In general, steel ratios in excess of 2.5 to 3 per cent are impractical for that reason.

Finally, for structures requiring maximum strength (long-span girders and the like), the table shows that new design methods will permit one to take full advantage of the combination of high-strength steels with high-strength concretes such as 60,000/5000. This combination will allow the development of large bending moments (as indicated by large values of $K$) by means of reasonably low steel percentages The present elastic theory makes this impossible in view of the low value of $p_t$ indicated for this combination.

In addition to these practical advantages, the adoption of plastic design methods will do away with the inconsistencies and contradictions in present codes, where design of beams is now carried out by the elastic theory, design of axially loaded columns by the plastic theory, and design of compression reinforced beams and of eccentrically loaded columns by a compromise between the two.

## PRESTRESSED CONCRETE

**12.6. Principles and Development of Prestressed Concrete.** Modern structural engineering tends to achieve more economical structures by improved methods of design and the use of higher-strength materials. This results in reduction of cross-sectional dimensions and consequent weight saving. Such a development is particularly important in the field of reinforced concrete where the dead load represents a large part of the total design load; also, in multistory buildings, any saving in depth of members, multiplied by the number of stories, can represent a considerable saving in total height, load on foundation, cubage of space which must be heated or air-conditioned, etc. In Art. 12.5 it was shown that such savings can be achieved if high-strength concretes and steels are used in conjunction with ultimate or plastic design.

However, in conventional reinforced concrete there are physical limitations to this development, in view of the following undesirable, but inherent features:

1. In flexural members all concrete below the neutral axis, roughly two-thirds of the total volume, does not participate in resisting bending moments and, in this sense, is wasted.

2. The size of members is often governed by shear, *i.e.*, diagonal tension, a type of stress to which concrete is not well suited in view of its low tensile strength.

3. Even though the use of high-strength steels in conjunction with

similar concretes results in smaller-size members, there is a limit beyond which one cannot go in this direction. This is so because hairline cracks on the tension side are inevitable. The width of these cracks is about proportional to the strain, and therefore to the stress in the steel. In order for these cracks not to be unsightly or even detrimental because of corrosion, the width of such cracks must be limited to about $\frac{1}{100}$ in. as an absolute maximum. For this reason reinforcement with yield points beyond about 60,000 psi, with correspondingly high working stresses, may become objectionable.

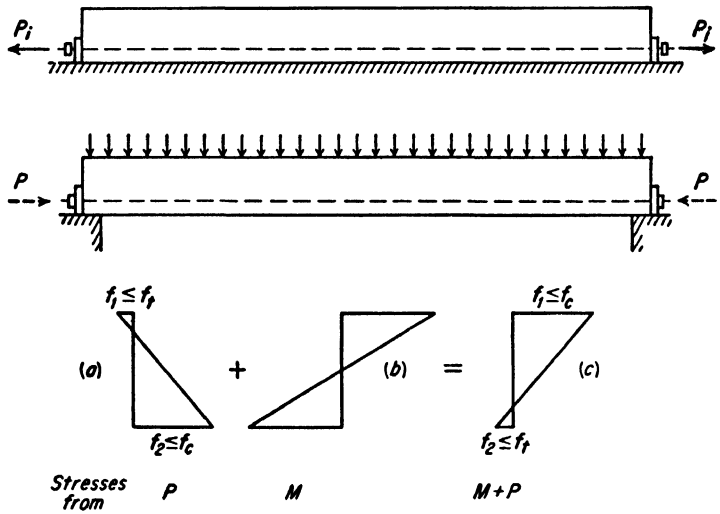



FIG. 12.9. Principle of prestressing.

These disadvantages can be, and have been, overcome by the use of prestressed concrete. The principle of this method is illustrated by Fig. 12.9.

Suppose a beam is fabricated in the following manner. First, high-strength reinforcing steel (with an ultimate tensile strength above 200,000 psi) is placed in the form and passed through holes in the two anchor plates. It is then prestressed by a force $P_i$, *the initial prestress*, which is applied by suitable means, such as jacks bearing against abutments. The concrete is then cast so that the anchor plates bear against the ends of the beam, and is allowed to harden. Next, the steel is secured against the anchor plates, and the jacks are removed. The steel tends to contract to its original length but is prevented from doing so by the concrete interposed between the anchor plates. Hence it remains under stress, and this stress is equivalent to a total compression force $P$ applied to the beam as shown.

This force $P$, after some time has elapsed, will be somewhat smaller

than the initial pretension $P$, for the following reason: (*a*) When the jacks are removed, the tension in the steel is resisted by an equal compression in the concrete. This compression produces a small shortening in the concrete, and thereby in the steel, with a corresponding reduction in steel stress. (*b*) A further shortening of the concrete is produced by its natural shrinkage and by creeping under constant load (see Art. 4.2). (*c*) If the steel is stressed beyond its creep limit (which, by some authorities, is not recommended in American practice but is widely done abroad), it will stretch somewhat under stress with a consequent stress reduction. These combined losses amount to about 12 to 20 per cent so that the *final prestress $P$ is about* 0.80 *to* 0.88 *of the initial prestress $P_i$*.

The eccentrically applied force $P$ produces stress distribution *a* in Fig. 12.9. It is so adjusted that the stress $f_2$ in the bottom fiber does not exceed the permissible concrete compression stress $f_c$, and the stress $f_1$ in the top fiber does not exceed the permissible concrete tension stress $f_t$, generally of the order of $f_c/10$. Hence, at this stage the concrete remains fully effective, since it is not permitted to crack in tension.

When the beam is subsequently loaded as shown, bending moments are set up which by themselves result in stress distribution *b*; these stresses superpose on distribution *a* and result in the final distribution *c*. The dimensions of the beam and the position and amount of prestress are so arranged that in this final distribution $f_1$ does not exceed $f_c$ and $f_2$ does not exceed $f_t$. Hence, at this stage, likewise, the concrete is not allowed to crack and thus performs throughout as a monolithic material, carrying its share of stress over the entire cross section, which is principally and often exclusively in compression.

Hence, cracking is avoided at any stage, and the entire concrete cross section is employed to resist moment. In conjunction with high-strength concrete, with $f'_c$ of about 5000 to 7000 psi, steel of very high strength can be used without undesirable consequences. It will be shown later that this process also results automatically in a very sizable reduction of diagonal tension due to shear.

The first suggestions for prestressing seem to have been made between 1886 and 1908 by the Americans P. H. Jackson and G. R. Steiner, the Austrian J. Mandl, and the German M. Koenen. The use of high-strength steel was first suggested by the Austrian F. von Emperger in 1923, while at about the same time the American R. H. Dill proposed "full prestressing" to eliminate cracks completely. While these proposals remained mainly on paper, the actual development of prestressed reinforced concrete structures abroad is chiefly due to E. Freyssinet (France, 1928), E. Hoyer (Germany, 1938), and most recently Prof. G. Magnel in Belgium. These methods have found very wide application in Europe for bridges up to spans of 200 ft and more, floors and roofs,

piles, railroad ties, etc. Moderate-size elements such as prestressed joists, beams, and slabs are successfully mass produced by precasting.

The development in this country has been slow, not for technical but for economical reasons. The European methods achieve great savings of materials, but the methods of prestressing require a great amount of highly qualified labor. This is no particular disadvantage under the European ratio of high material to low labor cost. The American ratio is exactly the reverse, which, so far, has made direct use of the European methods not too economical in this country. The first major such structure in the United States is the Walnut Lane Bridge in Philadelphia (1950) erected according to the Magnel system, with a main span of 160 ft, while a number of other structures have been constructed in recent years. Full economical use in this country has to await the development of laborsaving methods, to which much attention is now being paid. However, in a different field, an independent American method, originated by W. H. Hewitt in 1923, has found wide application. This consists in precompressing the cylindrical shells of water tanks and pipes by surrounding a concrete shell with hoops of steel which are tightened to a predetermined tension and then covered with a protective concrete layer.

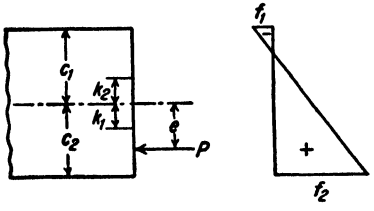



FIG. 12.10. Stress distribution due to prestress force $P$.

**12.7. Stresses in Beams.** In the following, $f_1$ and $f_2$, and $c_1$ and $c_2$, are the stresses in, and centroidal distances to, the top and bottom fibers, respectively, as shown in Fig. 12.10. Compression stresses are positive and tension stresses negative. As long as the concrete is prevented from cracking, stresses are computed from the equations pertaining to monolithic members, rather than from the formulas derived for cracked reinforced concrete sections.

If a force $P$ is applied to a monolithic section with an eccentricity $e$ measured from the centroid, the outer fiber stresses (Fig. 12.10) are obtained from

$$f_{1,2} = \frac{P}{A} \mp \frac{Mc_{1,2}}{I} = \frac{P}{A}\left(1 \mp c_{1,2}\frac{A}{I}\right)$$

since $M = Pe$. Introducing the core radius

$$k_{1,2} = \frac{I}{Ac_{1,2}}$$

this can be written as

$$f_{1,2} = \frac{P}{A}\left(1 \mp \frac{e}{k_{1,2}}\right) \tag{12.24}$$

It is seen from this equation that, if $e = k_1$, $f_1 = 0$. Hence, if the force is applied with an eccentricity equal to the core radius, the entire section is in compression, except that the stress at the opposite outer fiber is zero. This is the physical significance of the core radius.

The center section of a simply supported beam, or for that matter any section except those at the supports, is subject not only to the prestress force $P$ but also to the bending moment $M$ caused by the transverse load. Hence, at these span sections

$$f_1 = \frac{P}{A}\left(1 - \frac{e}{k_1}\right) + \frac{Mc_1}{I} \tag{12.25}$$

$$f_2 = \frac{P}{A}\left(1 + \frac{e}{k_2}\right) - \frac{Mc_2}{I} \tag{12.26}$$

At the supports, where $M = 0$,

$$f_1 = \frac{P}{A}\left(1 - \frac{e}{k_1}\right) \tag{12.27}$$

$$f_2 = \frac{P}{A}\left(1 + \frac{e}{k_2}\right) \tag{12.28}$$

In the uncracked section the maximum vertical shear stress at the centroid is (see Art. 3.1)

$$v = \frac{VQ}{Ib} \tag{12.29}$$

which, if $P$ were absent, would also be the magnitude of the maximum diagonal tension stress at that point. However, the presence of $P$ results in an additional longitudinal compression stress $P/A$ at the level of the centroidal axis. The inclined tension at the centroid is, therefore, computed from (see Art. 3.1)

$$f_{td} = \frac{P}{2A} - \sqrt{\left(\frac{P}{2A}\right)^2 + v^2} \tag{12.30}$$

which is seen to be always numerically smaller than the diagonal tension stress [that is, $v$ from Eq. (12.29)] which would act in the absence of $P$.

Equations (12.25) to (12.30) furnish all the information that is needed to compute *stresses in the uncracked section, i.e.*, up to loads usually considerably in excess of the design loads.

In addition it is usually desirable to compute *the ultimate or failure moment* of the beam. As the load is gradually increased to the failure load, cracking will occur on the tension side. To investigate the failure moment in prestressed beams after cracking, it is necessary to take account of the properties of high-strength steel which are different from those of ordinary reinforcing steel. Figure 12.11 shows the stress-strain

diagram of the cold-drawn wire used at the Walnut Lane Bridge, which is typical of American wires for use in prestressing. It is seen that this steel does not show the pronounced yield point of regular mild reinforcing steel. It has a proportional limit of about 142,000 psi, a 0.2 per cent offset yield strength of 215,000 psi, and an ultimate strength of 247,000 psi. On this bridge the final design prestress, after losses from shrinkage and creep, was 110,000 psi. It is seen that the strain corresponding to this stress is about 0.0039 in. per in. Now, if the beam is overloaded so that its steel stress is about 226,000 psi and is then unloaded, the figure shows that there will be a permanent steel elongation $e_p$ of 0.0039 (*i.e.*, the same amount as the elastic strain at the design stress), while the stress in the steel is reduced to zero. On reloading, consequently, the beam will perform the same as one without prestressing. Actually, no unloading is necessary for the effective prestress to be lost. If a beam is overloaded so that the plastic or permanent part $e_p$ of the total steel strain $e_t$ becomes equal to the elastic strain $e$ at prestress, the prestress has been dissipated, and at still higher loads the member performs like conventional reinforced concrete

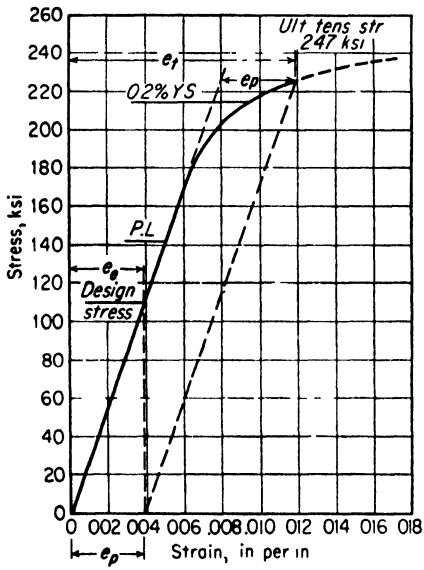



FIG. 12.11. Stress-strain diagram of steel wire used in Walnut Lane Bridge.

Since most prestressed beams are underreinforced ($p$ smaller than $p_0$, as given by the ultimate theory), failure will occur through rupture of the steel at its ultimate strength, *i.e.*, in the substantially horizontal part of the stress-strain diagram. Therefore, the ultimate moment is computed by Eq. (12.12) or Eq. (12.17), with the ultimate steel strength substituted for $f_y$.

**12.8. Design of Pretensioned Beams.** Prestress can be applied through pretensioning or posttensioning, depending upon whether the steel is stressed before or after the concrete is cast and hardened. In the former method, which will be discussed first, the steel, usually in the form of individual wires, is first stressed between fixed abutments (Fig. 12.12). The concrete is cast and allowed to harden, after which the wires are cut at the end faces of the beam Prestress is often maintained without special anchorage through the bond between concrete and steel

and also because the wire ends, when the wire is cut, expand slightly laterally in view of loss of stress and become conical, thus wedging against the concrete. This system is well suited for mass production, since casting beds can be made several hundred feet long, the entire length cast simultaneously, and individual beams cut to the desired length from the long casting.

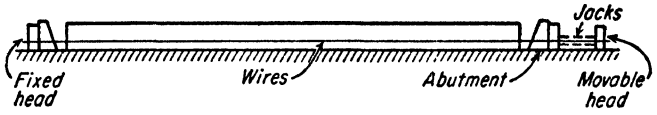



FIG. 12.12. Schematic view of producing pretensioning

The following conditions must be met in the beam at full design load: At the section of maximum moment $M$,

$$f_1 = \frac{P}{A}\left(1 - \frac{e}{k_1}\right) + \frac{Mc_1}{I} \leqq f_c \tag{12.31}$$

$$f_2 = \frac{P}{A}\left(1 + \frac{e}{k_2}\right) - \frac{Mc_2}{I} \geqq f \tag{12.32}$$

and at the supports,

$$f_1 = \frac{P}{A}\left(1 - \frac{e}{k_1}\right) \geqq f_t \tag{12.33}$$

$$f_2 = \frac{P}{A}\left(1 + \frac{e}{k_2}\right) \leqq f_c \tag{12.34}$$

The allowable compression stress $f_c$ is generally taken about 0.3 to $0.4f'_c$, while the permissible tension stress $f_t$, to prevent cracking under design load, should not be larger, numerically, than about $0.1f_c$ and is often taken as zero. Note that, by the indicated sign convention, $f_t$ is negative. Hence, Eqs. (12.32) and (12.33) stipulate that the respective fiber stresses, if tension, are not allowed to exceed $f_t$ in absolute value; they can be compression, if this presents an advantage.

Frequently one stipulates that $f_t = 0$. If, in addition, the section is symmetrical, that is, $k_1 = k_2 = k$, the following simplifications obtain. From Eq. (12.33),

$$e = k \tag{12.35}$$

From Eq. (12.31), since $(1 - e/k) = 0$, $Mc/I = f_c$, or the section modulus

$$S = \frac{I}{c} = \frac{M}{f_c} \tag{12.36}$$

and from Eq. (12.34), since $(1 + e/k) = 2$,

$$P = \frac{Af_c}{2} \tag{12.37}$$

Hence, for this case, one finds the required section modulus and the corresponding cross-sectional dimensions from Eq. (12.36), the required eccentricity of the prestress force from Eq. (12.35), and the final value of the prestress force from Eq. (12.37). To compensate for losses from creep, shrinkage, etc., the initial prestress is made equal to $P_i = P/0.85$ to $P/0.80$.

*Illustrative Example.* A slab bridge of 48 ft span is to be designed for a live load (including paving) of 400 psf; $f'_c = 6000$ psi,

$$f_c = 0.35 \times 6000 = 2100 \text{ psi}$$

$f_t = 0$, maximum diagonal tension in concrete $0.1 \times 2100 = -210$ psi. A steel with ultimate strength of 240,000 psi and with a proportional limit of 140,000 psi is to be used. To stay below the proportional limit, the initial prestress is limited to 135,000 psi, and the final value after shrinkage and creep to $0.8 \times 135{,}000 = 108{,}000$ psi. A section 12 in. wide will be designed.

Assuming a dead load of 300 psf, the maximum moment is

$$M = \frac{700 \times 18^2}{8} 12 = 2{,}410{,}000 \text{ in.-lb}$$

From Eq. (12.36), with $S = bh^2/6$, the required depth

$$h = \sqrt{\frac{6 \times 2{,}410{,}000}{2100 \times 12}} = 24.0 \text{ in.}$$

From Eq. (12.37), the required final prestress force

$$P = 12 \times 24 \times \frac{2100}{2} = 302{,}000 \text{ lb}$$

and from Eq. (12.35), since for a rectangle $k = h/6$, the eccentricity

$$e = {}^{24}{}_{6} = 4 \text{ in.}$$

The required steel area is $A_s = 302{,}000/108{,}000 = 2.8$ sq in.

Wires conforming to the specifications given above are manufactured in this country with diameters of 0.196, 0.236, and 0.276 in. and with areas, respectively, of 0.0302, 0.0434, and 0.0598 sq in. Taking the largest of these, the required number of 0.276-in. wires is

$$N = 2.8/0.0598 = 46.8$$

These could be arranged in one group whose centroid is at the specified distance of $e = 4$ in. below mid-depth, but this would not place any steel close to the tension face. However, it was pointed out that once cracking starts, the ultimate load for the prestressed beam is the same as if no prestress had been applied. From this it is clear that the more steel is

placed as closely as possible to the tension face, the larger the ultimate moment. In order to maintain the centroid of the wires at the required location, it then becomes necessary to place some of them near the compression face. With wires located at the level shown in Fig. 12.13, the number of wires in each group is determined: for the bottom group

$$N_1 = 46.8(13.5/18) = 35.1$$

say, 36 wires, and for the top group $N_2 = 46.8(4.5/18) = 11.7$, say 12 wires. These wires must be stressed equally and to a combined initial prestress of $P_i = 302{,}000/0.8 = 378{,}000$ lb.

In order to check the moment at which the first cracks are likely to form, these will be assumed to occur at a tension stress

$$f_t = 0.1f'_c = -600 \text{ psi.}$$

This moment $M_{cr}$ is obtained by equating $f_2$ from Eq. (12.26) to the above value of $f_t$, that is,

$$\frac{302{,}000}{288}\left(1 + \frac{4}{4}\right) - \frac{M_{cr}}{1152} = -600$$

from which $M_{cr} = 3\,120{,}000$ in.-lb. Since the design load of 700 psf produced a moment of 2,410,000 in.-lb, the load which produces $M_{cr}$ is

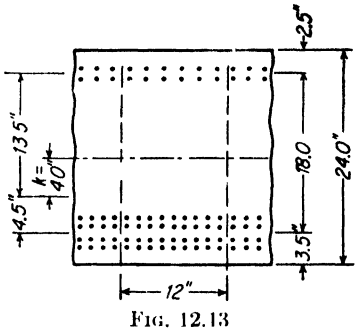


FIG. 12.13

$$\frac{700 \times 3.12}{2.41} = 905 \text{ psf}$$

or 1.29 times the design load. However, since the dead load of 300 psf is constant, this means that a live load of 605 psf, or 1.51 times the design live load, is required to produce the first cracks. This margin of safety against cracking is ample.

To check the carrying capacity of the slab, the ultimate moment will be determined according to C. S. Whitney (Art. 12.4). With

$$m = \frac{240{,}000}{0.85 \times 6000} = 46.8 \qquad \text{and} \qquad p = \frac{36 \times 0.0598}{12 \times 20.5} = 0.00875$$

from Eq. (12.16),

$$c = 20.5\left[1 - \left(0.00875 \times \frac{46.8}{2}\right)\right] = 16.32 \text{ in.}$$

and, from Eq. (12.17), the ultimate moment

$$M_u = 2.15 \times 240{,}000 \times 16.32 = 8{,}400{,}000 \text{ in.-lb}$$

Substitution in Eq. (12.21) shows that $M_u < M_c$, that is, ultimate strength is governed by rupture of steel rather than crushing of concrete.

The ultimate moment is seen to be about 3.5 times the design moment, a safety factor much larger than in conventional concrete structures.

To check diagonal tension, with a reaction $R = 700 \times 24 = 16{,}800$ lb, the maximum shear stress at the centroidal axis of the monolithic, rectangular section is $v = 1.5(V/A) = 1.5(16{,}800/288) = 87.5$ psi. From Eq. (12.30), with $P/A = 302{,}000/288 = 1045$ psi, the diagonal tension at the centroidal axis is $f_{td} = 522.5 - \sqrt{(522.5)^2 + (87.5)^2} = -8$ psi (tension). The value of this stress is seen to be only about one-tenth of the diagonal tension (equal to $-v$) which would act if the prestress $P$ were absent.

Although prestressed slab bridges of this type have been built abroad, the low value of the diagonal tension (as compared to the permissible value, assumed to be 210 psi) indicates that a considerable saving of material and load can be achieved by using beams of $I$ or box shape with relatively thin webs instead of solid slabs. These beams can be either cast in place or fabricated on the side and placed by cranes. For this reason such shapes, with webs just thick enough to resist the diagonal tension without web reinforcement, are widely used in prestressed concrete construction. The design of such beams is in no way different from that of the present example, which has been chosen chiefly in view of numerical simplicity.

**12.9. Design of Posttensioned Beams.** In posttensioning, wires are used in groups usually called "cables," although they are not in the usual form of cables. They consist of a number of individual parallel wires, not in contact, but with predetermined spacing from each other. In the Freyssinet method, the wires are enclosed in a sheath of corrugated sheet steel and placed in the forms, after which the concrete is cast. In the Magnel method, rubber inlays are inserted in the forms at the future location of the cables, the concrete is cast, and the inlays withdrawn after hardening. The preassembled cables are then pulled through the voids left by the inlays. Still other ways of placing are used or may be developed. In all these methods the wires are tensioned subsequently by jacking against the hardened concrete, and are fastened to anchorages at the ends of the beam. Grout is then forced under pressure to fill the remaining voids and bond the wires as well as possible to the concrete. One of the several advantages of posttensioning is that the reinforcement need not be straight, as is evidently necessary in pretensioning, but may be curved or bent up at predetermined points. Figure 12.14 shows a beam reinforced with three cables which are at the same elevation at mid-span. However, while the two outer cables are straight, the center cable may either be parabolically curved along its entire length, as shown on the right half, or may have a horizontal center portion while being sloped in the outer portions, as shown on the left.

Analysis and design of posttensioned beams are essentially the same as shown in Art. 12.8 for pretensioned beams, except for a few special features. In particular, Eqs. (12.31) to (12.34) also hold for posttensioned beams under design load; *i.e.*, the moment to be substituted in these equations is that caused by dead plus live load. However, as is evident from Fig. 12.14, in the center portion of the beam the wires are located far outside the core. Hence, when the prestress is applied, and before the beam is fully loaded, large tension stresses could result in the upper fibers which would cause cracking of the concrete. Such cracks are generally thought to be undesirable, even though they would probably close under full load. It is, therefore, necessary to check stresses at the stage when the prestress is applied.

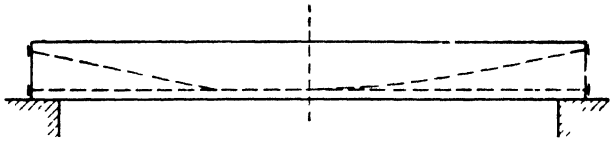


FIG. 12.14 Posttensioned beam with part of cables bent up (straight bent on left, parabolic bent on right).

When the wires are tensioned, they obviously produce a shortening of the bottom fibers and either a stretching or a smaller shortening of the top fibers. This means that in the process of tensioning the beam deflects upward and lifts off its bed. The stresses at mid-span are, therefore, never caused by the prestress force alone, but by that force and the simultaneously acting moment due to the weight of the beam. This upward deflection occurs both in pretensioned and in posttensioned beams. In the former, however, the design is not affected by this behavior. Since all wires are straight, the stress distribution due to prestress and dead weight need be checked only at the most unfavorable location, *i.e.*, at the support. For posttensioned beams the location of the reinforcement and the corresponding eccentricity $e$ at the center are different from those at the ends. Hence, the stresses due to prestress alone must be checked at both locations, using the pertinent $e$ in each case. In this case it is economical to take advantage of the counteracting moment at mid-span caused by the weight of the upward deflected beam.

In order to prevent cracking at any stage, in addition to Eqs. (12.31) to (12.34) the following conditions must be satisfied in the center portion of the beam:

$$f_{t1} = \frac{P_i}{A}\left(1 - \frac{e}{k_1}\right) + \frac{M_w c_1}{I} \geqq f_{tt} \tag{12.38}$$

$$f_{t2} = \frac{P_i}{A}\left(1 + \frac{e}{k_2}\right) - \frac{M_w c_2}{I} \leqq f_{tc} \tag{12.39}$$

In these equations, the subscript $i$ stands for "initial," while $w$ stands for the weight of the beam. That is, at the moment when full prestress $P_i$ has been reached and before losses from shrinkage and creep occur, the temporary initial fiber stresses $f_i$ are caused by the full initial prestress $P_i$ and the moment $M_w$ due to the weight of the beam. The permissible fiber stresses at that stage ($f_{ti}$ and $f_{ic}$) are often allowed to be somewhat larger numerically than those under full load ($f_t$ and $f_c$), since they are of temporary nature and refer to a stage where significant overloading is hardly possible. For this reason a satisfactory design is obtained in most cases if, in Eqs. (12.38) and (12.39), $P$, $f_t$, and $f_c$ are temporarily used instead of $P_i$, $f_{ti}$ and $f_{ic}$. After the design is completed, the actual stresses at the instant of full initial prestress are then checked by Eqs. (12.38) and (12.39), as written.

It is seen from Eqs. (12.38) and (12.39) that the stresses caused by $M_w$ relieve those due to the prestress $P$. It is, therefore, necessary to check these stresses at the most unfavorable location, which is that section where the prestress has its full value (all cables at bottom) while $M_u$ is a minimum. For cables bent up straight (left part of Fig. 12.14), this section is at the point of bend, while for curved cables (right part of Fig. 12.14) it is at mid-span.

If curved cables are used and if, as in Art. 12.8, $f_t = 0$ and, for symmetrical sections, $k_1 = k_2 = k$, from Eq. (12.38),

$$\frac{P}{A}\left(1 - \frac{e}{k}\right) = -\frac{M_w c}{I}$$

If this value is substituted in Eq. (12.31), one obtains

$$\frac{Mc}{I} - \frac{M_w c}{I} = f_c$$

or

$$S = \frac{I}{c} = \frac{M - M_w}{f_c} \tag{12.40}$$

Comparison with Eq. (12.36) shows one of the advantages of posttensioning: the required section modulus, instead of being governed by the full dead-load plus live-load moment $M$, depends merely on the smaller moment $(M - M_w)$. It can be said that the weight of the beam is completely carried by the prestress.

Once the cross-sectional dimensions are found from Eq. (12.40), reinforcement in the center portion of the beam is placed as closely to the bottom face as insulation will permit. The eccentricity $e$ is thereby determined. Then, from Eq. (12.32), with $f_t = 0$,

$$\frac{P}{A}\left(1 + \frac{e}{k}\right) = \frac{Mc}{I}$$

If $k = I/cA$ is substituted in this expression, it yields after some elementary transformation

$$P = \frac{M}{k + e} \tag{12.41}$$

This determines the value of the required prestress, after losses.

The losses in posttensioning are somewhat smaller than in pretensioning. This is so because part of the shrinkage has taken place before, and part of the creep during tension, while all the elastic compression of the concrete caused by $P$ occurs during rather than after the tensioning process. These deformations, therefore, do not cause a decrease in $P_i$. Consequently, for posttensioning, the required initial prestress force $P_i = P/0.88$ to $P/0.85$.

*Illustrative Example.* The same slab bridge as in Art. 12.8 is to be designed for posttensioning. The permissible stresses, at the instant of full initial tensioning, will be assumed as $f_{ic} = 0.40 \times 6000 = 2400$ psi and $f_{it} = -0.1 f_{ic} = -240$ psi.

Since the total load minus the weight of the slab is $700 - 300 = 400$ psf, the moment $M - M_w = 2{,}410{,}000(400/700) = 1{,}376{,}000$ in.-lb. From Eq. (12.40), with $S = bh^2/6$,

$$h = \sqrt{\frac{6 \times 1{,}376{,}000}{12 \times 2{,}100}} = 18.2 \text{ in., say, } 19 \text{ in.}$$

This results in a slab weight of $(19/12)150 = 248$ psf, or a total load of 648, instead of 700 psf, as was obtained with pretensioning. Hence $M = 2{,}410{,}000(648/700) = 2{,}230{,}000$ in.-lb. Assuming the center of the cables to be 2.5 in. from the bottom, $e = 9.5 - 2.5 = 7.0$ in., while $k = {}^{19}/_{6} = 3.17$ in. From Eq. (12.41) the required prestress force becomes

$$P = \frac{2{,}230{,}000}{3.17 + 7.0} = 219{,}000 \text{ lb}$$

and the required steel area is

$$A_s = \frac{219{,}000}{0.85 \times 135{,}000} = 1.91 \text{ sq in.}$$

This is only about two-thirds of the steel area previously determined for pretensioning.

The required number of 0.276-in. diameter wires is $1.91/0.0598 = 32$; three cables of 11 wires each will be used per foot of width of the slab.

It is now necessary to check $f_{it}$ and $f_{ic}$ at the instant of full prestress, from Eqs. (12.38) and (12.39), using $P_i = 219{,}000/0.85 = 258{,}000$ lb, and $M_w = 2{,}230{,}000 - 1{,}376{,}000 = 854{,}000$ in.-lb. If these values are substituted in the quoted equations, it is found that $f_{i1} = -190$ psi (ten-

sion) and $f_{t2} = +2445$ psi. The former is smaller than the temporarily permissible tension stress ($f_{tt} = -240$ psi) while the latter exceeds $f_{tc} = 2400$ psi by a mere 2 per cent, a negligible discrepancy.

At the supports no moment $M_w$ is available to counteract the stresses caused by $P_i$. If all cables were carried straight to the supports, large tension stresses and cracking would result at the top face in view of the large eccentricity For this reason every third cable will be bent up as shown in Fig. 12.14, their location at the supports being given in Fig. 12.15. The eccentricity $e$ of $P_i$ at the supports is then the distance from the centroid of all cables to the axis of the beam. It is easily verified that this arrangement results in $e < k$ so that no tension is caused in the top fibers. The compression $f_2$, if checked by Eq. (12.28), is found to be below the permissible value.

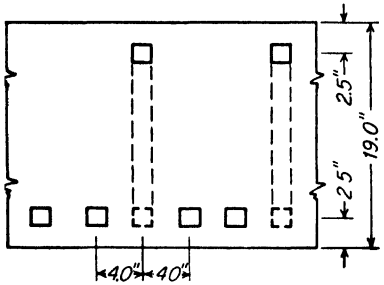


FIG. 12.15

The methods of determining the moment $M_{cr}$ which produces first cracking and the ultimate moment $M_{ult}$ are the same as for pretensioned beams (Art. 12.8).

**12.10. Additional Information References.** The discussion in the preceding articles is limited to the basic features of prestressed concrete design presented in their simplest form. A number of points of detail have been omitted, such as the following: determination of diagonal tension stresses in beams with bent cables; design of continuous, prestressed beams; partial versus full prestressing; specific information on bond, creep, and shrinkage; and almost all reference to details of erection and production procedures, mostly European, as well as to design of prestressed pipes and tanks, long and successfully used in this country. Except for the last-mentioned field it is certain that all the enumerated features will undergo considerable, independent development as prestressed concrete becomes more firmly established in the United States, while the basic principles of Arts. 12.6 to 12.9 are likely to remain substantially unchanged.

The following references may be consulted on the details of the present status of prestressed concrete design and technique:

1. "Principles and Practice of Prestressed Concrete," P. W. Abeles, Fr. Ungar Publ. Co., New York, 1949.
2. "Prestressed Concrete," G. Magnel, 2d ed., Concr. Publ. Ltd., London, 1950.
3. "Design of Prestressed Concrete," *Concrete Information* St74, Portland Cement Association.

4. "Typical Specifications for Prestressed Concrete Tanks," *Concrete Information* St58, Portland Cement Association.
5. "Prestressed Circular Concrete Tanks," J. L. Mason, Portland Cement Association, R/C No. 5.
6. "Design of Prestressed Tanks," J. M. Crom, *Proc. ASCE*, vol. 76, no. 37, September, 1950.
7. "Prestressed Concrete: Principles and Applications," E. Freyssinet, *J. Inst. Civ. Engrs. (London)*, vol. 33, p. 331, 1949.
8. The following articles in the *Journal of the American Concrete Institute:*

   *a.* "Prestressed Concrete Construction Procedures," T. Germundsson, vol. 21, p. 857, 1950.
   *b.* "Patents and Codes Relating to Prestressed Concrete," C. Dobell, vol. 21, p. 713, 1950.
   *c.* "Factors in Prestressed Girder Design" (Walnut Lane Bridge), M. Fornerod, vol. 22, p. 469, 1951.
   *d.* "Designing for Continuity in Prestressed Concrete Structures," A. L. Parme and G. H. Paris, vol. 23, p. 15, 1951.
   *e.* "Prototype Prestressed Beam Justifies Walnut Lane Bridge Design" (test results), G. Magnel, vol. 22, p. 301, 1950.
   *f.* A symposium of several papers on American developments in prestressed concrete in vol. 24, pp. 401–496, 1953.

Much scattered information has been gathered conveniently in a book by A. E. Komendant, "Prestressed Concrete Structures," McGraw-Hill Book Company, Inc., New York, 1952. However, even this book is already somewhat out of date, particularly in regard to recent American developments. The reader interested in this rapidly developing field should look for information in the latest periodical literature.

# APPENDIX A

# STANDARD NOTATION

The principal symbols used in the several sections have been collected here for convenience of reference. For additional notation for flat slabs, see Appendix C.

$A_c$ = area of core of a spirally reinforced column measured to the outside diameter of the spiral

$A_g$ = the over-all or gross area of spirally reinforced or tied columns

$A_s$ = effective cross-sectional area of metal reinforcement in tension in beams, and of longitudinal bars in columns

$A'_s$ = effective cross-sectional area of metal reinforcement in compression in beams

$A_v$ = total area of web reinforcement in tension within a distance $s$ (measured in a direction parallel to that of the main reinforcement), or the total area of all bars bent up in any one plane

$a$ = over-all depth of beam

$\alpha$ = angle between inclined web bars and axis of beam

$b$ = width of rectangular flexural member or width of flange in T and I sections

$b'$ = width of web in T and I flexural members

$C$ = total compressive stress in concrete; ratio of allowable concrete stress $f_a$ in axially loaded column to allowable fiber stress for concrete in flexure

$C'$ = total compressive stress in reinforcement

$D = l^2/2R^2$, = a factor, usually varying from 3 to 9. (The term R as used here is the radius of gyration of the whole column section.)

$d$ = depth from compression face of beam or slab to centroid of longitudinal tensile reinforcement; the least lateral dimension of a concrete column

$d'$ = depth from compression face of beam or slab to centroid of compression reinforcement

$E_c$ = modulus of elasticity of concrete in compression

$E_s$ = modulus of elasticity of steel in tension

$e$ = eccentricity of the resultant load on a column, measured from the gravity axis

$f_a$ = average allowable stress in the concrete of an axially loaded reinforced concrete column

$f_c$ = allowable fiber stress for concrete in flexure; computed concrete fiber stress in an eccentrically loaded column

$f'_c$ = compressive strength of concrete at age of 28 days, unless otherwise specified

$f_p$ = maximum allowable concrete fiber stress in an eccentrically loaded column

$f_s$ = tensile unit stress in longitudinal reinforcement in beams; nominal allowable stress in vertical column reinforcement

$f'_s$ = compressive unit stress in longitudinal reinforcement in beams

$f_v$ = tensile unit stress in web reinforcement

$h$ = unsupported length of column

$I$ = moment of inertia of a section about the neutral axis for bending

$i$ = nominal diameter of reinforcing bar

$j$ = ratio of lever arm of resisting couple to depth $d$

$jd = d - z$ = lever arm of resisting couple

$K = \frac{1}{2} f_c kj$ or $pf_s j$ in rectangular beams: the ratio $I/l$ or $I/h$ for beams and columns, respectively

$l$ = span length of beam or slab center to center of supports

$l'$ = clear span for positive moment and shear and the average of the two adjacent clear spans for negative moment

$\lambda_c$ = unit strain in concrete, in. per in.

$\lambda_s$ = unit strain in metal reinforcement, in. per in.

$N$ = axial load applied to reinforced concrete column

$n = \frac{E_s}{E_c} = \frac{30{,}000}{f'_c}$ = ratio of modulus of elasticity of steel to that of concrete

$\Sigma_0$ = sum of perimeters of bars in one set

$P$ = total allowable axial load on a column whose length does not exceed ten times its least cross-sectional dimension

$P'$ = total allowable axial load on a long column

$p$ = ratio of cross-sectional area of tensile reinforcement to effective compressive area of concrete in beams

$p'$ = ratio of cross-sectional area of compressive reinforcement to effective compressive area of concrete in beams: ratio of volume of spiral reinforcement to the volume of the concrete core (out to out of spirals) of a spirally reinforced concrete column

$p_g$ = ratio of the effective cross-sectional area of vertical reinforcement to the gross area $A_g$

$R$ = least radius of gyration of a section

$s$ = spacing of web reinforcement, measured in the direction of the longitudinal axis of the beam

$T$ = total tensile stress in longitudinal reinforcement

$t$ = thickness of flange of T beam: over-all depth of column section

$t_1$ = minimum total thickness of slab

$u$ = bond stress per unit of surface area of bar

$V$ = total shear

$V_c$ = total shear resisted by the concrete

$V' = V - V_c$

$v$ = shearing unit stress

$v_c$ = unit shear resisted by the concrete

$v' = v - v_c$

$w$ = uniformly distributed load per unit of length of beam or slab

$z$ = depth from compressive face of beam or slab to the resultant of the compressive stresses

# APPENDIX B

# ALLOWABLE UNIT STRESSES

## Allowable Unit Stresses in Concrete (ACI Code)

| Description | | Allowable unit stresses | | | | | |
|---|---|---|---|---|---|---|---|
| | | For any strength of concrete, Section 302, $n = \frac{30{,}000}{f'_c}$ | Maximum value, psi | For strength of concrete shown below | | | |
| | | | | $f'_c$ = 2000 psi $n$ = 15 | $f'_c$ = 2500 psi $n$ = 12 | $f'$ = 3000 psi $n$ = 10 | $f'_c$ = 3750 psi $n$ = 8 |
| Flexure* | | | | | | | |
| Extreme fiber stress in compression | $f_c$ | $0.45f'_c$ | | 900 | 1125 | 1350 | 1688 |
| Extreme fiber stress in tension in plain concrete footings | $f_c$ | $0.03f'_c$ | | 60 | 75 | 90 | 113 |
| Shear:* | | | | | | | |
| Beams with no web reinforcement | $v_c$ | $0.03f'_c$ | | 60 | 75 | 90 | 113 |
| Beams with properly designed web reinforcement | $v$ | $0.12f'_c$ | | 240 | 300 | 360 | 450 |
| Flat slabs at a distance $d$ from edge of column capital or drop panel | $v_c$ | $0.03f'_c$ | | 60 | 75 | 90 | 113 |
| Footings | $v_c$ | $0.03f'_c$ | 75 | 60 | 75 | 75 | 75 |
| Bond: | | | | | | | |
| Deformed bars | | | | | | | |
| Top bars† | $u$ | $0.07f'_c$ | 245 | 140 | 175 | 210 | 245 |
| In two-way footings (except top bars) | $u$ | $0.08f'_c$ | 280 | 160 | 200 | 240 | 280 |
| All others | $u$ | $0.10f'_c$ | 350 | 200 | 250 | 300 | 350 |
| Plain bars (must be hooked) | | | | | | | |
| Top bars | $u$ | $0.03f'_c$ | 105 | 60 | 75 | 90 | 105 |
| In two-way footings (except top bars) | $u$ | $0.036f'_c$ | 126 | 72 | 90 | 108 | 126 |
| All others | $u$ | $0.045f'_c$ | 158 | 90 | 113 | 135 | 158 |
| Bearing: | | | | | | | |
| On full area | $f_c$ | $0.25f'_c$ | | 500 | 625 | 750 | 938 |
| On one-third area or less‡ | $f_c$ | $0.375f'_c$ | | 750 | 938 | 1125 | 1405 |

* $V$ as a measure of diagonal tension.

† Top bars are horizontal bars so placed that more than 12 in. of concrete is cast in the member below the bar.

‡ The allowable bearing stress on an area greater than one-third but less than the full area shall be interpolated between the values given.

## Allowable Unit Stresses in Reinforcing Steel (ACI Code)

Tensile unit stress in longitudinal and web reinforcement:

20,000 psi for bars of intermediate and hard grade

18,000 psi for bars of structural grade

Tensile unit stress in main reinforcement in one-way slabs of not more than 12-ft span for bars $\frac{3}{8}$ in. or less in diameter:

50 per cent of the minimum yield point specified in the Standard Specifications of the American Society for Testing Materials, but in no case to exceed 30,000 psi

Compressive unit stress in vertical column reinforcement:

40 per cent of the minimum yield specified in the Standard Specifications of the American Society for Testing Materials, but in no case to exceed 30,000 psi.

# APPENDIX C

# FLAT SLABS WITH SQUARE OR RECTANGULAR PANELS

This appendix is a part of the report of the American Concrete Institute's Committee 318 on Building Code Requirements for Reinforced Concrete, adopted by the American Concrete Institute in 1951. Only those portions of the report which deal specifically with flat-slab design and construction are included. The notations used in the Code are defined as follows:

$A$ = The distance from the center line of the column, in the direction of any span, to the intersection of a 45-degree diagonal line from the center of the column to the bottom of the flat slab or drop panel, where such line lies wholly within the column, capital, or bracket, provided such capital or bracket is structurally capable of resisting shears and moments without excessive unit stress. In no case shall $A$ be greater than one-eighth the span in the direction considered.

$A_{av}$ = Average of the two values of $A$ for the two columns at the ends of a column strip, in the direction of the spans considered.

$C$ = Diameter or width of column capital at the under side of the slab or drop panel. No portion of the column capital shall be considered for structural purposes which lies outside the largest right circular cone, with 90 degrees vertex angle, that can be included within the outlines of the column capital.

$L$ = Span length of slab center to center of columns in the direction of which bending is considered.

$M_0$ = Sum of the positive and the average negative bending moments at the critical design sections of a flat slab panel. See Section 3(*b*).

$t_2$ = Thickness of flat slab without drop panels, or the thickness of flat slab through the drop panels where such are used.

$t_3$ = Thickness of flat slab (with drop panels) at points outside the drop panel.

$W$ = Total dead and live load uniformly distributed over a single panel area.

$W_{av}$ = The average of the total load on two adjacent panels.

$x$ = Coefficient of span $L$ which gives the distance from the center of column to the critical section for negative bending in design according to Section 2(*a*).

### 1. Scope

(*a*) The term flat slab shall mean a reinforced concrete slab supported by columns with or without flaring heads or column capitals, with or without depressed or drop panels and generally without beams or girders.

(*b*) Recesses or pockets in flat slab ceilings, located between reinforcing bars and forming cellular or two-way ribbed ceilings, whether left open or filled with permanent fillers, shall not prevent a slab from being considered a flat slab; but allowable unit stresses shall not be exceeded.

(*c*) This chapter provides for two methods of design of flat slab structures.

1. Any type of flat slab construction may be designed by application of the principles of continuity, using the method outlined in Section 2, or using other recognized methods of elastic analysis. In either case, the design must be subject to the provisions of Sections 5(*a*) and (*c*), 6, 8 and 9.
2. The common cases of flat slab construction described in Section 3 may be designed by the use of moment coefficients, given in Sections 3 and 4, and subject to the provisions of Sections 5, 6, 7, 8 and 9.

**2. Design of Flat Slabs as Continuous Frames**

(*a*) Except in the cases of flat slab construction where specified coefficients for bending may be used, as provided in Section 3, bending and shear in flat slabs and their supports shall be determined by an analysis of the structure as a continuous frame, and all sections shall be proportioned to resist the moments and shears thus obtained. In the analysis, the following assumptions may be made:

1. The structure may be considered divided into a number of bents, each consisting of a row of columns and strips of supported slabs, each strip bounded laterally by the center line of the panel on either side of the row of columns. The bents shall be taken longitudinally and transversely of the building.
2. Each such bent may be analyzed in its entirety; or each floor thereof and the roof may be analyzed separately with its adjacent columns above and below, the columns being assumed fixed at their remote ends. Where slabs are thus analyzed separately, in bents more than four panels long, it may be assumed in determining the bending at a given support that the slab is fixed at any support two panels distant therefrom beyond which the slab continues.
3. The joints between columns and slabs may be considered rigid and this rigidity may be assumed to extend in the slabs a distance $A$ from the center of the columns, and in the column to the intersection of the sides of the columns, and in the column to the intersection of the sides of the column and the 45 degree line defining $A$. The change in length of columns and slabs due to direct stress, and deflections due to shear, may be neglected. Where metal column capitals are used, account may be taken of their contributions to stiffness and resistance to bending and shear.
4. The supporting columns may be assumed free from settlement or lateral movement unless the amount thereof can be reasonably determined.
5. The moment of inertia of slab or column at any cross-section may be assumed to be that of the gross section of the concrete. Variation in the moments of inertia of the slabs and columns along their axes shall be taken into account.
6. Where the load to be supported is definitely known, the structure shall be analyzed for that load. Where the live load is variable but does not exceed three-quarters of the dead load, or the nature of the live load is such that all panels will be loaded simultaneously, the maximum bending may be assumed to obtain at all sections under full live load. Elsewhere, maximum positive bending near mid-span of a panel may be assumed to obtain under full live load in the panel and in alternate panels; and maximum negative bending at a support may be assumed to obtain under full live load in the adjacent panels only.
7. Where neither beams nor girders help to transfer the slab load to the supporting column, the critical section for negative bending may be assumed as not more than the distance $xL$ from the column center, where

$$x = 0.073 + 0.57\frac{A}{L} \tag{17}$$

In slabs supported by beams, girders, or walls, the critical section for negative bending shall be assumed at the face of such support.

8. The numerical sum of the maximum positive and the average maximum negative bending moments for which provision is made in the design in the direction of either side of a rectangular panel shall be assumed as not less than

$$\frac{1}{10} W_{av} L \left(1 - \frac{4A_{av}}{3L}\right)^2 \tag{18}$$

9. The bending at critical sections across the slabs of each bent may be apportioned between the column strip and middle strip, as defined in Section 5, in the ratio of the specified coefficients which affect such apportionment in the special cases of flat slabs provided in Section 3.
10. The maximum bending in columns may be assumed to obtain under full live load in alternate panels. Columns shall be proportioned to resist the maximum bending combined with the maximum direct load consistent therewith; and for maximum direct load combined with the bending under full load, the direct load subject to allowable reductions. In computing moments in columns at any floor, the far ends of the columns may be considered fixed.

(*b*) The foregoing provisions outline the method to be followed in analyzing and designing flat slabs in the general case. In all instances the design must conform to the requirements for panel strips and critical design sections, slab thickness and drop panels, capitals and brackets, arrangement of reinforcement and openings in flat slabs, as provided in Sections 5(*a*) and (*c*), 6, 8 and 9.

### 3. Design of Flat Slabs by Moment Coefficients

(*a*) In those cases of flat slab construction which fall within the following limitations as to continuity and dimensions, the bending moments at critical sections may be determined by the use of specified coefficients as provided in Section 4.

1. The ratio of length to width of panel does not exceed 1.33.
2. The slab is continuous for at least three panels in each direction.
3. The successive lengths in each direction differ by not more than twenty per cent of the shorter span.

(*b*) In such slabs, the numerical sum of the positive and negative bending moments in the direction of either side of an interior rectangular panel shall be assumed as not less than

$$M_0 = 0.09WL \left(1 - \frac{2c}{3L}\right)^2 \tag{19}$$

(*c*) Three-fourths of the width of the strip shall be taken as the width of the section in computing compression due to bending, except that on a section through a drop panel, three-fourths of the width of the drop panel shall be taken. Account shall be taken of any recesses which reduce the compressive area. Tension reinforcement distributed over the entire strip shall be included in the computations.

(*d*) The design of slabs under the procedure given in this section is subject to the provisions of all subsequent sections of this chapter (Sections 4 to 9).

### 4. Bending-moment Coefficients

(*a*) The bending moments at the critical sections of the middle and column strips of an interior panel shall be assumed as given in Table 4(*a*).

(*b*) The bending moments at critical sections of strips, in an exterior panel, at right angles to the discontinuous edge, where the exterior supports consist of reinforced concrete columns or reinforced concrete bearing walls integral with the slab, the ratio of stiffness of the support to that of the slab being at least as great as the ratio of the live load to the dead load and not less than one, shall be assumed as given in Table 4(*b*). Where a flat slab is so supported by a wall providing restraint at the discontinuous edge, the coefficient for negative bending at the edge shall be assumed more nearly equal in the column and middle strips, the sum remaining as given in Table 4(*b*), but that for the column strip shall not be less than $0.30\ M_0$. Bending in middle strips parallel to a discontinuous edge, except in a corner panel, shall be assumed the same as in an interior panel. $M_0$ shall be determined as provided in Section 3(*b*) for an interior panel.

**TABLE 4(*a*). Bending Moments in Interior Flat Slab Panel**

| | | |
|---|---|---|
| With drop panel: | | |
| Column strip | Negative moment | $0.50M_0$ |
| Column strip | Positive moment | $0.20M_0$ |
| Middle strip | Negative moment | $0.15M_0$ |
| Middle strip | Positive moment | $0.15M_0$ |
| Without drop panel: | | |
| Column strip | Negative moment | $0.46M_0$ |
| Column strip | Positive moment | $0.22M_0$ |
| Middle strip | Negative moment | $0.16M_0$ |
| Middle strip | Positive moment | $0\ 16M_0$ |

**TABLE 4(*b*). Bending Moments in Exterior Flat Slab Panel**

| | | |
|---|---|---|
| With drop panel: | | |
| Column strip | Exterior negative | $0.45M_0$ |
| Column strip | Positive moment | $0.25M$ |
| Column strip | Interior negative | $0\ 55M_0$ |
| Middle strip | Exterior negative | $0\ 10M_0$ |
| Middle strip | Positive moment | $0.19M_0$ |
| Middle strip | Interior negative | $0.165M_0$ |
| Without drop panel: | | |
| Column strip | Exterior negative | $0.41M_0$ |
| Column strip | Positive moment | $0.28M_0$ |
| Column strip | Interior negative | $0\ 50M_0$ |
| Middle strip | Exterior negative | $0.10M_0$ |
| Middle strip | Positive moment | $0.20M_0$ |
| Middle strip | Interior negative | $0.176M_0$ |

(*c*) The bending moments at critical sections of strips, in an exterior panel, at right angles to the discontinuous edge, where the exterior supports are masonry bearing walls or other construction which provide only negligible restraint to the slab, shall be assumed as given in Table 4(*b*) with the following modifications.

1. On critical sections at the face of the exterior support, negative bending in each strip shall be assumed as $0.05M_0$.
2. The coefficients for positive bending shall be increased by forty per cent.
3. The coefficients for negative bending at the first interior columns shall be increased thirty per cent.

**TABLE 4(c). Bending Moments in Panels with Marginal Beams or Walls**

| | | Marginal beams with depth greater than 1½ times the slab thickness; or bearing wall | | Marginal beams with depth 1½ times the slab thickness or less | |
|---|---|---|---|---|---|
| (a) Load to be carried by marginal beam or wall | | Loads directly superimposed upon it plus a uniform load equal to one-quarter of the total live and dead panel load | | Loads directly superimposed upon it exclusive of any panel load | |
| | | With drop | Without drop | With drop | Without drop |
| (b) Moment to be used in the design of half column strip adjacent and parallel to marginal beam or wall | Neg. | $0.125M_0$ | $0.115M_0$ | $0.25M_0$ | $0.23M_0$ |
| | Pos. | $0.05M_0$ | $0.055M_0$ | $0.10M_0$ | $0.11M_0$ |
| (c) Negative moment to be used in design of middle strip continuous across a beam or wall | Neg. | $0.195M_0$ | $0.208M_0$ | $0.15M_0$ | $0.16M_0$ |

(*d*) The bending moments in panels with marginal beams or walls, in the strips parallel and close thereto, and in the beams, shall be determined upon the basis of assumptions presented in Table 4(*c*).

(*e*) For design purposes any of the moment coefficients of Tables 4(*a*), 4(*b*), and 4(*c*) may be varied by not more than six per cent, but the numerical sum of the positive and negative moments in a panel shall not be taken as less than the amount specified.

(*f*) Panels supported by marginal beams on opposite edges shall be designed as solid one or two-way slabs to carry the entire panel load.

(*g*) The ratio of reinforcement in any strip shall not be less than 0.0025.

#### 5. Panel Strips and Critical Design Sections

(*a*) A flat slab panel shall be considered as consisting of strips in each direction as follows:

A middle strip one half panel in width, symmetrical about panel center line and extending through the panel in the direction of the span for bending.

A column strip consisting of the two adjacent quarter-panels either side of the column center lines.

(*b*) The critical sections for bending are located as follows:

Sections for negative bending shall be taken along the edges of the panel, on column center lines between capitals and around the perimeters of column capitals.

Sections for positive bending shall be taken at mid-span of the strips.

(*c*) Only the reinforcement which crosses a critical section within a strip may be considered effective to resist bending in the strip at that section. Reinforcement which crosses such section at an angle with the center line of the strip shall be assumed to contribute to the resistance of bending only its effective area in the direction of the strip, as defined in Section 4.

**6. Slab Thickness and Drop Panels**

(*a*) The thickness of a flat slab and the size and thickness of the drop panel, where used, shall be such that the compressive stress due to bending at the critical sections of any strip and the shear about the column capital and the drop panel shall not exceed the unit stresses allowed in concrete of the quality used.

(*b*) The shearing unit stress on a vertical section which lies at a distance $t_2 - 1\frac{1}{2}$ in. beyond the edge of the column capital and parallel or concentric with it, shall not exceed the following values with $d$ taken as $t_2 - 1\frac{1}{2}$ in.:

1. $0.03f'_c$, when at least 50 per cent of the total negative reinforcement in the column strip passes directly over the column capital.
2. $0.025f'_c$, when 25 per cent or less of the total negative reinforcement in the column strip passes directly over the column capital.
3. For intermediate percentages, intermediate values of the shearing unit stress shall be used.

(*c*) The shearing unit stress on vertical sections which lie at a distance $t_3 - 1\frac{1}{2}$ in. beyond the edge of the drop panel and parallel with them shall not exceed $0.03f'_c$ with $d$ taken as $t_3 - 1\frac{1}{2}$ in. At least 50 per cent of the cross-sectional area of the negative reinforcement in the column strip must be within the width of strip directly above the drop panel.

(*d*) Slab thickness shall not, however, be less than $L/40$ with drop panels or $L/36$ without drop panels.

(*e*) The thickness of the drop panel below the slab shall not be more than one-fourth the distance from the edge of the column capital to the edge of the drop panel.

**7. Capitals and Brackets**

(*a*) Where a column is without a flaring concrete capital the distance $c$ shall be taken as the diameter of the column. Structural metal embedded in the slab or drop panel may be regarded as contributing to resistance in bending and shear.

(*b*) Where a reinforced concrete beam frames into a column without capital or bracket on the same side with the beam, the value of $c$ may be taken as the width of the column plus twice the projection of the beam above or below the slab or drop panel for computing bending in strips parallel to the beam.

(*c*) Brackets capable of transmitting the negative bending and the shear in the column strips to the columns without excessive unit stress may be substituted for column capitals at exterior columns. The value of $c$ where brackets are used shall be taken as twice the distance from the center of the column to a point where the bracket is $1\frac{1}{2}$ inches thick, but not more than the thickness of the column plus twice the depth of the bracket.

(*d*) The average of the diameters $c$ of the column capitals at the four corners of a panel shall be used in determining the bending in the middle strips of the panel. The average of the diameters $c$ of the two column capitals at the ends of the column strip shall be used in determining bending in the strip.

**8. Arrangement of Reinforcement**

(*a*) Slab reinforcement shall be provided to resist the bending and bond stresses not only at critical sections, but also at intermediate sections.

(*b*) Bars shall be spaced evenly across strips or bands and the spacing shall not exceed three times the slab thickness.

(*c*) In exterior panels the reinforcement perpendicular to the discontinuous edge for positive bending, shall extend to the edge and have embedment of at least six

inches in spandrel beams, walls or columns. All such reinforcement for negative bending shall be bent, hooked or otherwise anchored in spandrel beams, walls or columns.

### 9. Openings in Flat Slabs

Openings of any size may be cut through a flat slab if provision is made for the total positive and negative resisting moments, as required in Sections 2 or 3, without exceeding the allowable stresses.

# APPENDIX D

# TABLES AND DIAGRAMS

**TABLE 1. Designations, Areas, Perimeters, and Weights of Standard Bars**

| Bar designation* | Diameter, in. | Cross-sectional area, sq in. | Perimeter, in. | Unit wt per ft, lb |
|---|---|---|---|---|
| No. 2 | 1/4 = 0.250 | 0.05 | 0.79 | 0.167 |
| No. 3 | 3/8 = 0.375 | 0.11 | 1.18 | 0.376 |
| No. 4 | 1/2 = 0.500 | 0.20 | 1.57 | 0.668 |
| No. 5 | 5/8 = 0.625 | 0.31 | 1.96 | 1.043 |
| No. 6 | 3/4 = 0.750 | 0.44 | 2.36 | 1.502 |
| No. 7 | 7/8 = 0.875 | 0.60 | 2.75 | 2.044 |
| No. 8 | 1 = 1.000 | 0.79 | 3.14 | 2.670 |
| No. 9 | 1 1/8† = 1.128 | 1.00 | 3.54 | 3.400 |
| No. 10 | 1 1/4† = 1.270 | 1.27 | 3.99 | 4.303 |
| No. 11 | 1 3/8† = 1.410 | 1.56 | 4.43 | 5.313 |

* Based on the number of eighths of an inch included in the nominal diameter of the bars. The nominal diameter of a deformed bar is equivalent to the diameter of a plain bar having the same weight per foot as the deformed bar. Bar No. 2 in plain rounds only. All others in deformed rounds.

† Approximate to the nearest 1/8 in.

**TABLE 2. Areas of Groups of Standard Bars, Square Inches**

| Bar designation | Number of bars | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| No. 4 | 0.39 | 0.58 | 0.78 | 0.98 | 1.18 | 1.37 | 1.57 | 1.77 | 1 96 | 2.16 | 2.36 | 2.55 | 2.75 |
| No. 5 | 0.61 | 0.91 | 1.23 | 1.53 | 1.84 | 2.15 | 2.45 | 2.76 | 3.07 | 3.37 | 3.68 | 3.99 | 4.30 |
| No. 6 | 0.88 | 1.32 | 1.77 | 2.21 | 2.65 | 3.09 | 3.53 | 3.98 | 4.42 | 4.86 | 5.30 | 5.74 | 6.19 |
| No. 7 | 1.20 | 1.80 | 2.41 | 3.01 | 3.61 | 4.21 | 4.81 | 5.41 | 6.01 | 6.61 | 7.22 | 7.82 | 8.42 |
| No. 8 | 1.57 | 2.35 | 3 14 | 3.93 | 4.71 | 5.50 | 6.28 | 7.07 | 7.85 | 8.64 | 9.43 | 10.21 | 11.00 |
| No. 9 | 2.00 | 3.00 | 4.00 | 5.00 | 6.00 | 7.00 | 8.00 | 9.00 | 10.00 | 11.00 | 12.00 | 13.00 | 14.00 |
| No. 10 | 2.53 | 3.79 | 5.06 | 6.33 | 7.59 | 8.86 | 10.12 | 11.39 | 12.66 | 13.92 | 15.19 | 16.45 | 17.72 |
| No. 11 | 3.12 | 4.68 | 6.25 | 7.81 | 9.37 | 10.94 | 12.50 | 14.06 | 15.62 | 17.19 | 18.75 | 20.31 | 21.87 |

## TABLE 3. Perimeters of Groups of Standard Bars, Inches

| Bar designation | Number of bars | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| No. 4 | 3.1 | 4.7 | 6.2 | 7.8 | 9.4 | 11.0 | 12.6 | 14.1 | 15.7 | 17.3 | 18.8 | 20.4 | 22.0 |
| No. 5 | 3.9 | 5.9 | 7.8 | 9.8 | 11.8 | 13.7 | 15.7 | 17.7 | 19.5 | 21.6 | 23.6 | 25.5 | 27.5 |
| No. 6 | 4.7 | 7.1 | 9.4 | 11.8 | 14.1 | 16.5 | 18.8 | 21.2 | 23.6 | 25.9 | 28.3 | 30.6 | 33.0 |
| No. 7 | 5.5 | 8.2 | 11.0 | 12.7 | 16.5 | 19.2 | 22.0 | 24.7 | 27.5 | 30.2 | 33.0 | 35.7 | 38.5 |
| No. 8 | 6.3 | 9.4 | 12.6 | 15.7 | 18.9 | 22.0 | 25.1 | 28.3 | 31.4 | 34.6 | 37.7 | 40.9 | 44.0 |
| No. 9 | 7.1 | 10.6 | 14.2 | 17.7 | 21.3 | 24.8 | 28.4 | 31.9 | 35.4 | 39.0 | 42.5 | 46.0 | 49.6 |
| No. 10 | 8.0 | 12.0 | 16.0 | 20.0 | 23.9 | 27 9 | 31.9 | 35.9 | 39.9 | 43.9 | 47.9 | 51.9 | 55.9 |
| No. 11 | 8.9 | 13.3 | 17.7 | 22.2 | 26.6 | 31.0 | 35.4 | 39.9 | 44.3 | 48.7 | 53.2 | 57.6 | 62.0 |

## TABLE 4. Areas of Bars in Slabs, Square Inches Per Foot

| Spacing, in. | Bar designation | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | No. 3 | No. 4 | No. 5 | No. 6 | No. 7 | No. 8 | No. 9 | No. 10 | No. 11 |
| 3 | 0.44 | 0.78 | 1.23 | 1.77 | 2.40 | 3.14 | 4.00 | 5.06 | 6.25 |
| 3½ | 0.38 | 0.67 | 1.05 | 1.51 | 2.06 | 2.69 | 3.43 | 4.34 | 5.36 |
| 4 | 0.33 | 0.59 | 0.92 | 1.32 | 1.80 | 2.36 | 3.00 | 3.80 | 4.68 |
| 4½ | 0.29 | 0.52 | 0.82 | 1.18 | 1.60 | 2.09 | 2.67 | 3.37 | 4.17 |
| 5 | 0.26 | 0.47 | 0.74 | 1.06 | 1.44 | 1.88 | 2.40 | 3.04 | 3.75 |
| 5½ | 0.24 | 0.43 | 0.67 | 0.96 | 1.31 | 1.71 | 2.18 | 2.76 | 3.41 |
| 6 | 0.22 | 0.39 | 0.61 | 0.88 | 1.20 | 1.57 | 2.00 | 2.53 | 3.12 |
| 6½ | 0.20 | 0.36 | 0.57 | 0.82 | 1.11 | 1.45 | 1.85 | 2.34 | 2.89 |
| 7 | 0.19 | 0.34 | 0.53 | 0.76 | 1.03 | 1.35 | 1.71 | 2.17 | 2.68 |
| 7½ | 0.18 | 0.31 | 0.49 | 0.71 | 0.96 | 1.26 | 1.60 | 2.02 | 2.50 |
| 8 | 0.17 | 0.29 | 0.46 | 0.66 | 0.90 | 1.18 | 1.50 | 1.89 | 2.34 |
| 9 | 0.15 | 0.26 | 0.41 | 0.59 | 0.80 | 1.05 | 1.33 | 1.69 | 2.08 |
| 10 | 0.13 | 0.24 | 0.37 | 0.53 | 0.72 | 0.94 | 1.20 | 1.52 | 1.87 |
| 12 | 0.11 | 0.20 | 0.31 | 0.44 | 0.60 | 0.78 | 1.00 | 1.27 | 1.56 |

## TABLE 5. Design of Rectangular Beams and Slabs

$k = \frac{n}{n+r}$ $\quad j = 1 - \frac{k}{3}$ $\quad p = \frac{n}{2r(n+r)}$ $\quad K = \frac{1f_c}{2} kj$ or $pf_s j$

| $n$ and $f'_c$ | $f_s$ | $f_c$ | $k$ | $j$ | $p$ | $K$ |
|---|---|---|---|---|---|---|
| 8 (3750) | 18,000 | 1250 | 0.357 | 0.881 | 0.0124 | 197 |
| | | 1500 | 0.400 | 0.867 | 0.0167 | 260 |
| | | 1688 | 0.429 | 0.857 | 0.0201 | 310 |
| | 20,000 | 1250 | 0.333 | 0.889 | 0.0104 | 185 |
| | | 1500 | 0.375 | 0.875 | 0.0141 | 246 |
| | | 1688 | 0.403 | 0.866 | 0.0170 | 294 |
| 10 (3000) | 18,000 | 1000 | 0.357 | 0.881 | 0.0099 | 157 |
| | | 1200 | 0.400 | 0.867 | 0.0133 | 208 |
| | | 1350 | 0.428 | 0.857 | 0.0161 | 248 |
| | 20,000 | 1000 | 0.333 | 0.889 | 0.0083 | 148 |
| | | 1200 | 0.375 | 0.875 | 0.0113 | 197 |
| | | 1350 | 0.403 | 0.866 | 0.0136 | 235 |
| 12 (2500) | 16,000 | 833 | 0.385 | 0.872 | 0.0100 | 140 |
| | | 1000 | 0.429 | 0.857 | 0.0134 | 184 |
| | | 1125 | 0.457 | 0.848 | 0.0161 | 218 |
| | 18,000 | 833 | 0.357 | 0.881 | 0.0083 | 131 |
| | | 1000 | 0.400 | 0.867 | 0.0111 | 173 |
| | | 1125 | 0.428 | 0.857 | 0.0134 | 207 |
| | 20,000 | 833 | 0.333 | 0.889 | 0.0069 | 123 |
| | | 1000 | 0.375 | 0.875 | 0.0094 | 164 |
| | | 1125 | 0.403 | 0.866 | 0.0113 | 196 |
| 15 (2000) | 16,000 | 667 | 0.385 | 0.872 | 0.0080 | 112 |
| | | 800 | 0.429 | 0.857 | 0.0107 | 147 |
| | 18,000 | 667 | 0.357 | 0.881 | 0.0066 | 105 |
| | | 800 | 0.400 | 0.867 | 0.0089 | 139 |
| | | 900 | 0.429 | 0.857 | 0.0107 | 165 |
| | 20,000 | 667 | 0.333 | 0.889 | 0.0055 | 99 |
| | | 800 | 0.375 | 0.875 | 0.0075 | 131 |
| | | 900 | 0.403 | 0.866 | 0.0091 | 157 |

## TABLE 6. Review of Rectangular Beams and Slabs

$k = \sqrt{2pn + (pn)^2} - pn$ $\quad j = 1 - \frac{1}{3}k$

| $p$ | $n = 8$ | | $n = 10$ | | $n = 12$ | | $n = 15$ | |
|---|---|---|---|---|---|---|---|---|
| | $k$ | $j$ | $k$ | $j$ | $k$ | $j$ | $k$ | $j$ |
| 0.0010 | 0.119 | 0.960 | 0.132 | 0.956 | 0.145 | 0.952 | 0.158 | 0.947 |
| 0.0020 | 0.164 | 0.945 | 0.181 | 0.940 | 0.196 | 0.935 | 0.217 | 0.928 |
| 0.0030 | 0.196 | 0.935 | 0.217 | 0.928 | 0.235 | 0.922 | 0.258 | 0.914 |
| 0.0040 | 0.223 | 0.926 | 0.246 | 0.918 | 0.266 | 0.911 | 0.292 | 0.903 |
| 0.0050 | 0.246 | 0.918 | 0.270 | 0.910 | 0 291 | 0.903 | 0.320 | 0.893 |
| 0.0054 | 0.254 | 0.915 | 0.279 | 0.907 | 0.300 | 0.900 | 0.329 | 0.891 |
| 0.0058 | 0.262 | 0.913 | 0.287 | 0.904 | 0.309 | 0.897 | 0.337 | 0.888 |
| 0.0062 | 0.269 | 0.910 | 0.296 | 0.901 | 0.317 | 0.894 | 0.348 | 0.884 |
| 0.0066 | 0.276 | 0.908 | 0.304 | 0.899 | 0.325 | 0.892 | 0.356 | 0.881 |
| 0.0070 | 0.283 | 0.906 | 0.311 | 0.896 | 0.334 | 0.889 | 0.365 | 0.878 |
| 0.0072 | 0.286 | 0.905 | 0.314 | 0.895 | 0.338 | 0.887 | 0.369 | 0.877 |
| 0.0074 | 0.290 | 0.903 | 0.318 | 0.894 | 0.342 | 0.886 | 0.372 | 0 876 |
| 0.0076 | 0.293 | 0.902 | 0.321 | 0.893 | 0.345 | 0.885 | 0.376 | 0 875 |
| 0.0078 | 0.297 | 0.901 | 0.325 | 0.892 | 0.349 | 0.884 | 0.380 | 0.873 |
| 0.0080 | 0.300 | 0.900 | 0.328 | 0.891 | 0.353 | 0.882 | 0.384 | 0.872 |
| 0.0082 | 0.303 | 0.899 | 0.332 | 0.889 | 0.356 | 0.881 | 0.387 | 0.871 |
| 0.0084 | 0.306 | 0.898 | 0.336 | 0.888 | 0.360 | 0.880 | 0.390 | 0.870 |
| 0.0086 | 0.309 | 0.897 | 0.338 | 0.887 | 0.363 | 0.879 | 0.394 | 0 869 |
| 0.0088 | 0.312 | 0.896 | 0.341 | 0.886 | 0.366 | 0.878 | 0.398 | 0.867 |
| 0.0090 | 0.314 | 0.895 | 0.344 | 0.885 | 0.370 | 0.877 | 0.402 | 0.866 |
| 0.0092 | 0.317 | 0.894 | 0.347 | 0.884 | 0.373 | 0.876 | 0.405 | 0.865 |
| 0.0094 | 0.320 | 0.893 | 0.350 | 0.883 | 0.376 | 0.875 | 0.407 | 0 864 |
| 0.0096 | 0.322 | 0.893 | 0.353 | 0.882 | 0.379 | 0.874 | 0 411 | 0 863 |
| 0.0098 | 0.325 | 0.892 | 0.356 | 0.881 | 0.381 | 0.873 | 0.414 | 0 862 |
| 0.0100 | 0.328 | 0.891 | 0.358 | 0.881 | 0.385 | 0.872 | 0.418 | 0.861 |
| 0.0104 | 0.333 | 0.889 | 0.363 | 0.879 | 0.391 | 0.870 | 0.423 | 0.859 |
| 0.0108 | 0.339 | 0.887 | 0.369 | 0.877 | 0.396 | 0.868 | 0.429 | 0 857 |
| 0.0112 | 0.343 | 0.886 | 0.375 | 0.875 | 0.402 | 0.866 | 0.434 | 0 855 |
| 0.0116 | 0.348 | 0.884 | 0.380 | 0.873 | 0.407 | 0.864 | 0.440 | 0.853 |
| 0.0120 | 0.353 | 0.882 | 0.384 | 0.872 | 0.412 | 0.863 | 0.446 | 0.851 |
| 0.0124 | 0.357 | 0.881 | 0.389 | 0.870 | 0.417 | 0.861 | 0.451 | 0.850 |
| 0.0128 | 0.362 | 0.879 | 0.394 | 0.869 | 0.422 | 0.859 | 0.457 | 0.848 |
| 0.0132 | 0.366 | 0.878 | 0.398 | 0.867 | 0.427 | 0.858 | 0.461 | 0.846 |
| 0.0136 | 0.370 | 0.877 | 0.403 | 0.866 | 0.432 | 0.856 | 0.466 | 0.845 |
| 0.0140 | 0.374 | 0.875 | 0.407 | 0.864 | 0.436 | 0.855 | 0.471 | 0.843 |
| 0.0144 | 0.379 | 0.874 | 0.412 | 0.863 | 0.440 | 0.853 | 0.475 | 0.842 |
| 0.0148 | 0.383 | 0.872 | 0.416 | 0.861 | 0.444 | 0.852 | 0.479 | 0.840 |
| 0.0152 | 0.386 | 0.871 | 0.420 | 0.860 | 0.449 | 0.850 | 0.483 | 0.839 |
| 0.0156 | 0.390 | 0.870 | 0.424 | 0.859 | 0.453 | 0.849 | 0.487 | 0.838 |
| 0.0160 | 0.394 | 0.869 | 0.428 | 0.857 | 0.457 | 0.848 | 0.493 | 0.836 |
| 0.0170 | 0.403 | 0.866 | 0.437 | 0.854 | 0.467 | 0.845 | 0.502 | 0.833 |
| 0.0180 | 0.412 | 0.863 | 0.446 | 0.851 | 0.476 | 0.841 | 0.513 | 0.829 |
| 0.0190 | 0.420 | 0.860 | 0.455 | 0.848 | 0.485 | 0.838 | 0.522 | 0.826 |
| 0.0200 | 0.428 | 0.857 | 0.463 | 0.846 | 0.493 | 0.836 | 0.531 | 0.823 |

## TABLE 7. Columns with Longitudinal Bars and Spirals

Loads in thousand pounds

$P = 0.225f'_cA_g + f_sA_s$

| Diameter of column, in. | Gross area $A_g$, sq in. | Load on bars, $f_sA_s$ for $p_g = 0.01$ | | Load on concrete, $0.225f'_cA_g$ | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $f_s$ | | $f'_c$ | | | | |
| | | Intermediate grade 16,000 | Hard grade 20,000 | 2000 | 2500 | 3000 | 3750 | 5000 |
| 12* | 113 | 14 | 18 | 41 | 51 | 61 | 76 | 102 |
| 13* | 133 | 17 | 22 | 48 | 60 | 72 | 90 | 119 |
| 14 | 154 | 25 | 31 | 69 | 87 | 104 | 130 | 173 |
| 15 | 177 | 28 | 35 | 80 | 99 | 119 | 149 | 199 |
| 16 | 201 | 32 | 40 | 91 | 113 | 136 | 170 | 227 |
| 17 | 227 | 36 | 45 | 102 | 128 | 153 | 192 | 256 |
| 18 | 255 | 41 | 51 | 114 | 143 | 172 | 215 | 287 |
| 19 | 284 | 45 | 57 | 128 | 159 | 191 | 239 | 319 |
| 20 | 314 | 50 | 63 | 141 | 177 | 212 | 265 | 353 |
| 21 | 346 | 55 | 69 | 156 | 195 | 234 | 292 | 390 |
| 22 | 380 | 61 | 76 | 171 | 214 | 257 | 321 | 428 |
| 23 | 416 | 66 | 83 | 187 | 234 | 280 | 350 | 467 |
| 24 | 452 | 72 | 90 | 204 | 254 | 305 | 382 | 510 |
| 25 | 491 | 79 | 98 | 221 | 276 | 331 | 414 | 552 |
| 26 | 531 | 85 | 106 | 239 | 299 | 358 | 448 | 598 |
| 27 | 573 | 92 | 115 | 258 | 322 | 387 | 483 | 644 |
| 28 | 616 | 98 | 123 | 277 | 346 | 416 | 519 | 693 |
| 29 | 661 | 106 | 132 | 297 | 372 | 446 | 557 | 743 |
| 30 | 707 | 113 | 141 | 318 | 398 | 477 | 596 | 795 |

* Spirals for these are excessive and seldom available. The loads given in the table are for circular columns with lateral ties.

## TABLE 8. Columns with Longitudinal Bars and Lateral Ties

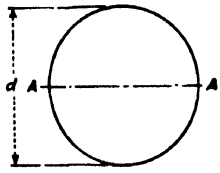
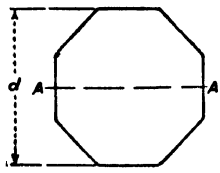
Loads in thousand pounds

$$P = 0.18f'_cA_g + 0.8f_sA_s$$

| Dimensions of column, in. | | Gross area $A_g$, sq in. | Load on bars, $0.8f_sA_s$ for $p_g = 0.01$ | | Load on concrete, $0.18f'_cA_g$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | $f'_c$ | | | | |
| | | | Intermediate grade 16,000 | Hard grade 20,000 | 2000 | 2500 | 3000 | 3750 | 5000 |
| 10 | 12 | 120 | 15 | 19 | 43 | 54 | 65 | 81 | 108 |
| | 14 | 140 | 18 | 22 | 50 | 63 | 76 | 95 | 127 |
| | 16 | 160 | 20 | 26 | 58 | 72 | 86 | 108 | 144 |
| 12 | 12 | 144 | 18 | 23 | 52 | 65 | 78 | 97 | 129 |
| | 14 | 168 | 22 | 27 | 60 | 76 | 91 | 113 | 151 |
| | 16 | 192 | 25 | 31 | 69 | 86 | 104 | 130 | 173 |
| 14 | 14 | 196 | 25 | 31 | 71 | 88 | 106 | 132 | 176 |
| | 16 | 224 | 29 | 36 | 81 | 101 | 121 | 151 | 201 |
| | 18 | 252 | 32 | 40 | 91 | 113 | 136 | 170 | 227 |
| 16 | 16 | 256 | 33 | 41 | 92 | 115 | 138 | 173 | 301 |
| | 18 | 288 | 37 | 46 | 104 | 130 | 156 | 194 | 259 |
| | 20 | 320 | 41 | 51 | 115 | 144 | 173 | 216 | 275 |
| | 22 | 352 | 45 | 56 | 127 | 158 | 190 | 238 | 317 |
| 18 | 18 | 324 | 41 | 52 | 117 | 146 | 175 | 219 | 292 |
| | 20 | 360 | 46 | 58 | 130 | 162 | 194 | 243 | 324 |
| | 22 | 396 | 51 | 63 | 143 | 178 | 214 | 267 | 356 |
| | 24 | 432 | 55 | 69 | 156 | 194 | 233 | 292 | 389 |
| 20 | 20 | 400 | 51 | 64 | 144 | 180 | 216 | 270 | 360 |
| | 22 | 440 | 56 | 70 | 158 | 198 | 238 | 297 | 396 |
| | 24 | 480 | 61 | 77 | 173 | 216 | 259 | 324 | 432 |
| | 26 | 520 | 67 | 83 | 187 | 234 | 281 | 351 | 468 |
| 22 | 22 | 484 | 62 | 77 | 174 | 218 | 261 | 327 | 436 |
| | 24 | 528 | 68 | 84 | 190 | 238 | 285 | 356 | 475 |
| | 26 | 572 | 73 | 92 | 206 | 257 | 309 | 386 | 515 |
| | 28 | 616 | 79 | 99 | 222 | 277 | 333 | 416 | 555 |
| 24 | 24 | 576 | 74 | 92 | 207 | 259 | 311 | 389 | 518 |
| | 26 | 624 | 80 | 100 | 225 | 281 | 337 | 421 | 561 |
| | 28 | 672 | 86 | 108 | 242 | 302 | 363 | 454 | 605 |
| 26 | 26 | 676 | 87 | 108 | 243 | 304 | 365 | 456 | 608 |
| | 28 | 728 | 93 | 116 | 262 | 328 | 393 | 491 | 654 |
| 28 | 28 | 784 | 100 | 125 | 282 | 353 | 423 | 529 | 705 |
| 30 | 30 | 900 | 115 | 144 | 324 | 405 | 486 | 608 | 810 |

## TABLE 9. Areas, Weights, and Moments of Inertia

Moments of inertia about the axis *A-A*

| *d* |  | | |  | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Area, sq. in. | Weight per ft., lb. | $I$, in.$^4$ | Area, sq. in. | Weight per ft., lb. | $I$, in.$^4$ | Area, sq. in. | Weight per ft., lb. | $I$, in.$^4$ |
| 12 | 113 | 118 | 1,018 | 119 | 124 | 1,136 | 12.0 | 12.5 | 144 |
| 13 | 133 | 138 | 1,402 | 140 | 146 | 1,565 | 13.0 | 13.5 | 183 |
| 14 | 154 | 160 | 1,886 | 162 | 169 | 2,105 | 14.0 | 14.6 | 229 |
| 15 | 177 | 184 | 2,485 | 186 | 194 | 2,775 | 15.0 | 15.6 | 281 |
| 16 | 201 | 210 | 3,217 | 212 | 221 | 3,591 | 16.0 | 16.7 | 341 |
| 17 | 227 | 237 | 4,100 | 239 | 249 | 4,577 | 17.0 | 17.7 | 409 |
| 18 | 255 | 265 | 5,153 | 268 | 280 | 5,753 | 18.0 | 18.8 | 486 |
| 19 | 284 | 295 | 6,397 | 299 | 312 | 7,142 | 19.0 | 19.8 | 572 |
| 20 | 314 | 327 | 7,854 | 331 | 345 | 8,768 | 20.0 | 20.8 | 667 |
| 21 | 346 | 361 | 9,547 | 365 | 381 | 10,658 | 21.0 | 21.9 | 772 |
| 22 | 380 | 396 | 11,499 | 401 | 418 | 12,837 | 22.0 | 22.9 | 887 |
| 23 | 416 | 433 | 13,737 | 438 | 457 | 15,335 | 23.0 | 24.0 | 1,014 |
| 24 | 452 | 471 | 16,286 | 477 | 497 | 18,181 | 24.0 | 25.0 | 1,152 |
| 25 | 491 | 511 | 19,175 | 518 | 539 | 21,406 | 25.0 | 26.1 | 1,302 |
| 26 | 531 | 553 | 22,432 | 560 | 583 | 25,042 | 26.0 | 27.1 | 1,465 |
| 27 | 573 | 597 | 26,087 | 604 | 629 | 29,123 | 27.0 | 28.1 | 1,640 |
| 28 | 616 | 642 | 30,172 | 650 | 677 | 33,683 | 28.0 | 29.2 | 1,829 |
| 29 | 661 | 688 | 34,719 | 697 | 726 | 38,759 | 29.0 | 30.2 | 2,032 |
| 30 | 707 | 736 | 39,761 | 746 | 777 | 44,388 | 30.0 | 31.2 | 2,250 |

## TABLE 10. Size and Pitch of Spirals, ACI Code

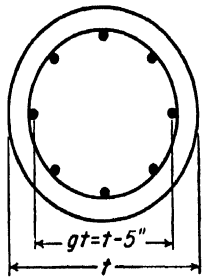
| Diameter of column, in. | Out to out of spiral, in | $f'_c$ 2500 | $f'_c$ 3000 | $f'_c$ 3750 | $f'_c$ 5000 |
|---|---|---|---|---|---|
| Hot-rolled Intermediate-grade | | | | | |
| 14, 15 | 11, 12 | 3/8–2 | 3/8–1 3/4 | 1/2–2 1/2 | 1/2 1 3/4 |
| 16 | 13 | 3/8–2 | 3/8–1 1/4 | 1/2–2 1/2 | 1/2–2 |
| 17–19 | 14–16 | 3/8–2 1/4 | 3/8–1 3/4 | 1/2–2 1/2 | 1/2–2 |
| 20–23 | 17–20 | 3/8–2 1/4 | 3/8 1 3/4 | 1/2–2 3/4 | 1/2–2 |
| 24–30 | | 3/8–2 1/4 | 3/8–2 | 1/2–2 3/4 | 1/2 2 |
| Cold-drawn Wire | | | | | |
| 14, 15 | 11, 12 | 1/4–1 3/4 | 3/8 2 3/4 | 3/8–2 | 1/2–2 3/4 |
| 16 23 | 13–20 | 1/4–1 3/4 | 3/8 2 3/4 | 3/8 2 1/4 | 1/2–3 |
| 24–29 | 21–26 | 1/4 1 3/4 | 3/8–3 | 3/8 2 1/4 | 1/2–3 |
| 30 | 27 | 1/4 1 3/4 | 3/8–3 | 3/8–2 1/4 | 1/2–3 1/4 |

## TABLE 11. Values of $f_a$, the Average Allowable Stress in Axially Loaded Columns

| $f'_c$ | $n$ | Values of $p$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0 010 | 0.015 | 0 020 | 0.025 | 0.030 | 0.035 | 0.040 | 0.050 | 0.060 | 0.070 | 0.080 |
| Tied Columns, $f_s$ = 16,000 | | | | | | | | | | | | |
| 2500 | 12 | 521 | 551 | 579 | 604 | 627 | 648 | 668 | 704 | 744 | 760 | 781 |
| 3000 | 10 | 613 | 645 | 675 | 702 | 728 | 751 | 774 | 814 | 850 | 881 | 910 |
| 3750 | 8 | 750 | 785 | 817 | 847 | 875 | 902 | 927 | 974 | 1016 | 1055 | 1089 |
| 5000 | 6 | 979 | 1016 | 1051 | 1084 | 1117 | 1147 | 1177 | 1233 | 1284 | 1331 | 1375 |
| Spiral Columns, $f_s$ = 16,000 | | | | | | | | | | | | |
| 2500 | 12 | 651 | 689 | 723 | 755 | 784 | 810 | 835 | 879 | 917 | 950 | 980 |
| 3000 | 10 | 766 | 806 | 843 | 878 | 909 | 939 | 967 | 1017 | 1062 | 1101 | 1137 |
| 3750 | 8 | 938 | 981 | 1021 | 1059 | 1094 | 1127 | 1159 | 1218 | 1270 | 1318 | 1361 |
| 5000 | 6 | 1224 | 1270 | 1314 | 1356 | 1396 | 1434 | 1471 | 1540 | 1604 | 1663 | 1718 |
| Tied Columns, $f_s$ = 20,000 | | | | | | | | | | | | |
| 2500 | 12 | 550 | 592 | 631 | 667 | 699 | 729 | 757 | 806 | 850 | 887 | 920 |
| 3000 | 10 | 642 | 687 | 729 | 767 | 803 | 836 | 868 | 924 | 974 | 1019 | 1059 |
| 3750 | 8 | 780 | 828 | 873 | 915 | 955 | 992 | 1027 | 1094 | 1152 | 1205 | 1254 |
| 5000 | 6 | 1010 | 1060 | 1109 | 1156 | 1200 | 1243 | 1283 | 1360 | 1430 | 1496 | 1557 |
| Spiral Columns, $f_s$ = 20,000 | | | | | | | | | | | | |
| 2500 | 12 | 687 | 740 | 789 | 833 | 874 | 912 | 946 | 1008 | 1062 | 1109 | 1150 |
| 3000 | 10 | 803 | 859 | 911 | 959 | 1004 | 1046 | 1085 | 1155 | 1218 | 1273 | 1323 |
| 3750 | 8 | 975 | 1035 | 1091 | 1144 | 1193 | 1240 | 1284 | 1366 | 1439 | 1506 | 1567 |
| 5000 | 6 | 1262 | 1326 | 1386 | 1444 | 1500 | 1553 | 1604 | 1700 | 1788 | 1870 | 1946 |

## TABLE 12. Moment of Inertia of Column Verticals Arranged in a Circle of Diameter 5 In. Less than Diameter of Column

Values of $(n - 1)I_s$ in inches$^4$ for $p_g = 0.01$



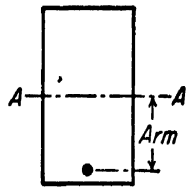

| Diameter of column $t$, in. | Diameter of steel circle $gt$, in. | $A_s$ where $p_g = 0.01$ | $f'_c$ 2000 | 2500 | 3000 | 3750 | 5000 |
|---|---|---|---|---|---|---|---|
| 12 | 7 | 1.13 | 97 | 76 | 62 | 49 | 35 |
| 13 | 8 | 1.33 | 149 | 117 | 96 | 74 | 53 |
| 14 | 9 | 1.54 | 218 | 172 | 140 | 109 | 78 |
| 15 | 10 | 1.77 | 310 | 243 | 199 | 155 | 111 |
| 16 | 11 | 2.01 | 426 | 335 | 274 | 213 | 152 |
| 17 | 12 | 2 27 | 572 | 449 | 368 | 286 | 204 |
| 18 | 13 | 2 55 | 754 | 593 | 485 | 377 | 271 |
| 19 | 14 | 2.84 | 974 | 765 | 626 | 487 | 348 |
| 20 | 15 | 3.14 | 1166 | 916 | 750 | 583 | 416 |
| 21 | 16 | 3.46 | 1550 | 1218 | 996 | 775 | 554 |
| 22 | 17 | 3.80 | 1922 | 1510 | 1236 | 961 | 686 |
| 23 | 18 | 4.16 | 2359 | 1853 | 1516 | 1179 | 842 |
| 24 | 19 | 4.52 | 2856 | 2244 | 1836 | 1428 | 1020 |
| 25 | 20 | 4.91 | 3437 | 2701 | 2210 | 1719 | 1227 |
| 26 | 21 | 5.31 | 4098 | 3220 | 2635 | 2049 | 1464 |
| 27 | 22 | 5.73 | 4853 | 3813 | 3120 | 2427 | 1733 |
| 28 | 23 | 6 16 | 5703 | 4481 | 3666 | 2852 | 2037 |
| 29 | 24 | 6.61 | 6663 | 5235 | 4283 | 3331 | 2380 |
| 30 | 25 | 7.07 | 7733 | 6076 | 4971 | 3867 | 2762 |

The bars are assumed transformed into a thin-walled cylinder having the same sectional area as the bars. Then $I_s = A_s s^2 \div 8$.

For other values of $p_g$, multiply the value from the table by $100p_g$.

## TABLE 13. Moments of Inertia of Bars in Inches[4]

For various distances from an axis *AA*




| Arm, in. | Bar number | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 2 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 6 |
| 2½ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 10 |
| 3 | 2 | 3 | 4 | 5 | 7 | 9 | 12 | 14 |
| 3½ | 2 | 4 | 5 | 7 | 10 | 12 | 16 | 19 |
| 4 | 3 | 5 | 7 | 10 | 13 | 16 | 20 | 25 |
| 4½ | 4 | 6 | 9 | 12 | 16 | 20 | 26 | 32 |
| 5 | 5 | 8 | 11 | 15 | 20 | 25 | 32 | 39 |
| 5½ | 6 | 9 | 13 | 18 | 24 | 30 | 38 | 47 |
| 6 | 7 | 11 | 16 | 22 | 28 | 36 | 46 | 56 |
| 6½ | 8 | 13 | 19 | 25 | 33 | 42 | 54 | 66 |
| 7 | 10 | 15 | 22 | 29 | 39 | 49 | 62 | 77 |
| 7½ | 11 | 17 | 25 | 34 | 44 | 56 | 71 | 88 |
| 8 | 13 | 20 | 28 | 39 | 50 | 64 | 81 | 100 |
| 8½ | 14 | 22 | 32 | 43 | 57 | 72 | 92 | 113 |
| 9 | 16 | 25 | 36 | 49 | 64 | 81 | 103 | 127 |
| 9½ | 18 | 28 | 40 | 54 | 71 | 90 | 114 | 141 |
| 10 | 20 | 31 | 44 | 60 | 79 | 100 | 127 | 156 |
| 10½ | 22 | 34 | 49 | 66 | 87 | 110 | 149 | 172 |
| 11 | 24 | 37 | 53 | 73 | 95 | 121 | 153 | 189 |
| 11½ | 26 | 41 | 58 | 80 | 104 | 132 | 168 | 207 |
| 12 | 28 | 44 | 64 | 87 | 113 | 144 | 182 | 225 |
| 13 | 33 | 52 | 75 | 102 | 133 | 169 | 214 | 264 |

## Table 14. Values of $D = \frac{t^2}{2R^2}$ for Eccentrically Loaded Columns

| $g$ | $(n-1)p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 |
| Square or Rectangular Sections with Ties | | | | | | | | |
| 0.80 | 5.53 | 5.20 | 4.95 | 4.75 | 4.59 | 4.46 | 4.35 | 4.26 |
| 0.70 | 5.75 | 5.56 | 5.41 | 5.29 | 5.19 | 5.10 | 5.03 | 4.96 |
| 0.60 | 5.95 | 5.92 | 5.89 | 5.86 | 5.84 | 5.82 | 5 80 | 5.79 |
| Square Sections with Spirals | | | | | | | | |
| 0.80 | 6.01 | 6.03 | 6.05 | 6.07 | 6 08 | 6 09 | 6.10 | 6.00 |
| 0.70 | 6.15 | 6.27 | 6.39 | 6.49 | 6 58 | 6.66 | 6.74 | 6.80 |
| 0.60 | 6.25 | 6.49 | 6.70 | 6.90 | 7.08 | 7.24 | 7.39 | 7.53 |
| Round Sections with Spirals | | | | | | | | |
| 0.80 | 7.80 | 7.64 | 7.51 | 7.41 | 7 32 | 7 21 | 7.17 | 7.11 |
| 0.70 | 8.02 | 8.03 | 8 04 | 8 05 | 8 06 | 8 06 | 8.07 | 8.08 |
| 0.60 | 8.21 | 8.39 | 8.55 | 8.70 | 8.82 | 8 91 | 9.01 | 9.14 |

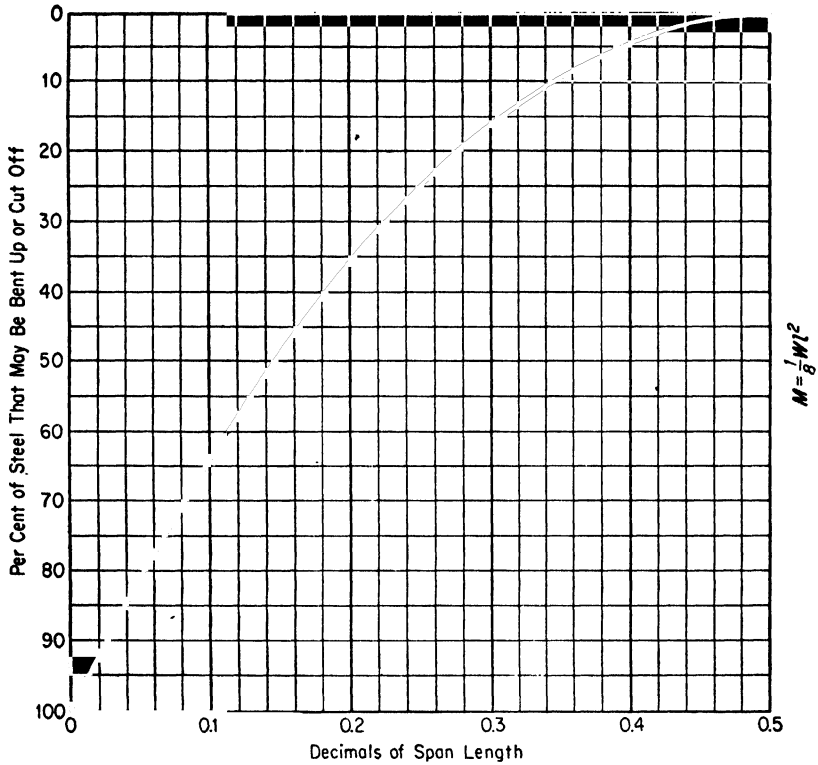



DIAGRAM 1. Location of points where bars may be bent up for simply supported beams uniformly loaded.

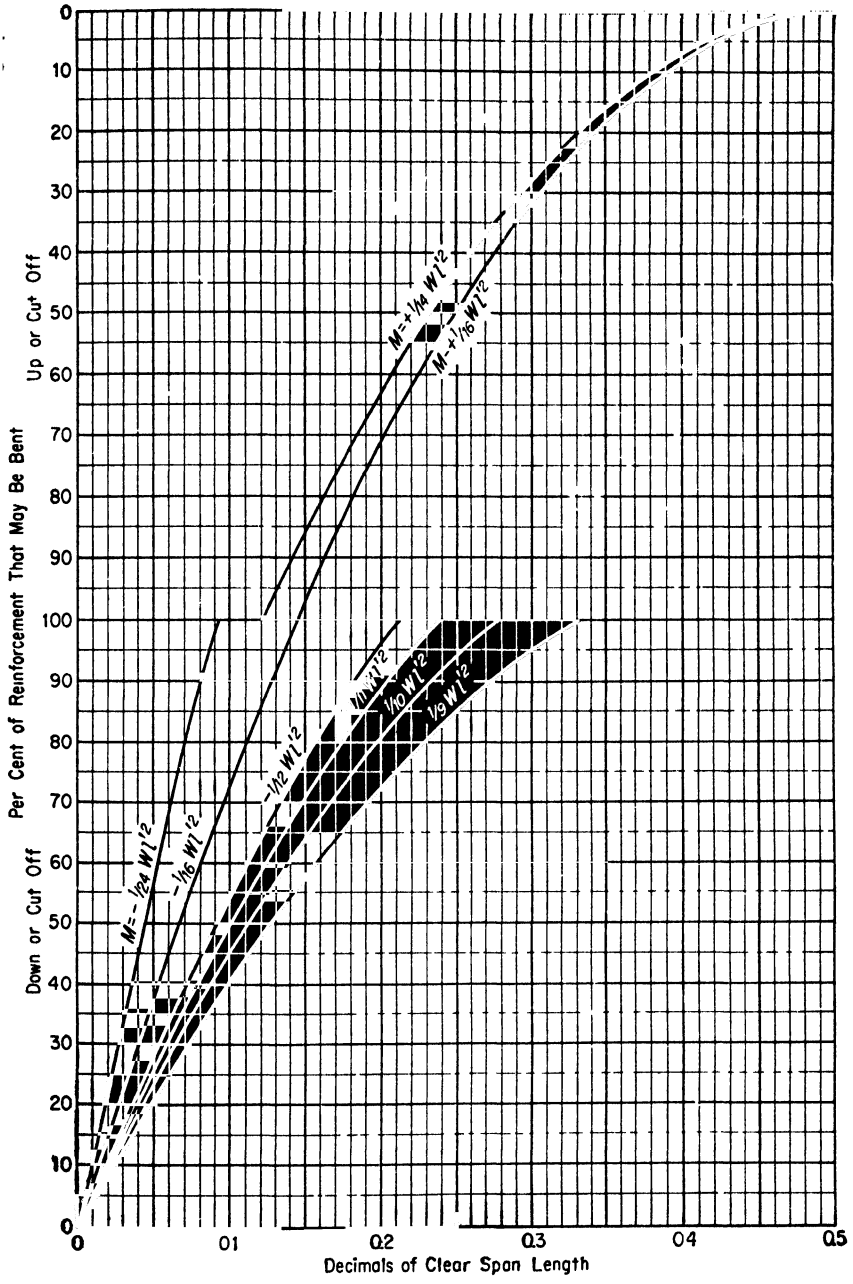



DIAGRAM 2. Approximate location of points where bars may be bent up or down or cut off for continuous beams uniformly loaded and built integrally with their supports according to the coefficients of the ACI Code.

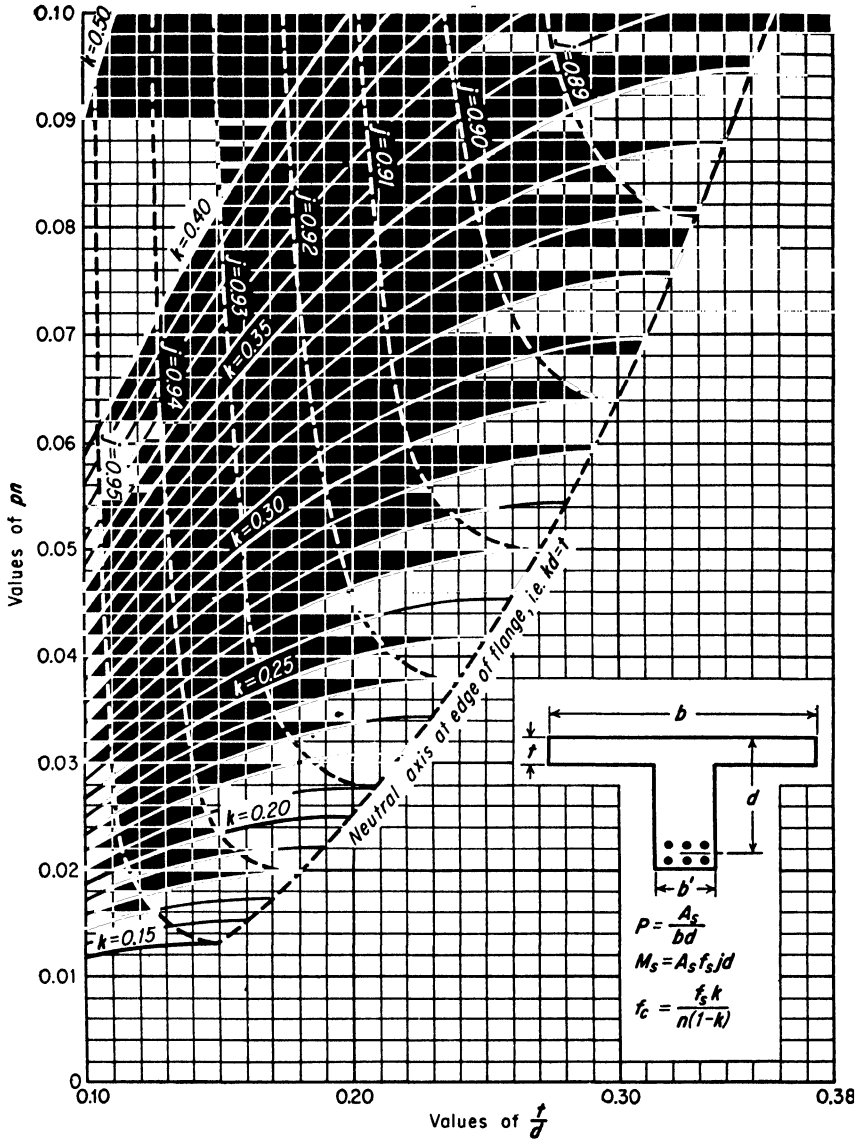



DIAGRAM 3. T-beam review.

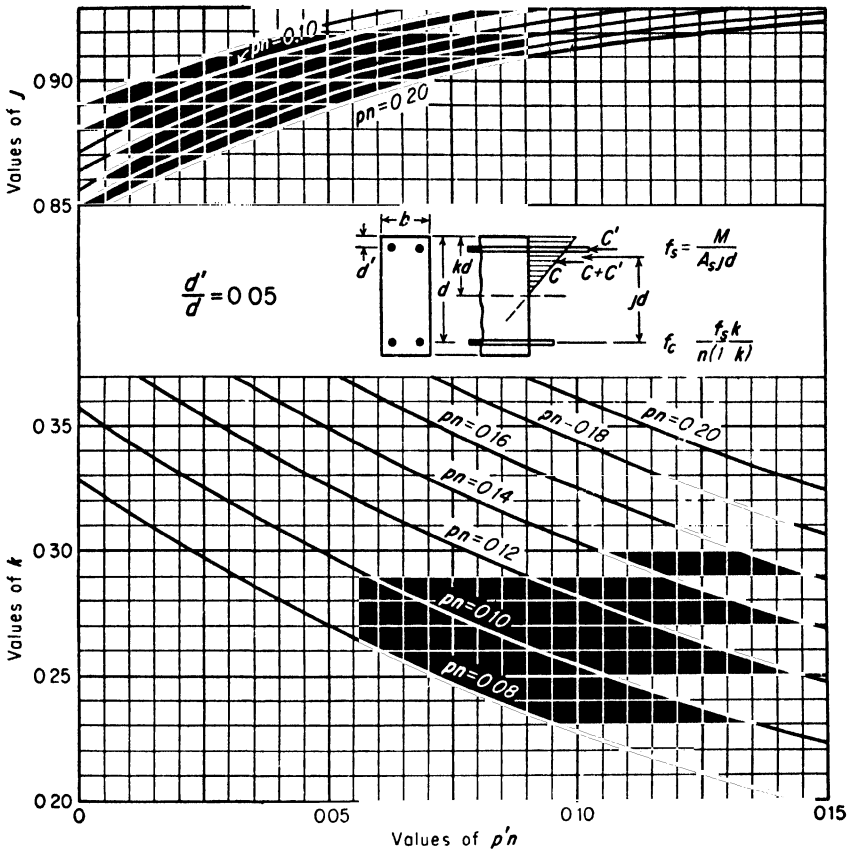



DIAGRAM 1 Rectangular beams reinforced for compression

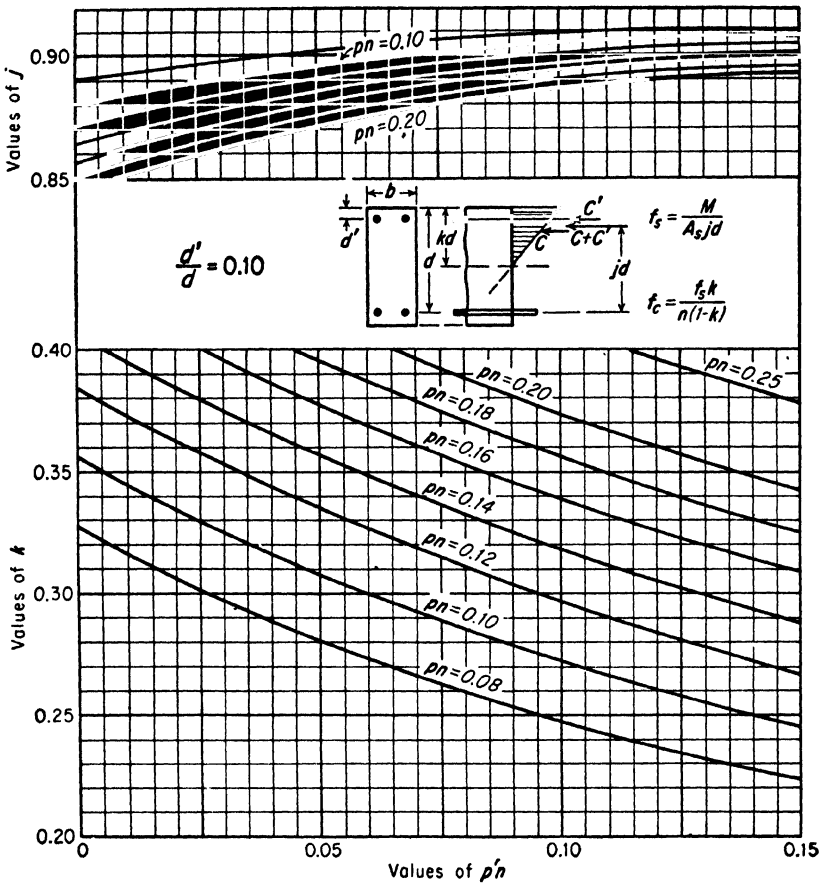



DIAGRAM 5. Rectangular beams reinforced for compression.

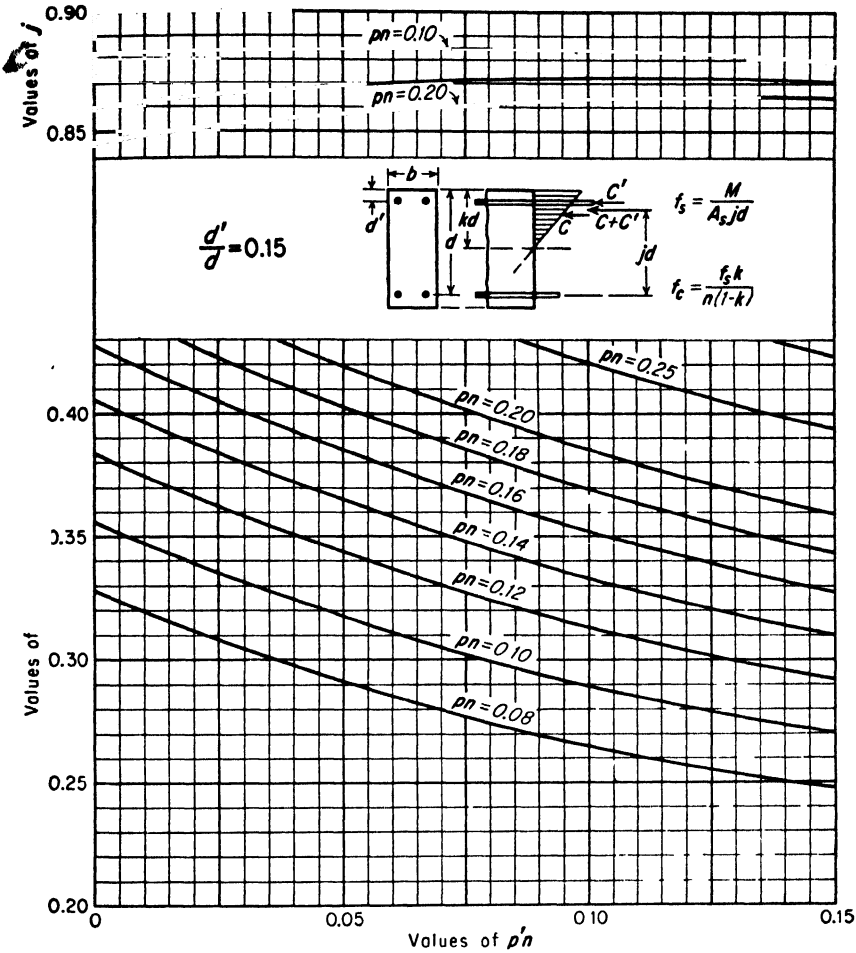



DIAGRAM 6. Rectangular beams reinforced for compression.

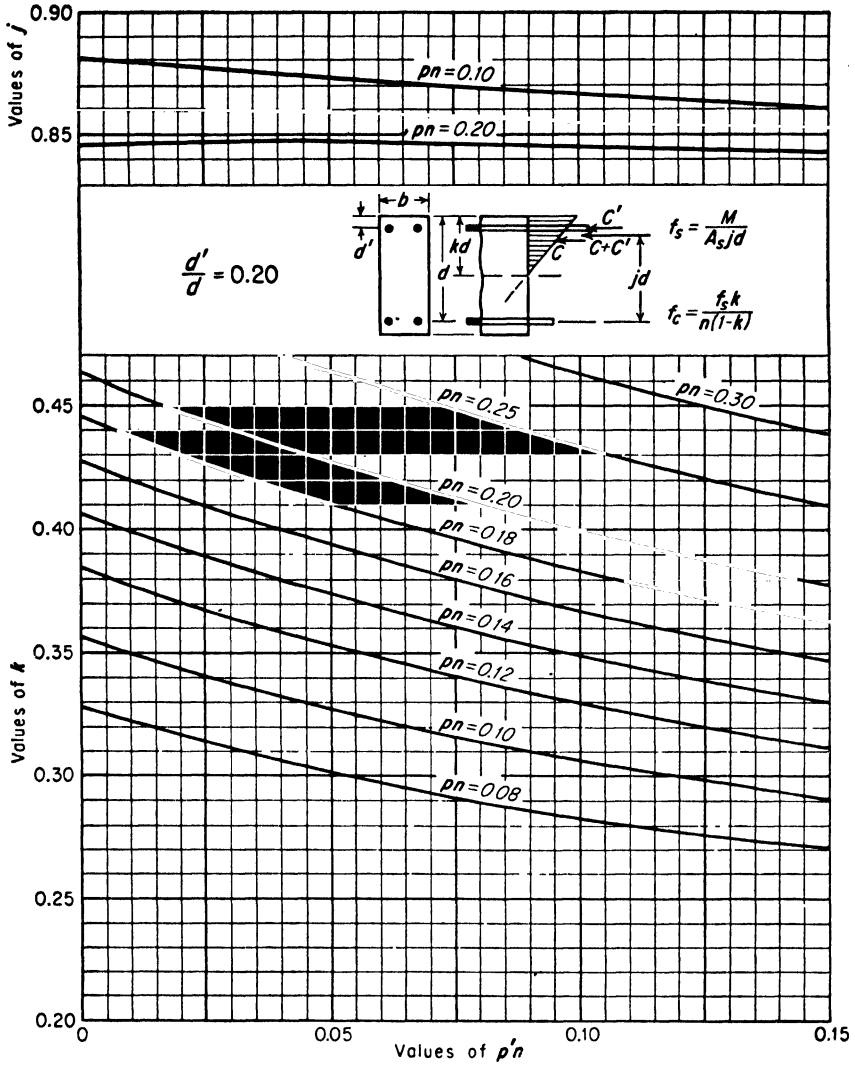



DIAGRAM 7. Rectangular beams reinforced for compression.

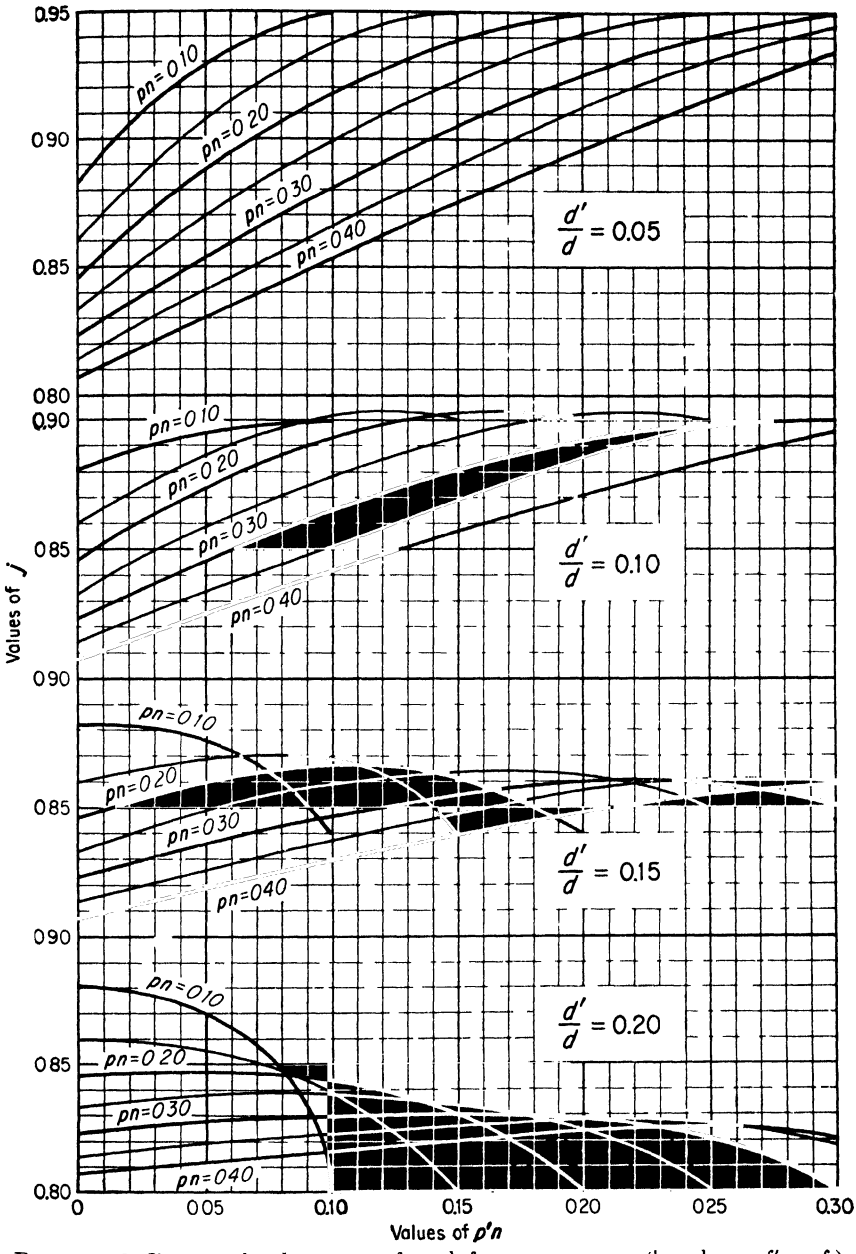



DIAGRAM 8 Rectangular beams reinforced for compression (based on $f'_s = f_s$).

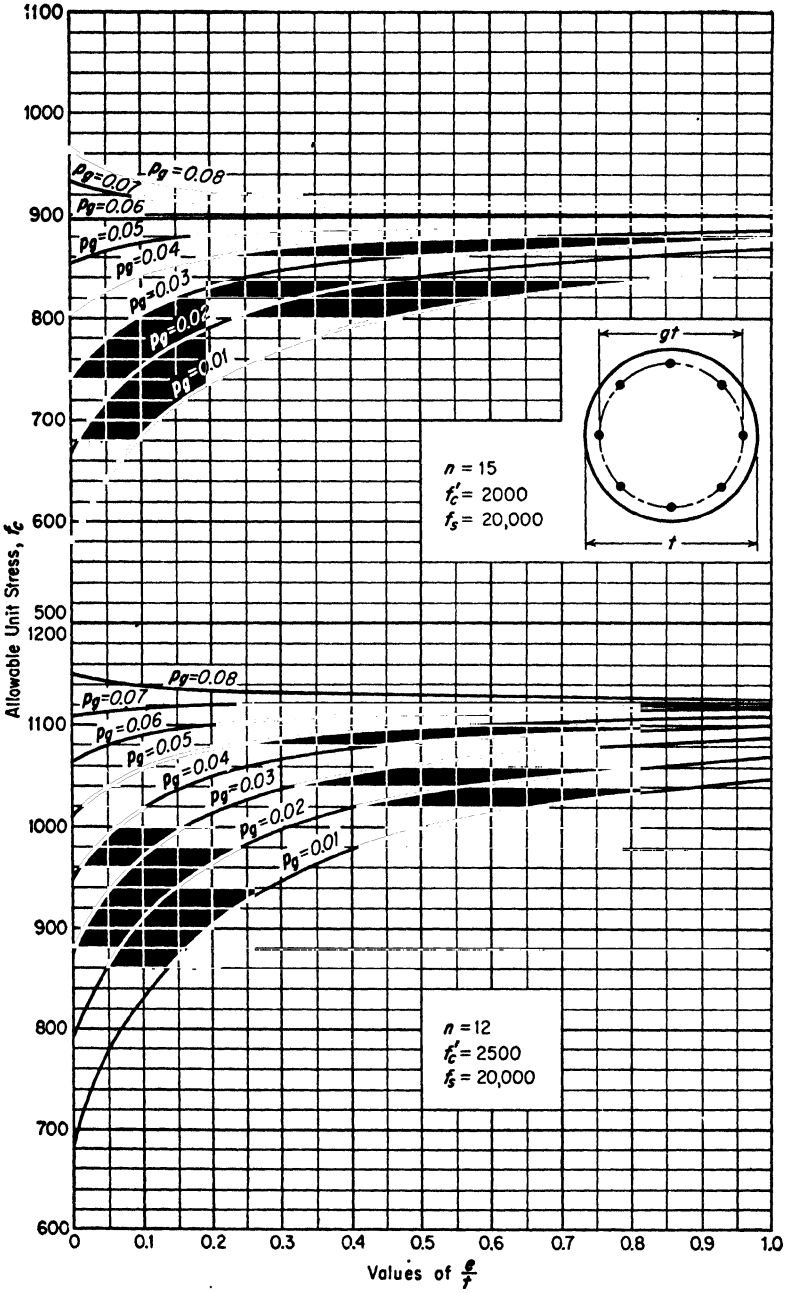



DIAGRAM 9. Allowable unit stresses in eccentrically loaded circular spiral columns.

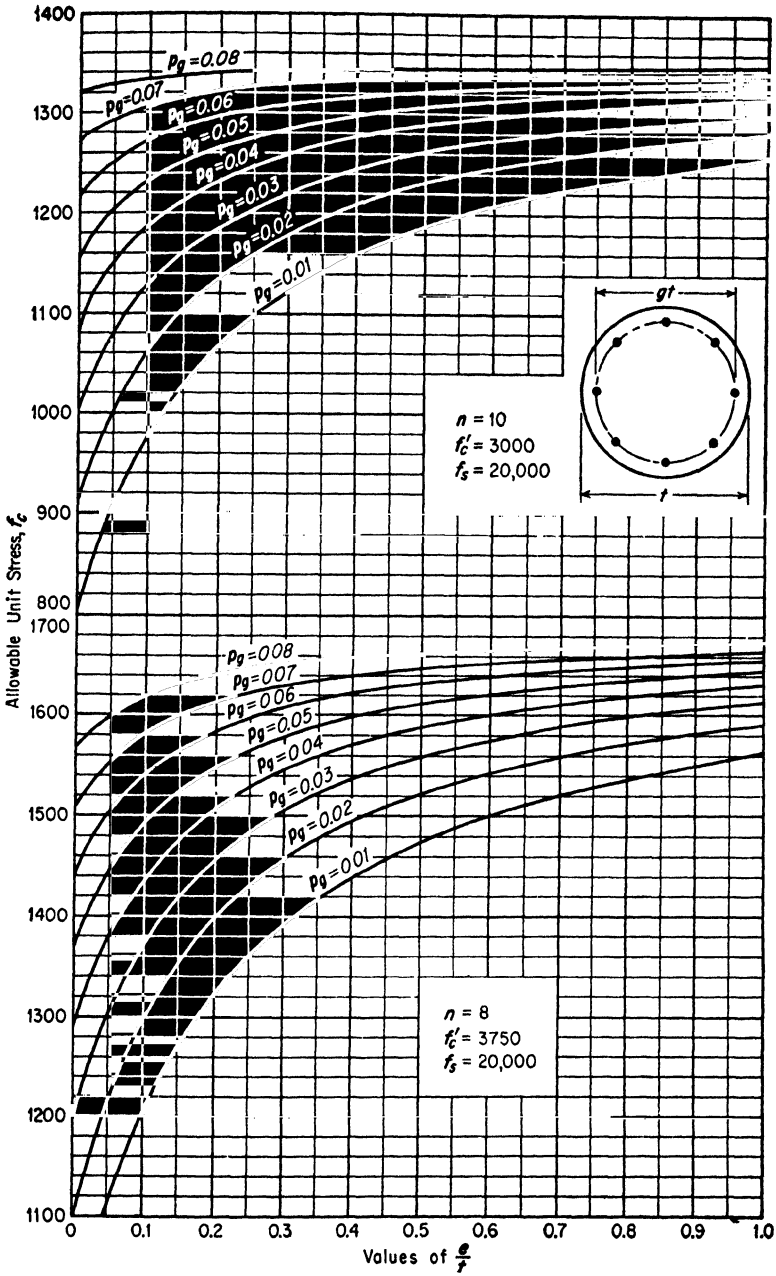



DIAGRAM 10. Allowable unit stresses in eccentrically loaded circular spiral columns.

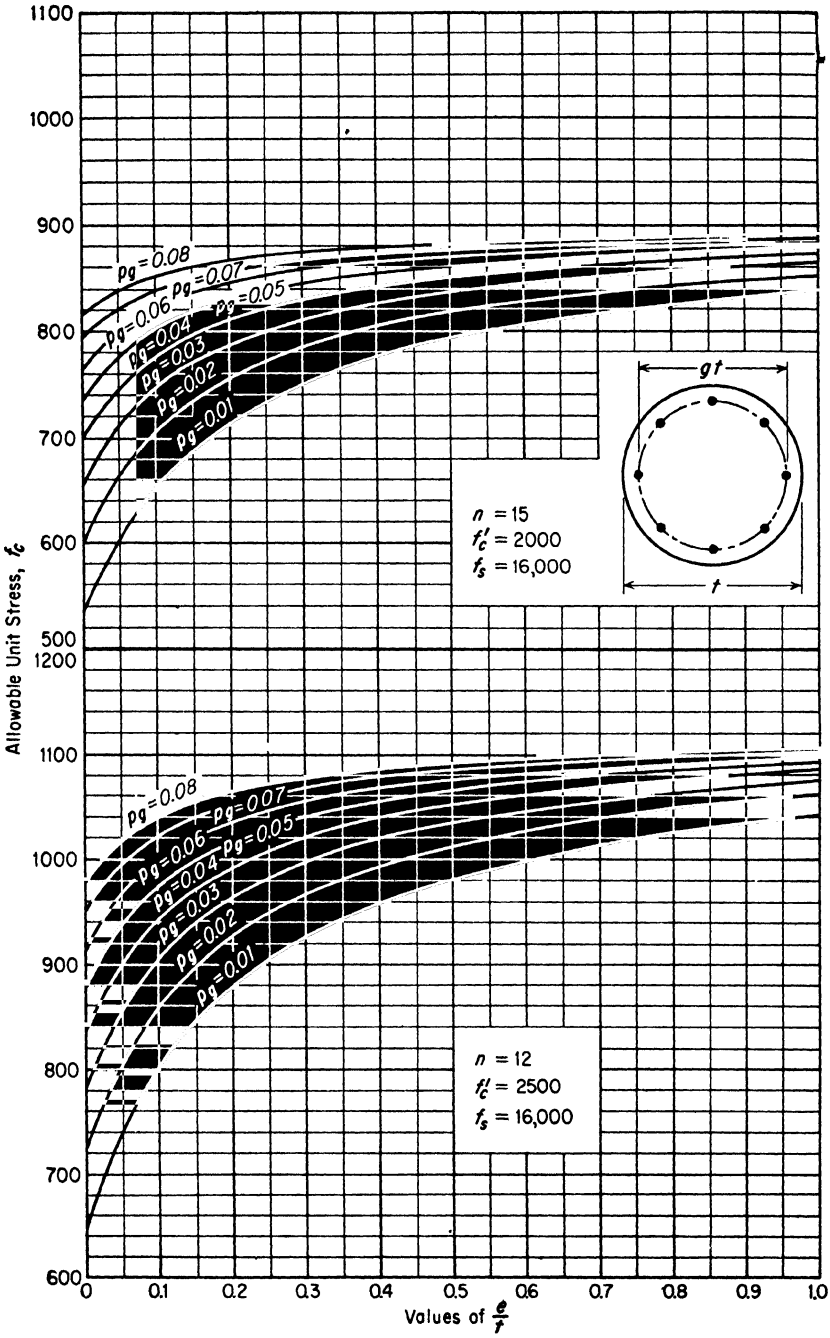



DIAGRAM 11. Allowable unit stresses in eccentrically loaded circular spiral columns.

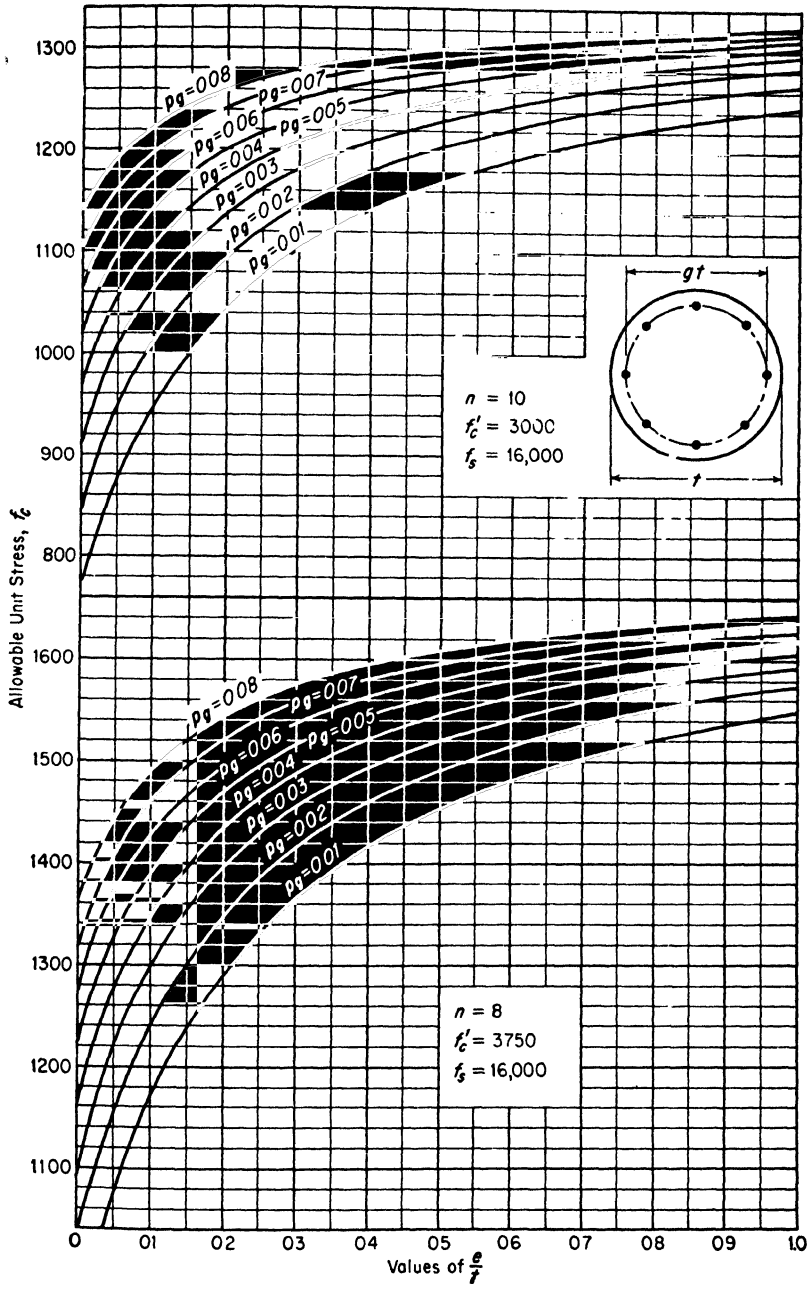



DIAGRAM 12 Allowable unit stresses in eccentrically loaded circular spiral columns.

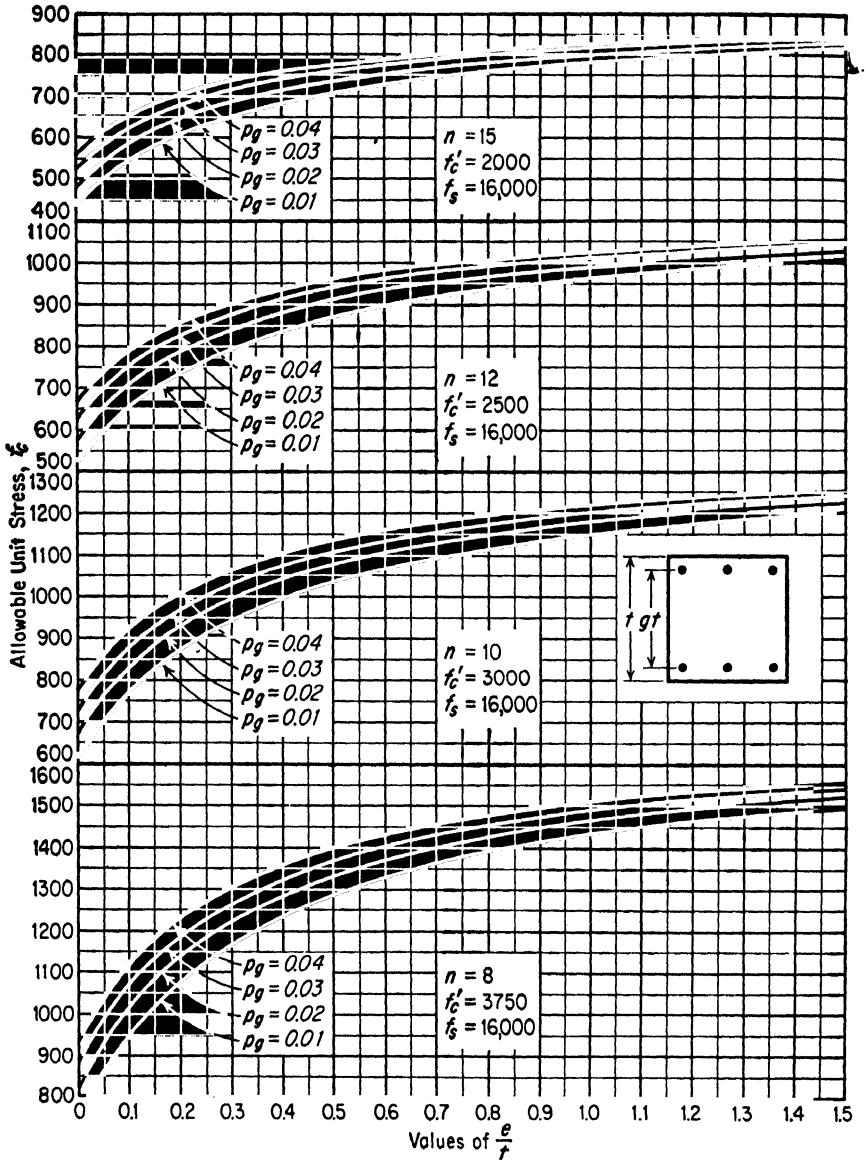



DIAGRAM 13. Allowable unit stresses in eccentrically loaded tied columns.

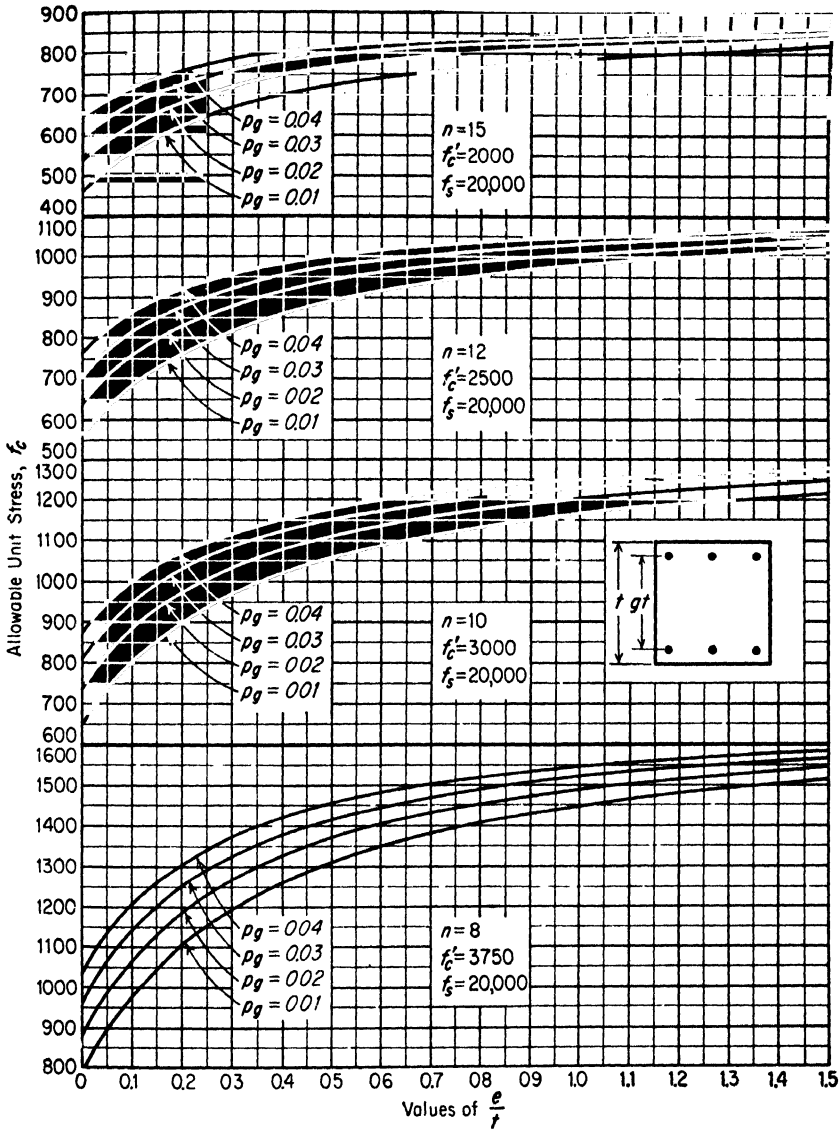



DIAGRAM 11. Allowable unit stresses in eccentrically loaded tied columns.

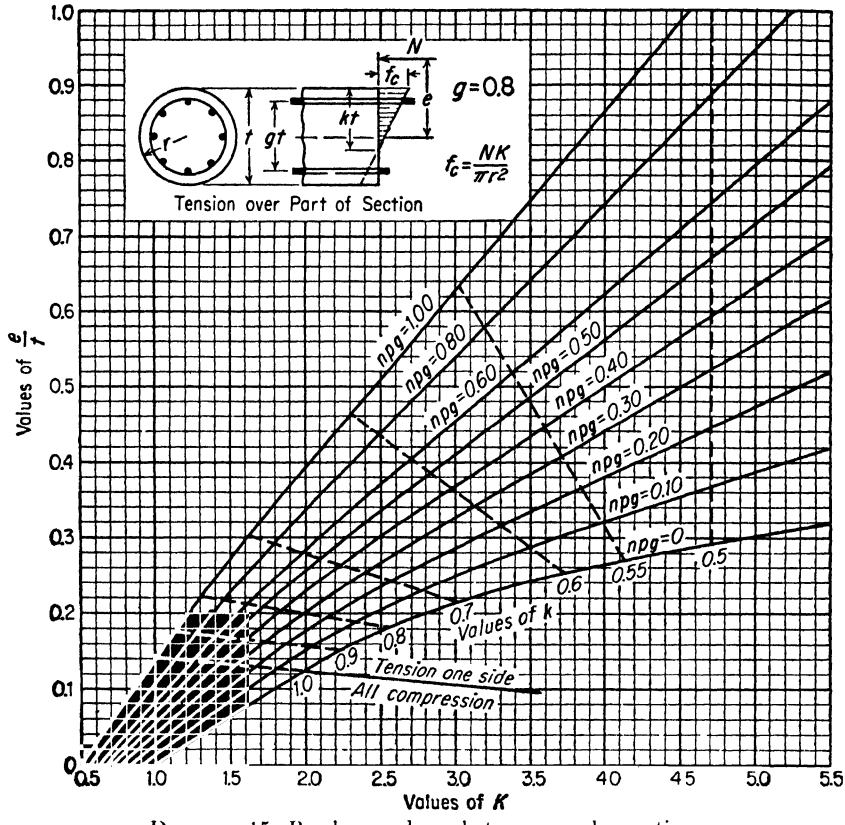



DIAGRAM 15. Bending and axial stress, circular sections.

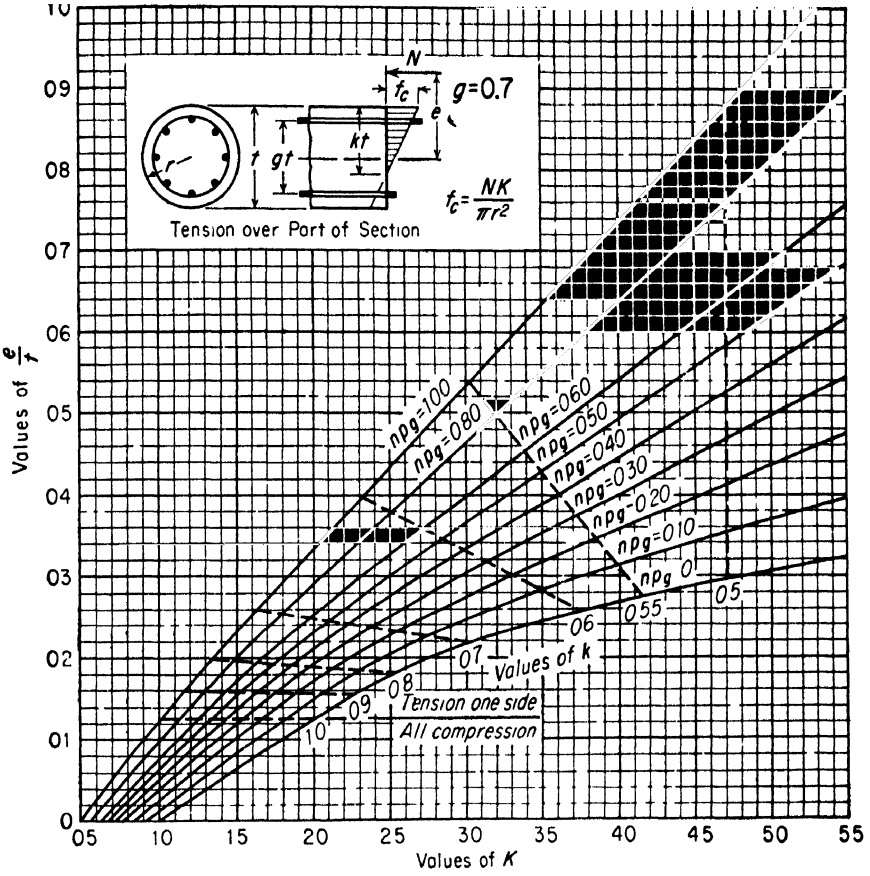



DIAGRAM 16 Bending and axial stress, circular sections

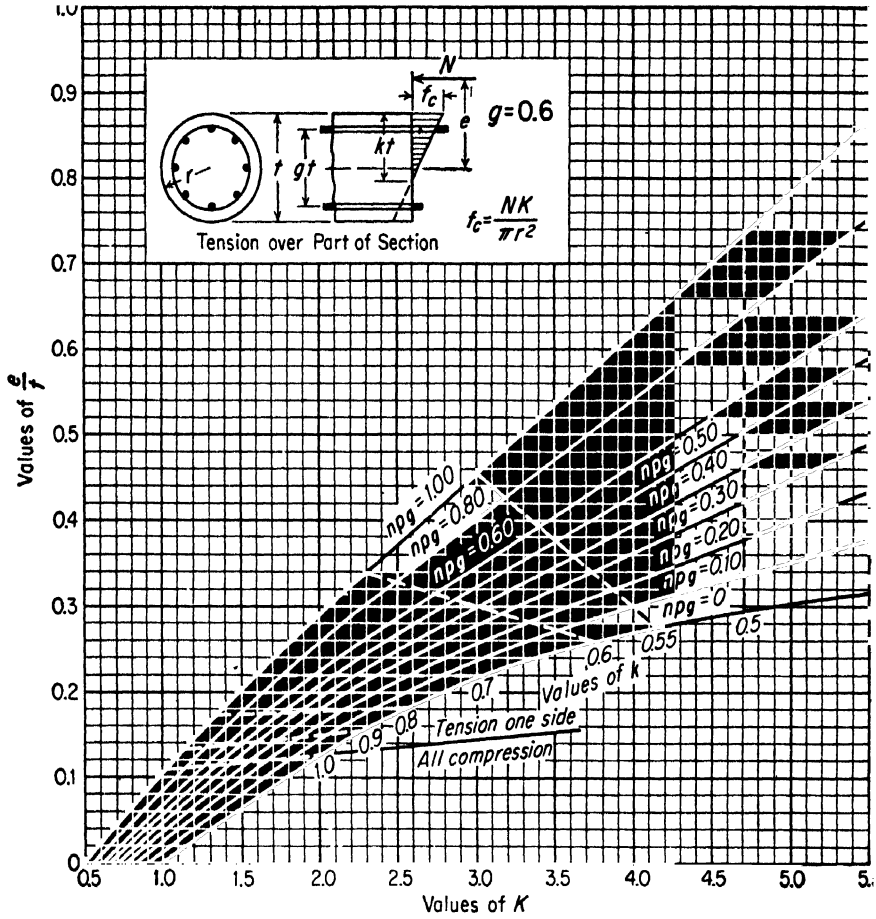



DIAGRAM 17. Bending and axial stress, circular sections.

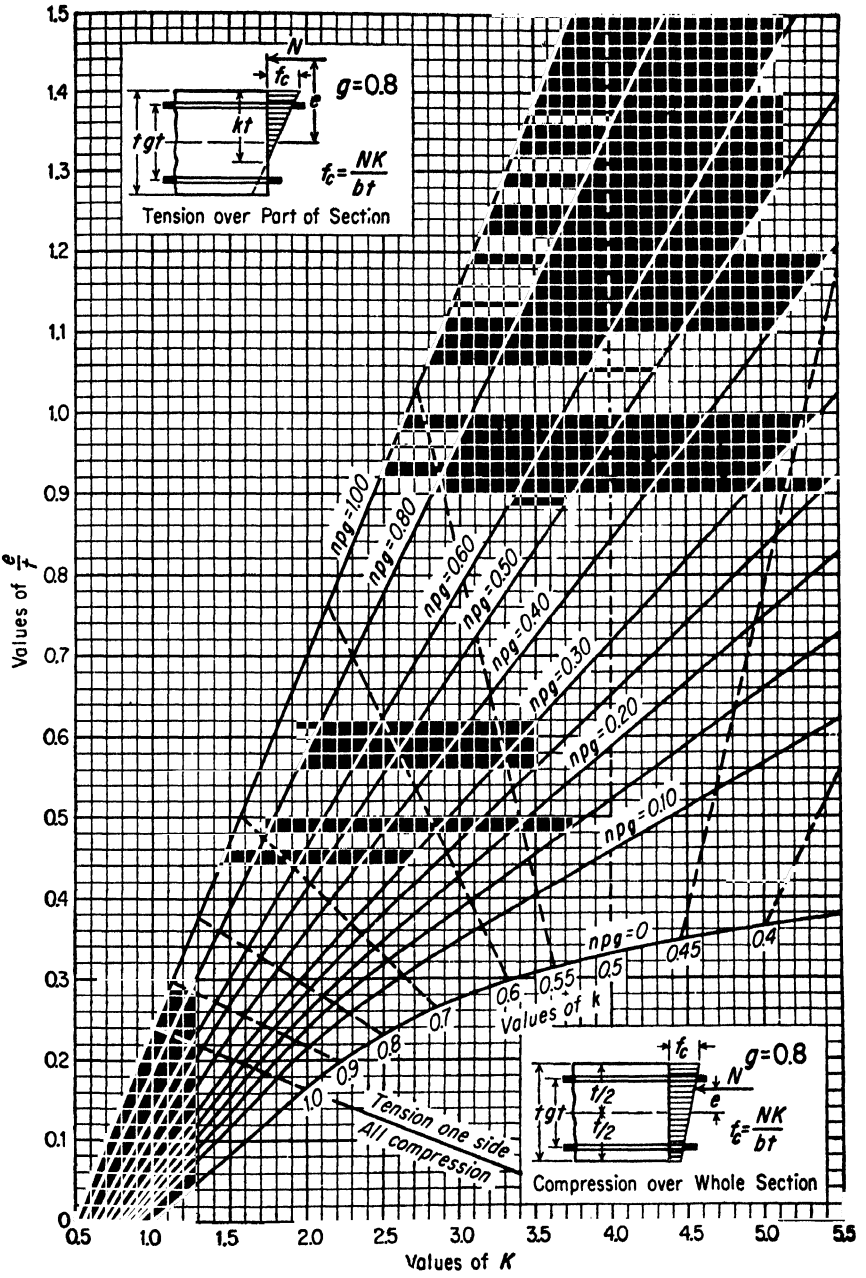



DIAGRAM 18. Bending and axial stress, rectangular sections.

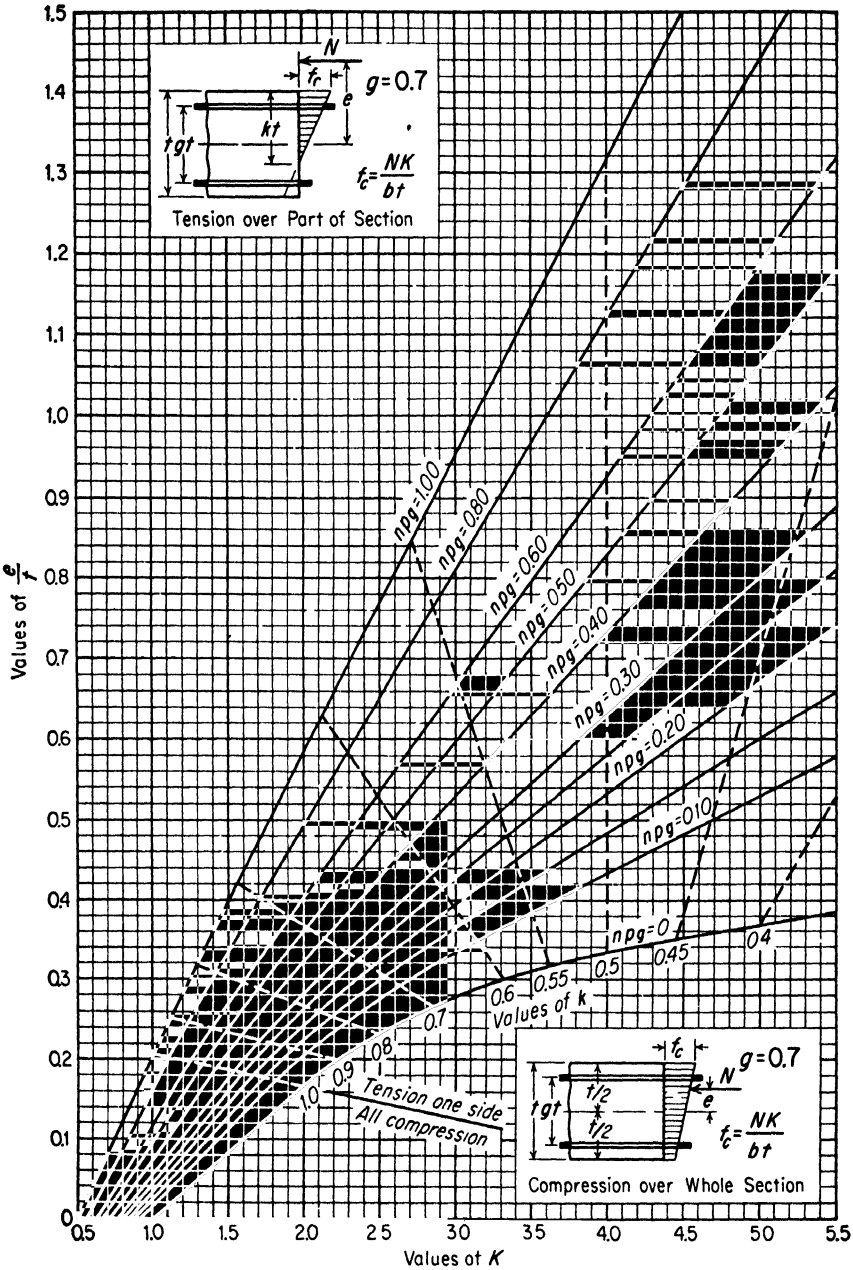



DIAGRAM 19. Bending and axial stress, rectangular sections.

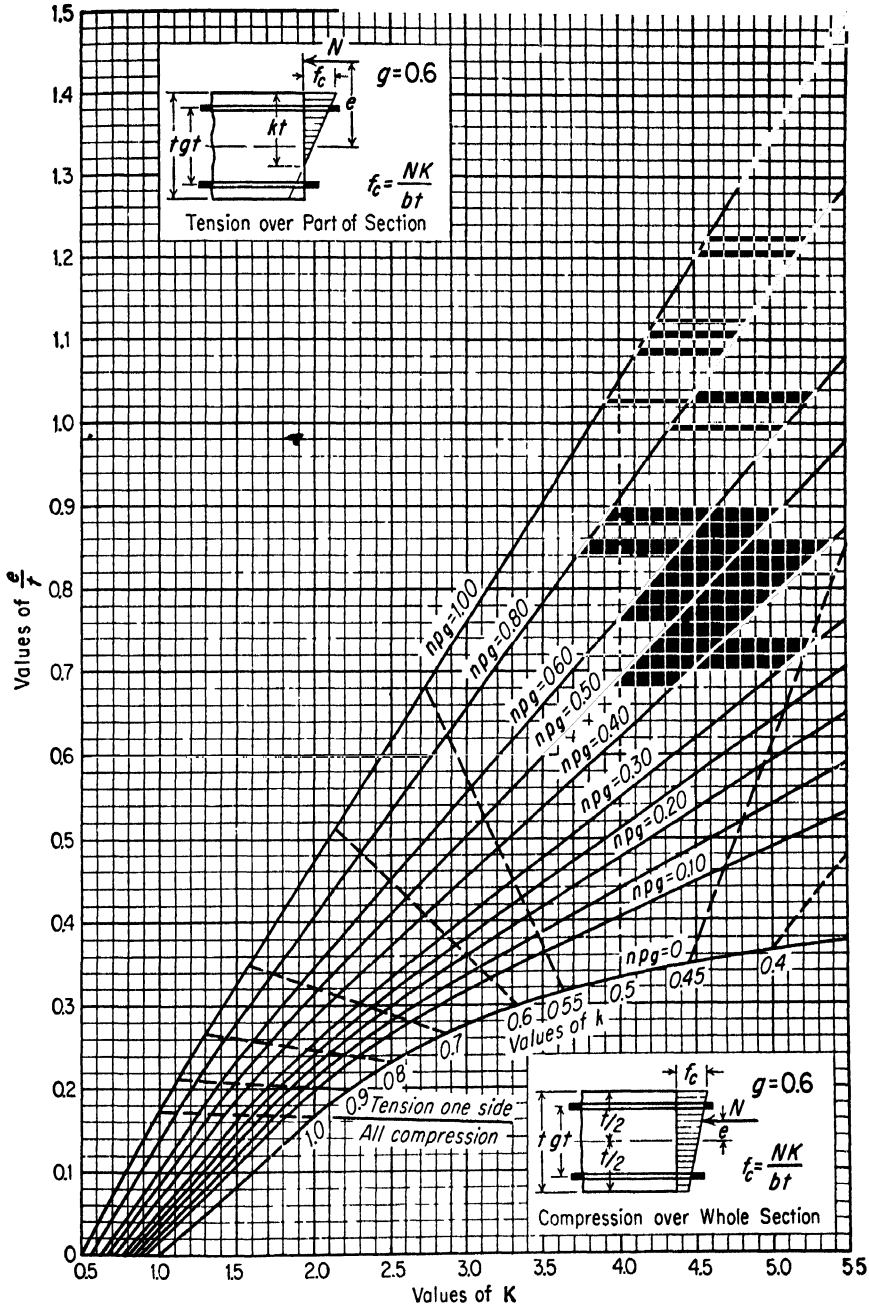



DIAGRAM 20. Bending and axial stress, rectangular sections.

# INDEX


A

Abutments, highway bridge, 397
Aggregates, coarse, 4
  determination of unit weight, 11
  fine, 3
  fineness modulus, 8
  grading, 6, 7
  maximum size, 5, 7
  minimum size, 6
  moisture absorbed by, 10, 15, 17
  moisture content, 18
  organic impurities in, 8
  surface area, 3, 8
  water absorbed by, 15, 18
Allowable stresses, 52, 85, 140, 459, 488 493
  bond, 39, 459
  for combined compression and bending, 140, 488-493
Anchor plate for arch ties, 385
Anchorage of web reinforcement, 92
Arches, advantages of, 350
  axis, length of, 372, 374
    shape of, 351, 367
      for bridge arches, 368
      parabolic, 367
      for roof arches, 367
  barrel, 352
  bridge, open-spandrel, 351
    spandrel-filled, 351
  buckling of, 386
  critical load positions, 371
  economical steel percentages, 353
  fixed (*see* Fixed arches)
  hinges in, 353, 375
  for long-span roofs, 350 352
  moments of inertia, 367
  preliminary design, 367
  rib shortening, 356, 375
  shrinkage, 357
Arches, structural action of, 350
  temperature, 357, 375
  three-hinged, 350
  two-hinged (*see* Two-hinged arches)

B

Bars, areas, perimeters, and weights of, 468, 469
  size extras for, 33
  standard hooks for, 92
  standard sizes of, 468
Beams, arrangement of reinforcement, 51
  continuous, 116, 152
    moment-distribution method, 178, 180
    moments in (table), 153
    slope-deflection method, 162, 165
    theorem of three moments, 152
  in flat-slab buildings, design of, 315
  flexure, 44 62
    allowable unit stresses, 52
    assumptions in theory of, 46
    formulas, 48
  floor, design of, 99, 116, 269, 272, 282
    joist, 278, 286
    slab beam-and-girder, 266
  framing stair well, design of, 265, 320
  highway bridge, 406, 410, 415, 420
  moments in, 49–51
  plain concrete, 43
    diagonal tension in, 82
    shearing stress in, 78
  prestressed concrete, postensioned, 450–453
    pretensioned, 446, 448
  rectangular, with tension reinforcement, 44
    illustrative examples, 53, 56
    tables for, 56, 470, 471
    transformed section, 58

Beams, reinforced for compression, 106
diagrams for review, 112, 483–487
flexure, 106 115
formulas, for design, 107
for review, 109
illustrative examples, 113
shearing stresses in, 85, 459
stresses in homogeneous, 41
T (*see* T beams)
Bearing capacity of soils, 207, 208
Bending and axial stress 134
allowable stresses for combined compression and, 140, 488 493
circular sections, compression over whole section, 138
diagrams for, 194 196
tension over part of section, 140
illustrative examples, 145
rectangular sections compression over whole section, 136
diagrams, 197 199
tension over part of section, 137
Bond between concrete and steel 35
length of embedment required for, 39
new, to old concrete, 28
Bond stresses, 36
allowable, 39, 459
Bridges, arches, 351
axis for 368
highway (*see* Highway bridges)
Building codes, ACI flat slab regulations, 461
allowable bearing pressures, soil, 208
loads, 244
stairways 265
walls, 260–262
Building frames, 186 201
horizontal loads 197
preliminary analysis, 204
sidesway in, 189
vertical loads, 186
approximate analysis, 189
Buildings, 243
flat-slab, 256, 287
advantages of, 288
columns in (*see* Columns)
design of, 298–326
engineering drawings, 324 326
floor systems, 245 258
(*See also* Floors)
steel vs concrete, relative merits, 243
Buildings, ties, live-load reduction in, 209
Bulking factor, 10

## C

Cement, 1
high-early-strength, 2
portland, 2
Cinders, 6
Clark, A P , 38
Color metric test, 8
Column footings (*see* Footings)
Column sections, areas of, 472 474
moment of inertia of, 474
weights of, 474
Column spirals, pitches and sizes, 131, 475
Columns, combination, 128
composite, 128
dimensions of, 129
eccentric loads on 130
exterior design of 147
in flat slab buildings 302 311
bending in, 304 310
design of 302 307
flexural stresses in 130
allowable 140 488 493
illustrative examples 145
interior design of 148
lateral ties size and spacing of 128
long 129
with longitudinal and spiral reinforcement 122
allowable loads 126, 472
diagrams for eccentrically loaded 194 196
with longitudinal steel and lateral ties, 127
allowable loads, 127, 473
diagrams for eccentrically loaded, 497 499
moments in, 130
pipe, 128
tables for design of, 472, 473
types of, 122
ultimate design of, 123 126, 437
unsupported length, 129
Composite beams (*see* Highway bridges, composite)
Concrete 1
abrasive resistance, 30
cinder, 6

Concrete, coefficient of expansion, 28, 35
compressive strength, 25
consistency, 14
contraction, 28
curing, 21
steam in, 22
effect on, of acids, 32
of air-entraining agents, 23
of alkali, 31
of calcium chloride, 24
of freezing, 22
of hydrated lime, 24
of oils, 32
of sea water, 31
of sewage, 32
of time of mixing, 20
of waterproofing compounds, 24
elastic limit, 27
elasticity, modulus of, 26, 429
expansion, 28
mixing, control of, 19
design of, 15
workability of, 13, 14
yield of, 13
permeability, 11
placing of, 20
plastic flow of, 27, 124
plasticity of (*see* Plasticity)
prestressed (*see* Prestressed concrete)
proportioning of, 9
reinforced, 35, 40
resistance to fire, 29
shearing strength, 25
soil friction, 331
stress-strain diagram, 26, 124, 426, 428
structural properties, 25
tensile strength, 25, 40
thermal conductivity, 29
transverse strength, 25
weathering qualities, 30
weight, 29
Continuity, 150
Continuous beams (*see* Beams)
Cross, Hardy, 173
Crushed stone, 4

D

Design of concrete mixtures, 15–16
(*See also* Preliminary design; Ultimate design)
Diagonal tension, 81
distribution of, 87
in plain concrete beams, 82
provision for, 83
in reinforced concrete beams, 82
Diagrams, beams reinforced for compression, 483–487
bending and axial stress, 488 499
bending of reinforcement, 480, 481
T-beam review, 482

E

Earth pressure, 328
active, 329
for common conditions of loading, 332
determination of, 328–334
effect of groundwater on, 333
passive 329
rest, 329
with surcharge, 332
Earthquake forces, 197
Effective span length, 201

F

Fineness modulus, 8
Fire protection, walls, 263
Fixed arches, 351
critical load positions, 371
moments and reactions (tables), 370, 373
preliminary design, 366, 367
integral evaluation, 366
theory of, 359
elastic center, 362, 364
rib shortening, 365, 375
shrinkage, 363
temperature, 363, 375
thickness variation (table), 374
Fixed-end moments, 159
for far end hinged, 160, 163, 164, 177
for moment-distribution method, 171, 177
table of, 163, 164
Flat-slab floors (*see* Floors)
Flexure, beams reinforced for compression, 106 115
rectangular beams, 44–62
T beams, 93–105
Floors, 245

Floors, beam-and-girder, 246
design of, 266
flat-plate, 256
flat-slab, 256, 287
ACI code requirements, 461
advantages of, 288
bending moments in, 289–294, 299, 301
design of, 298
factors in, 295
engineering drawings, 324–326
methods of reinforcing, 294, 324–326
ribbed, 256
thickness of, 294
joist (*see* ribbed, *below*)
loads on, 244
precast, 253
ribbed, 247
with clay-tile fillers, 250
design of, 286
reinforcement, 285
with steel-pan fillers, 248
design of, 278
two-way slab, 254
weights of, 279, 286
slab-band, 247
steel deck, 254
steel joists, 251
surfaces, 258
two-way slab, 254
ribbed, 254
Footings, 207
area proportioning, 209
cantilever, 224
design of, 235
combined, 213, 224–240
design of, 225, 230
concrete, factors affecting design of, 214
plain, 221
connected, 224
design of, 235
design loads of, 209
live-load reduction for, 209
multiple-column, 224
on piles, 240
pressure distribution under, 212
single-column (*see* Single-column footings)
size, effect of, on settlement, 211
strap, 213, 224
Footings, strap, design of, 235
types of, 213
wall, 213, 215
Foundations, 207
mat, 213
raft, 213
spread, 207
types of, 213
Frames, building (*see* Building frames)
rigid, 151
hinges for, 375
moment-distribution method, 181
preliminary design, 204
slope-deflection method, 166, 168, 171
Freezing prevention, 22
Frost action on soils, 330
Frost depth, 214

G

Girders (*see* Beams)
Gravel, 5

H

Highway bridges, 387
AASHO requirements, 389
abutments, 397
beams, 406, 410, 415, 416
bearings, 400
composite, 388, 418
properties of, 420
shear connectors, 419, 423
deck-girder, 387
design of, 404
details of, 412
construction, 398–402, 413, 418
impact, 393
joints in, 400
live loads, 389–393
distribution of, 393, 396
moments in, 394, 403, 405, 406, 414
prestressed, 388, 448, 453
slab, 387
design of, 402–404
details of, 404–405, 414
through-girder, 387
design of, 413
details of, 417
types, 387
wheel-load distribution, 393, 396

Hinges, 375
in arches and rigid frames, 353, 375
Considère, 376
function of, 375
Mesnager, 375
design of, 375–377
steel, 375

I

Impact, 393

J

Jensen, V. P., 428
Joints, expansion and contraction, in retaining walls, 338
in highway bridges, 400
Joists (*see* Beams; Floors, ribbed)

L

Laitance, 20
Lateral ties, arrangement of, 128
size and spacing for columns, 128
Live-load reduction in tier buildings, 209
Loads, building frames, horizontal, 197
vertical, 186, 189
design, of footings, 209
on floor and roof, 244
on highway bridges, 389-393
truck, 389, 393, 396
ultimate, theory of, 445, 449, 454
on walls, 260
weight of ribbed floors, 279, 286

M

Modulus of elasticity, of concrete, 26, 429
of steel, 429
Moisture absorbed by aggregates, 10, 15, 17
Moment-area principles, 153
Moment-distribution method, 173–185
carry-over factors, 175
continuous beams, 178, 180
distribution factors, 175, 178
fixed-end moments, 174, 177
rigid frames, 181
sidesway, 173, 181, 185
unbalanced moments, 174
Moments, in beams, 49–51
Moments, bending, in flat-slab floors, 289–294, 299, 301
in continuous beams, 152, 153
fixed-end (*see* Fixed-end moments)
in highway bridges, 394, 403, 405, 406, 414
of inertia, arches, 367
column sections, 474
column verticals, 477, 478
effective, 202
two-hinged arches (table), 369
Mortar mixtures, density of, 12
design of, 10
permeability of, 11
strength of, 11
workability of, 13
yield of, 13

P

Partitions (*see* Walls)
Piles, footings on, 240
Plastic design (*see* Ultimate design)
Plasticity, in arches, influence of, 357
of concrete, in beams with compression reinforcement, 106
in column design, 123
relief, by, of overstress, 205
of settlement stresses, 209
Plasticity ratio in ultimate design, 429
Portal method, 198
Postensioning (*see* Prestressed concrete)
Precasting, 444
floors, 253
Preliminary design, 204
of arches, 367
fixed, 366, 367
two-hinged, 379
of building frames, 205
of single-story frames, 206
Pressure distribution in soils, 212
Prestressed concrete, 441–455
allowable stresses, 447
cracking, 443, 449, 451
creep, 443
final prestress, 443
highway bridges, 388
high-strength steel, 446, 448
history of, 443
initial prestress, 442
postensioned beams, 450
advantages of, 451

Prestressed concrete, postensioned beams, anchorage, 450
bond, 450
design of, 453
Freyssinet method, 450
grouting, 450
Magnel method, 450
theory of, 451
precasting, 444, 447
pretensioned beams, bond, 446
design of, 448
theory of, 446
principles of, 441
safety factor in, 449, 450
shrinkage, 443
stress distribution, 443
theory of, 444
bending stresses, 445
diagonal tension, 415, 450
shear stresses, 415
ultimate loads, 445, 419, 454
wires, sizes and properties of, 448
Prestressed highway bridges, 388, 448, 453
Pretensioning (*see* Prestressed concrete)

R

Reinforced concrete, advantages of, 35
in tension, 40
Reinforcement, anchoring of, 92
coefficient of expansion, 35
column, 126, 127
moment of inertia of, 177, 478
deformed bars, 37
deformations of, 39
length, quantity, and size extras, 33
tables, 468, 469
embedment for bond, 39
expanded metal, 34
in flat-slab floors, 294, 324–326
grade of steel, 34
modulus of elasticity, 35
placing of, in beams, 51
in slabs, 64
in ribbed floors, 285
in single-column footings, 218
in slab-beam-and-girder floors, 276
temperature, 40
ultimate strength of, 35
web (*see* Web reinforcement)
Reinforcement, wire fabric, 34
Retaining walls, 327
bearing pressure, 208, 336, 338
bridge abutments, 397
cantilever, 327
design of, 340
counterfort, 327
design of, 346
drainage, 327
importance of, 330
provision for, 337
expansion and contraction joints in, 338
external stability of, 334
overturning, 337
sliding, 334
gravity, 327
design of, 338
types of, 327
Rigid frames (*see* Frames)
Roofs, arch, 350–352
design of, 377
shape of axis, 367
general requirements, 258
loads on, 244
(*See also* Floors)

S

Safety factor, in conventional design, 425
in prestressed concrete, 449, 450
in ultimate design, 425
Sand, 3
selection of, 6
standard Ottawa, 8
tests of, 6
Settlement, 207
differential, 209
effect of footing size on, 211
uniform, 209
design for, 210
Shearing stresses, 78
allowable unit, 85, 459
in haunched members, 349
in nonprismatic members, 349
in plain concrete beams, 78
in reinforced concrete beams, 78
in wedge-shaped members, 349
Sidesway, 169
in building frames, 189

Sidesway, moment-distribution method, 173, 181, 185
slope-deflection method, 169–172
Single-column footings, 217–224
bending moments, 218
bond, 219
cap, 217
design of square footing, 222, 311
dowels, 217, 221
under octagonal columns, 221
pedestal, 217, 218
plain concrete, 221
rectangular, 313
for rectangular columns, 313
reinforcement, 218
under round columns, 221, 311
shear, 219
sloped, 217
stepped, 217
stress at base of column, 221
weight of, 217
Slabs, 62
flat (*see* Floors)
highway bridges, 402, 405, 414
one-way, design of, 267
stairway, design of, 319
supported on four sides, 69
design of, 71, 74
moment coefficients for, 73
supported on three sides, moments in, 346
supported on two sides, 63
design of, 65
temperature reinforcement in, 64
under triangular and trapezoidal load, 346
Slag, 5
Slope-deflection method, 154–172
continuous beams, 162, 165
restraint conditions, 159
table of, 161
sign convention, 156, 158
Slump test, 14
Soils, 207
angle, of internal friction, 329
table of, 331
of repose, 328
bearing capacity, 207
table of allowable, 208
cohesion, 328, 330
friction with concrete, table, 331
frost action, 330
Soils, frost depth, 214
pressure distribution in, 212
unit weight, table, 331
Span, effective, 201
Stairways, beams around, 265, 320
building-code requirements, 265
construction details, 266
layout, 264
Standard hook, 92
Steel, modulus of elasticity, 429
stress-strain diagram, of prestressing, 446
of reinforcing, 124, 428
(*See also* Reinforcement)
Stirrups, inclined, size of, 89
spacing of, 88
vertical, size of, 89
spacing of, 88, 89
Stress-strain diagram, of concrete, 26, 124, 426, 428
of prestressing steel, 446
of reinforcing steel, 124, 428
Stresses, allowable (*see* Allowable stresses)
bending (*see* Bending and axial stress)
bond, 36, 39, 459
flexural, in columns (*see* Columns)
in homogeneous beams, 11
settlement, relief by plasticity, 209
shearing (*see* Shearing stresses)
in single-column footings, 218
at base of column, 221
Substructures, 207
Support conditions, 203
Surface area, 3, 8

## T

T beams, 93
design of, 99, 100
in bridges, 406
in buildings, 270, 273
over supports, 116
diagrams for review, 98, 482
flexure, 93–105
formulas for, 95
proportions of, 98
shearing strength of, 97
types of, 98
Tables, bars, area, perimeters, and weights, 468, 469
columns, allowable concentric loads, 131, 472, 473

Tables, columns, areas, weights, and moment of inertia, 131, 472–474
pitch and size of spirals, 131, 475
rectangular beam, design, 56, 470
review, 56, 471
Talbot, A. N., 424
Three moments, theorem of, 152
Transformed section, members subject to bending and direct stress, 135
rectangular beams, 58
design, 61
review, 60
Truck loads, 389
distribution of, 393, 396
Two-hinged arches, 351
critical load positions, 371
design of roof arch, 377
anchor plate, 385
final, 381
hinges, 386
preliminary, 379
moments and reactions (table), 369
preliminary design, 367
theory of, 354
integral evaluation, 366
rib shortening, 356, 358, 375
shrinkage, 357
temperature changes, 357, 375, 381
tie rods, 352, 358, 381
tied, 352

U

Ultimate design, 424–441
columns, 123–126
eccentrically loaded, 437
difference between elastic design and, 432, 433
Jensen-PCA method, 428
plasticity ratio, 429
practical significance of, 438
compression steel, 438
high-strength concrete, 439
high-strength steel, 440, 441
safety factor in, 425
Ultimate design, stress and strain tribution, 427, 430, 432, 434
Whitney method, for beams, 434
design chart, 437
for eccentrically loaded columns,
Unit stresses (*see* Allowable stresses)

V

Vibrators, 20

W

Wall footings, 213, 215
design of, 215
Walls, basement, 262
bearing, 261
building code requirements, 260–26
covering, 263
loads on, 260
panel and curtain, 259
parapet, 262
partitions, 263
retaining (*see* Retaining walls)
Water absorbed by aggregates, 15, 18
Water-cement ratio, 13
Web reinforcement, 83
anchorage of, 92
bent bars, 89
arrangement of, 89
design of, 87
stirrups, inclined, 88, 89
vertical, 88, 89
types of, 83
where not required, 85
Whitney, C. S., 124, 434, 357
Wind-load analysis, 197, 198
Workability of mixtures, 13–15
Working stresses (*see* Allowable stress

Y

Yield, of concrete mixtures, 17
of mortar mixtures, 13
Young's modulus, 26, 429